Hermitian Inner Product Space (3B)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Complex Vector Inner Product

Hermitian inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^H \cdot \boldsymbol{y} = \sum x_i^* y_i$$

$\boldsymbol{x}^H$: conjugate transpose

Norm of Hermitian inner products

$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} = \sqrt{\boldsymbol{x}^H \cdot \boldsymbol{x}} = \sqrt{\sum x_i^* x_i}$$

the length of a vector

$$\boldsymbol{x} = \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix}$$

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = \boldsymbol{x}^H \cdot \boldsymbol{x} = \sum x_i^* x_i$$

$$\begin{pmatrix} a_1 - j b_1 & a_2 - j b_2 & \cdots & a_n - j b_n \end{pmatrix} \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix} = \sum_{i=1}^{n} a_i^2 + b_i^2$$

Cauchy-Schwartz Inequality

For all vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of an inner product space

$$\left|\langle \boldsymbol{x}, \boldsymbol{y}\rangle\right|^2 \le \langle \boldsymbol{x}, \boldsymbol{x}\rangle \cdot \langle \boldsymbol{y}, \boldsymbol{y}\rangle$$

$$\left|\langle \boldsymbol{x}, \boldsymbol{y}\rangle\right| \le \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\|$$

The equality holds if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly dependent ⇨ maximum

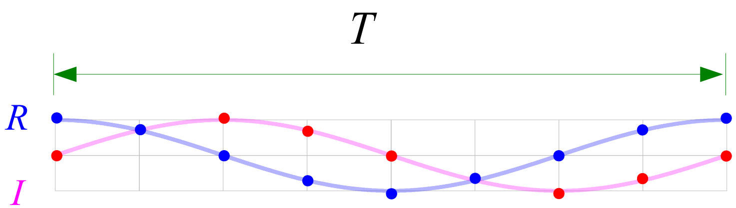
$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \boldsymbol{x}^H \cdot \boldsymbol{y} \le \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\| \qquad \boldsymbol{x} = \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix} \qquad \boldsymbol{y} = k \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix}$$

Inner product is maximum
when $\boldsymbol{y} = k\,\boldsymbol{x}$

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle \le k\left(\sum_{i=1}^{n} a_i^2 + b_i^2\right)$$

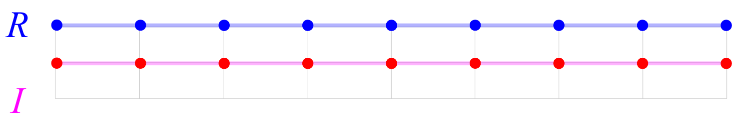
Inner Product Examples (1)

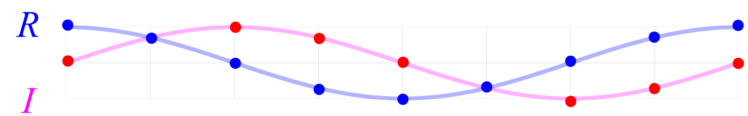
$f_0 = 1/T \qquad \omega_0 = 2\pi/T$



$e^{j\mathbf{1}\omega_0 t}$

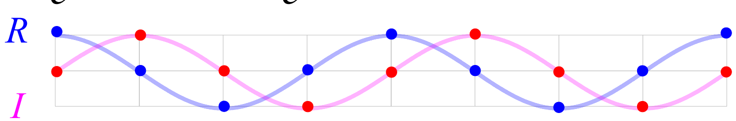
$e^{+j(\mathbf{1}-\mathbf{1})\omega_0 t} = e^{+j\mathbf{0}\omega_0 t}$

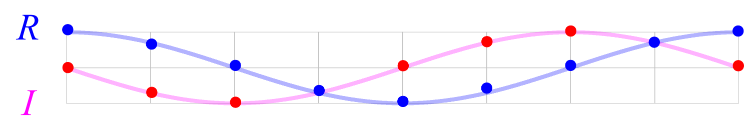




$e^{j\mathbf{1}\omega_0 t}$

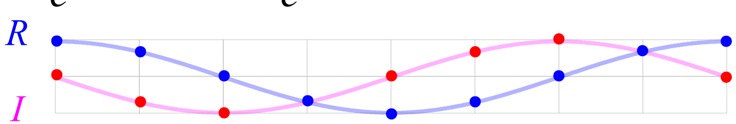
$e^{+j(\mathbf{1}+\mathbf{1})\omega_0 t} = e^{+j\mathbf{2}\omega_0 t}$

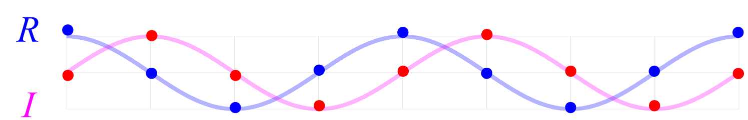




$e^{j(-\mathbf{1})\omega_0 t}$

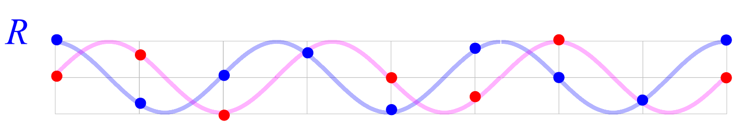
$e^{+j(\mathbf{1}-\mathbf{2})\omega_0 t} = e^{+j(-\mathbf{1})\omega_0 t}$

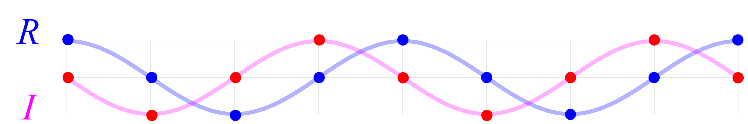




$e^{j\mathbf{2}\omega_0 t}$

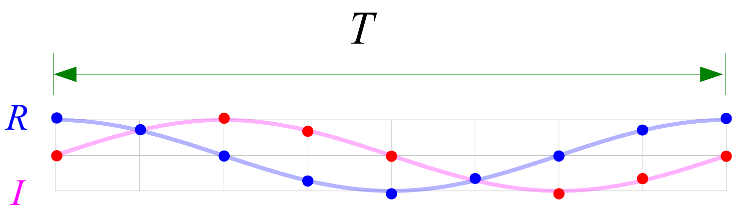
$e^{+j(\mathbf{1}+\mathbf{2})\omega_0 t} = e^{+j\mathbf{3}\omega_0 t}$





$e^{j(-\mathbf{2})\omega_0 t}$

Inner Product Examples (2)



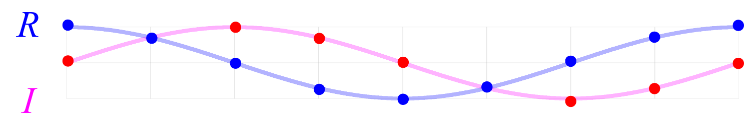
$f_0 = 1/T \qquad \omega_0 = 2\pi/T$

$e^{j\mathbf{1}\omega_0 t}$

$\langle \boldsymbol{r}_1, \boldsymbol{r}_1 \rangle = \boldsymbol{r}_1^H \cdot \boldsymbol{r}_1 = 8$

$\boldsymbol{r}_1^H = (1 \quad \frac{1+j}{\sqrt{2}} \quad +j \quad \frac{-1+j}{\sqrt{2}} \quad -1 \quad \frac{-1-j}{\sqrt{2}} \quad -j \quad \frac{1-j}{\sqrt{2}})$

$\boldsymbol{r}_1 = (1 \quad \frac{1-j}{\sqrt{2}} \quad -j \quad \frac{-1-j}{\sqrt{2}} \quad -1 \quad \frac{-1+j}{\sqrt{2}} \quad +j \quad \frac{1+j}{\sqrt{2}})^T$

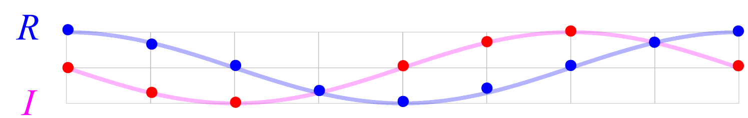


$e^{j\mathbf{1}\omega_0 t}$

$\langle \boldsymbol{r}_1, \boldsymbol{r}_{-1} \rangle = \boldsymbol{r}_1^H \cdot \boldsymbol{r}_{-1} = 0$

$\boldsymbol{r}_1^H = (1 \quad \frac{1+j}{\sqrt{2}} \quad +j \quad \frac{-1+j}{\sqrt{2}} \quad -1 \quad \frac{-1-j}{\sqrt{2}} \quad -j \quad \frac{1-j}{\sqrt{2}})$

$\boldsymbol{r}_1 = (1 \quad \frac{1+j}{\sqrt{2}} \quad +j \quad \frac{-1+j}{\sqrt{2}} \quad -1 \quad \frac{-1-j}{\sqrt{2}} \quad -j \quad \frac{1-j}{\sqrt{2}})^T$

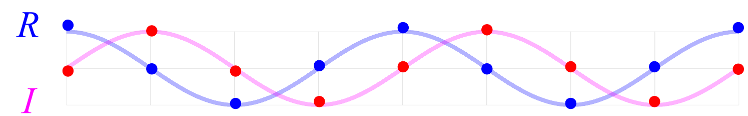


$e^{j(\mathbf{-1})\omega_0 t}$

$\langle \boldsymbol{r}_1, \boldsymbol{r}_2 \rangle = \boldsymbol{r}_1^H \cdot \boldsymbol{r}_2 = 0$

$\boldsymbol{r}_1^H = (1 \quad \frac{1+j}{\sqrt{2}} \quad +j \quad \frac{-1+j}{\sqrt{2}} \quad -1 \quad \frac{-1-j}{\sqrt{2}} \quad -j \quad \frac{1-j}{\sqrt{2}})$

$\boldsymbol{r}_2 = (1 \quad +j \quad -1 \quad -j \quad +1 \quad +j \quad -1 \quad -j)^T$

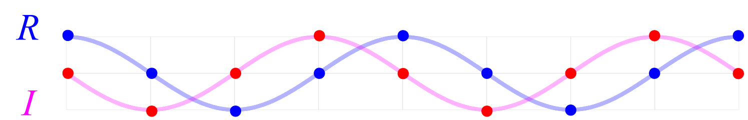


$e^{j\mathbf{2}\omega_0 t}$

$\langle \boldsymbol{r}_1, \boldsymbol{r}_{-2} \rangle = \boldsymbol{r}_1^H \cdot \boldsymbol{r}_{-2} = 0$

$\boldsymbol{r}_1^H = (1 \quad \frac{1+j}{\sqrt{2}} \quad +j \quad \frac{-1+j}{\sqrt{2}} \quad -1 \quad \frac{-1-j}{\sqrt{2}} \quad -j \quad \frac{1-j}{\sqrt{2}})$

$\boldsymbol{r}_2 = (1 \quad -j \quad -1 \quad +j \quad +1 \quad -j \quad -1 \quad +j)^T$



$e^{j(\mathbf{-2})\omega_0 t}$

N=8 DFT Matrix in Cosine and Sine Terms

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} = \cos\left(\frac{\pi}{4}\cdot k\cdot n\right) - j\sin\left(\frac{\pi}{4}\cdot k\cdot n\right)$$

$$\begin{bmatrix}
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 \\
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 1 - j\sin(\pi/4)\cdot 1 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 & \cos(\pi/4)\cdot 3 - j\sin(\pi/4)\cdot 3 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 5 - j\sin(\pi/4)\cdot 5 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 & \cos(\pi/4)\cdot 7 - j\sin(\pi/4)\cdot 7 \\
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 \\
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 3 - j\sin(\pi/4)\cdot 3 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 & \cos(\pi/4)\cdot 1 - j\sin(\pi/4)\cdot 1 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 7 - j\sin(\pi/4)\cdot 7 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 & \cos(\pi/4)\cdot 5 - j\sin(\pi/4)\cdot 5 \\
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 \\
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 5 - j\sin(\pi/4)\cdot 5 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 & \cos(\pi/4)\cdot 7 - j\sin(\pi/4)\cdot 7 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 1 - j\sin(\pi/4)\cdot 1 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 & \cos(\pi/4)\cdot 3 - j\sin(\pi/4)\cdot 3 \\
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 \\
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 7 - j\sin(\pi/4)\cdot 7 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 & \cos(\pi/4)\cdot 5 - j\sin(\pi/4)\cdot 5 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 3 - j\sin(\pi/4)\cdot 3 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 & \cos(\pi/4)\cdot 1 - j\sin(\pi/4)\cdot 1
\end{bmatrix}$$

N=8 DFT Matrix Real and Imaginary Terms

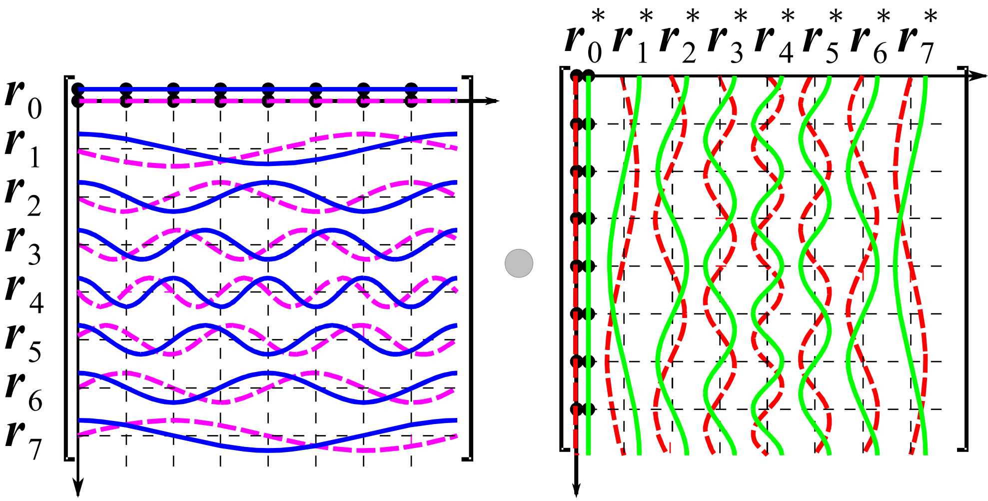
$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} = \cos\left(\frac{\pi}{4}\cdot k\cdot n\right) - j\,\sin\left(\frac{\pi}{4}\cdot k\cdot n\right)$$

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & +\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & +j & +\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} \\
1 & -j & -1 & +j & 1 & -j & -1 & +j \\
1 & -\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & +j & +\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & -1 & +\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & -j & +\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & -1 & +\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & +j & -\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} \\
1 & +j & -1 & -j & 1 & +j & -1 & -j \\
1 & +\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & +j & -\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & -j & +\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}
\end{bmatrix}
\begin{matrix}
\leftarrow \boldsymbol{r}_0 \\
\leftarrow \boldsymbol{r}_1 \\
\leftarrow \boldsymbol{r}_2 \\
\leftarrow \boldsymbol{r}_3 \\
\leftarrow \boldsymbol{r}_4 \\
\leftarrow \boldsymbol{r}_5 = \boldsymbol{r}_{-3} \\
\leftarrow \boldsymbol{r}_4 = \boldsymbol{r}_{-2} \\
\leftarrow \boldsymbol{r}_5 = \boldsymbol{r}_{-1}
\end{matrix}$$

Orthogonality

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$

$$\begin{cases} \boldsymbol{A}^H = \boldsymbol{B} \\ \boldsymbol{B}^H = \boldsymbol{A} \end{cases} \quad \begin{cases} \boldsymbol{AB} = N\,\boldsymbol{I} \\ \boldsymbol{BA} = N\,\boldsymbol{I} \end{cases} \quad \Rightarrow \quad \begin{cases} \boldsymbol{A}^H\boldsymbol{A} = \boldsymbol{A}\,\boldsymbol{A}^H = N\,\boldsymbol{I} \\ \boldsymbol{B}^H\boldsymbol{B} = \boldsymbol{B}\,\boldsymbol{B}^H = N\,\boldsymbol{I} \end{cases}$$



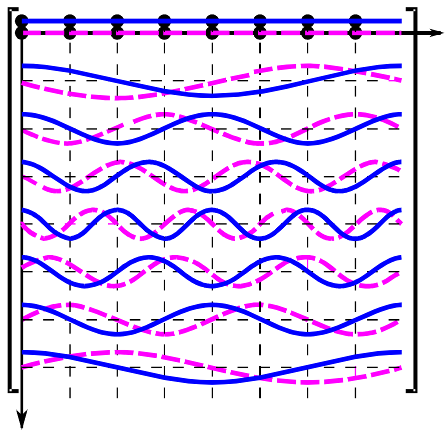
$$= \begin{bmatrix}
r_0\cdot r_0^* & r_0\cdot r_1^* & r_0\cdot r_2^* & r_0\cdot r_3^* & r_0\cdot r_4^* & r_0\cdot r_5^* & r_0\cdot r_6^* & r_0\cdot r_7^* \\
r_1\cdot r_0^* & r_1\cdot r_1^* & r_1\cdot r_2^* & r_1\cdot r_3^* & r_1\cdot r_4^* & r_1\cdot r_5^* & r_1\cdot r_6^* & r_1\cdot r_7^* \\
r_2\cdot r_0^* & r_2\cdot r_1^* & r_2\cdot r_2^* & r_2\cdot r_3^* & r_2\cdot r_4^* & r_2\cdot r_5^* & r_2\cdot r_6^* & r_2\cdot r_7^* \\
r_3\cdot r_0^* & r_3\cdot r_1^* & r_3\cdot r_2^* & r_3\cdot r_3^* & r_3\cdot r_4^* & r_3\cdot r_5^* & r_3\cdot r_6^* & r_3\cdot r_7^* \\
r_4\cdot r_0^* & r_4\cdot r_1^* & r_4\cdot r_2^* & r_4\cdot r_3^* & r_4\cdot r_4^* & r_4\cdot r_5^* & r_4\cdot r_6^* & r_4\cdot r_7^* \\
r_5\cdot r_0^* & r_5\cdot r_1^* & r_5\cdot r_2^* & r_5\cdot r_3^* & r_5\cdot r_4^* & r_5\cdot r_5^* & r_5\cdot r_6^* & r_5\cdot r_7^* \\
r_6\cdot r_0^* & r_6\cdot r_1^* & r_6\cdot r_2^* & r_6\cdot r_3^* & r_6\cdot r_4^* & r_6\cdot r_5^* & r_6\cdot r_6^* & r_6\cdot r_7^* \\
r_7\cdot r_0^* & r_7\cdot r_1^* & r_7\cdot r_2^* & r_7\cdot r_3^* & r_7\cdot r_4^* & r_7\cdot r_5^* & r_7\cdot r_6^* & r_7\cdot r_7^*
\end{bmatrix}$$

$$\langle \boldsymbol{r}_i^H, \boldsymbol{r}_i^H \rangle = \boldsymbol{r}_i \cdot \boldsymbol{r}_i^* = N$$

$$\langle \boldsymbol{r}_i^H, \boldsymbol{r}_j^H \rangle = \boldsymbol{r}_i \cdot \boldsymbol{r}_j^* = 0 \quad (i \neq j)$$

N=8 DFT : Inner Product X[0]

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix} =$$



$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

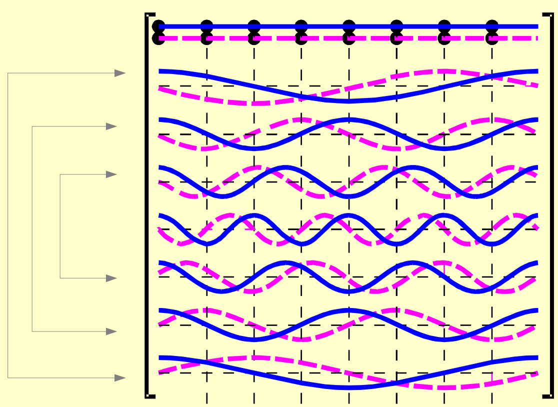
$\boldsymbol{x}$

X[0] measures "0 cycle" component in x

$$\langle \boldsymbol{r}_0^H, \boldsymbol{x} \rangle = \boldsymbol{r}_0 \cdot \boldsymbol{x} \le \|\boldsymbol{r}_0^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_0^H$

When $\boldsymbol{x}$ *looks like this,* $\boldsymbol{X}[0]$ *is max.*

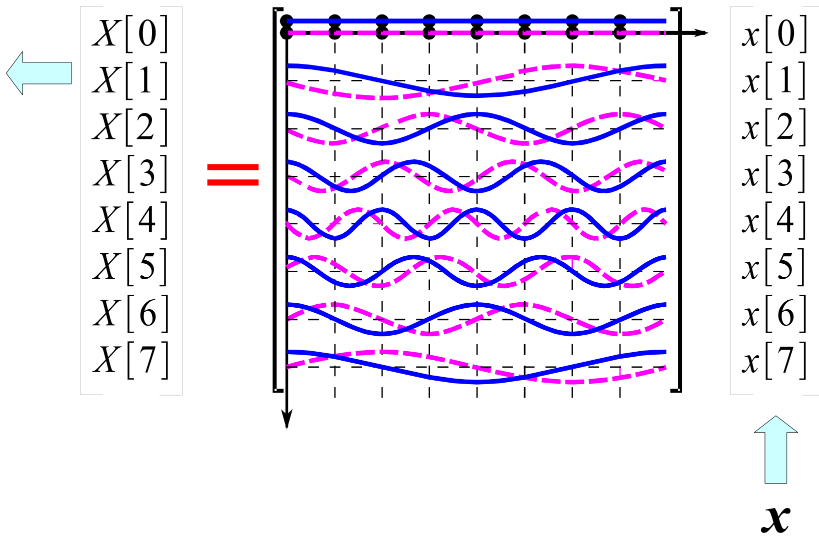


0 cycle
-1 cycles
-2 cycles
-3 cycles
-4 cycles
+3 cycles
+2 cycles
+1 cycles

$$Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

N=8 DFT : Inner Product X[1]

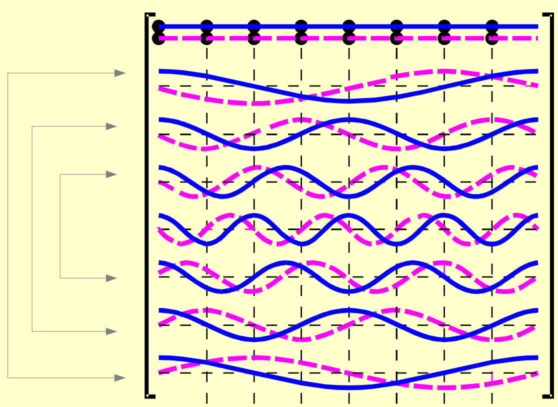


X[1] measures "+1 cycle" component in x

$$\langle \boldsymbol{r}_1^H, \boldsymbol{x} \rangle = \boldsymbol{r}_1 \cdot \boldsymbol{x} \le \|\boldsymbol{r}_1^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_1^H$

*When **x** looks like this, **X**[1] is max.*

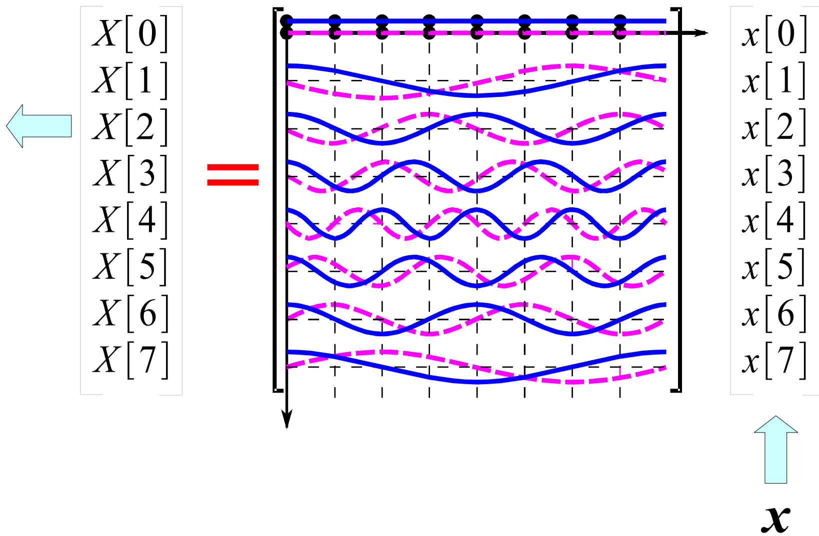


0 cycle
-1 cycles
-2 cycles
-3 cycles
-4 cycles
+3 cycles
+2 cycles
+1 cycles

$$Re\left\{e^{-j\frac{2\pi}{8}kn}\right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{e^{-j\frac{2\pi}{8}kn}\right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

N=8 DFT : Inner Product X[2]

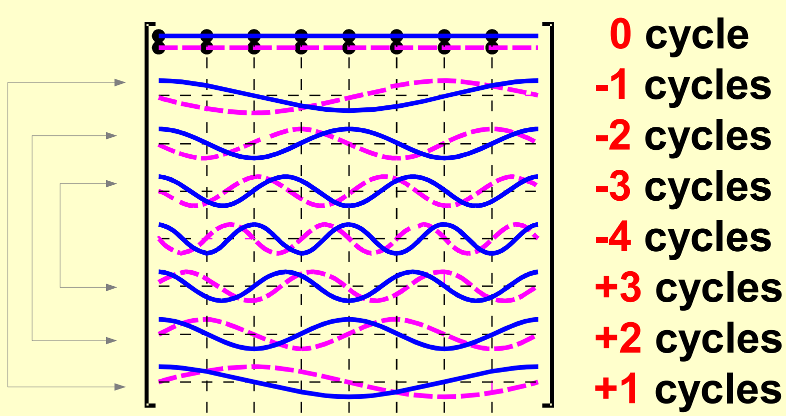


X[2] measures "+2 cycle" component in x

$$\langle \boldsymbol{r}_2^H, \boldsymbol{x} \rangle = \boldsymbol{r}_2 \cdot \boldsymbol{x} \;\leq\; \|\boldsymbol{r}_2^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_2^H$

*When **x** looks like this, **X**[2] is max.*



$$Re\left\{e^{-j\frac{2\pi}{8}kn}\right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{e^{-j\frac{2\pi}{8}kn}\right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

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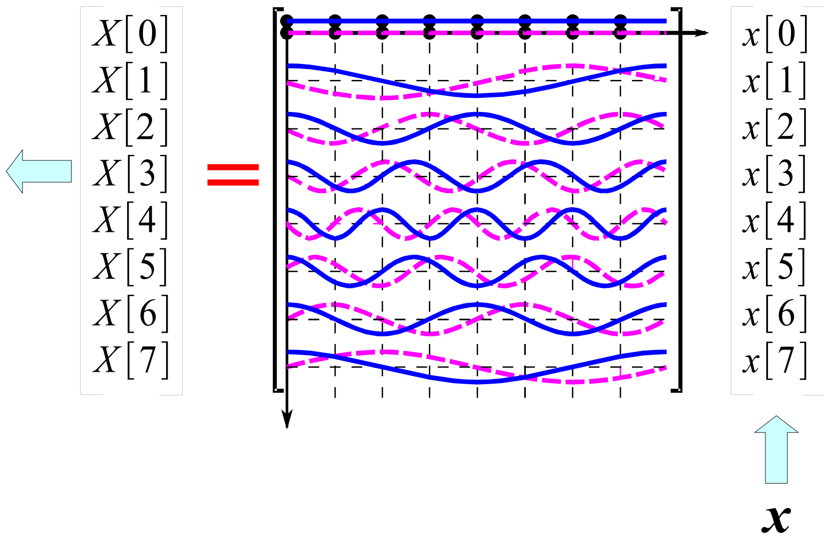
N=8 DFT : Inner Product X[3]

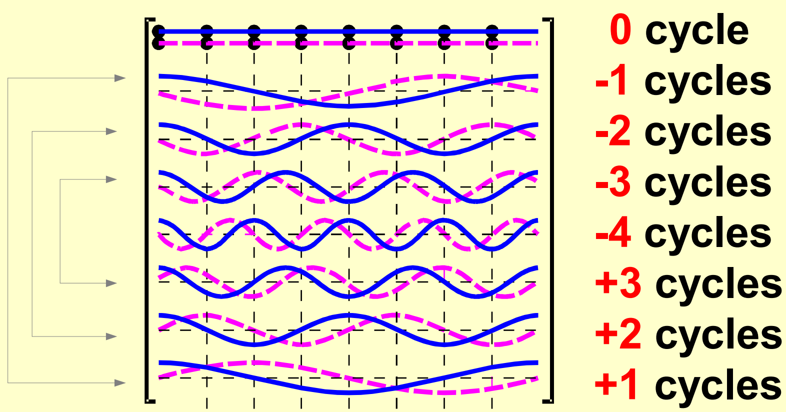


X[3] measures "+3 cycle" component in x

$$\langle \boldsymbol{r}_3^H, \boldsymbol{x} \rangle = \boldsymbol{r}_3 \cdot \boldsymbol{x} \;\le\; \|\boldsymbol{r}_3^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_3^H$

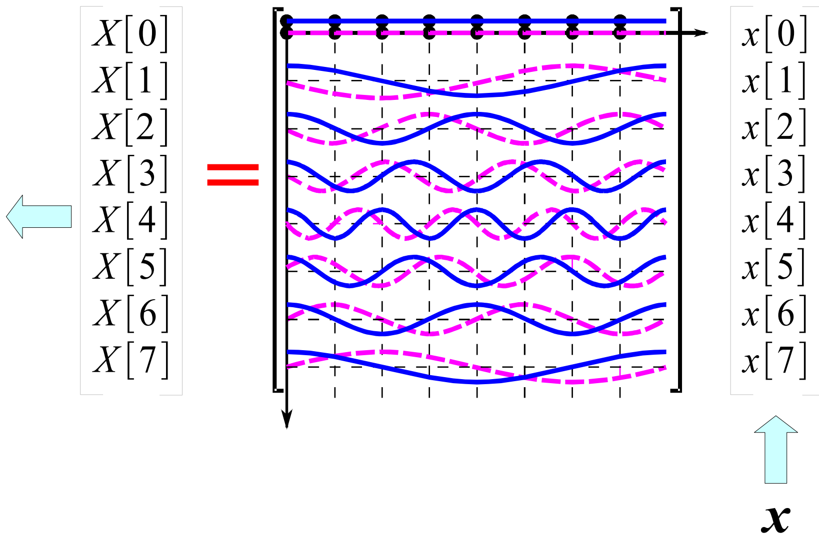
*When **x** looks like this, **X**[3] is max.*



$$Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

N=8 DFT : Inner Product X[4]

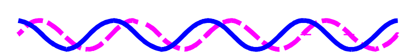


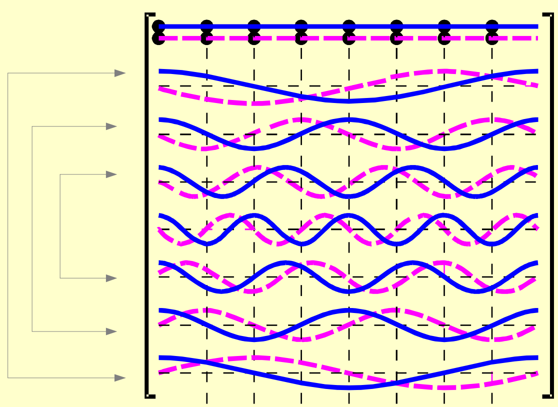
X[4] measures "+4 cycle" component in x

$$\langle \boldsymbol{r}_4^H , \boldsymbol{x} \rangle = \boldsymbol{r}_4 \cdot \boldsymbol{x} \ \leq \ \|\boldsymbol{r}_4^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\, \boldsymbol{r}_4^H$

*When **x** looks like this, **X**[4] is max.*



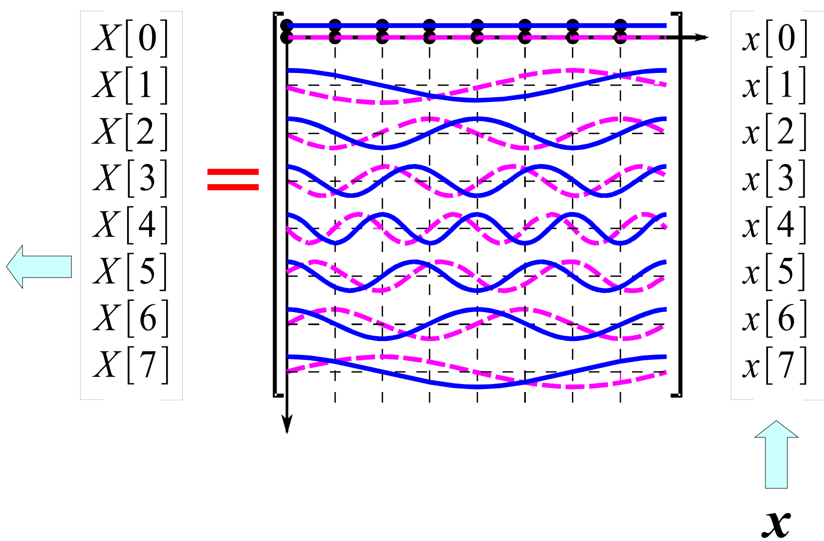


0 cycle
-1 cycles
-2 cycles
-3 cycles
-4 cycles
+3 cycles
+2 cycles
+1 cycles

(solid line) $Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

(dashed line) $Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[5]

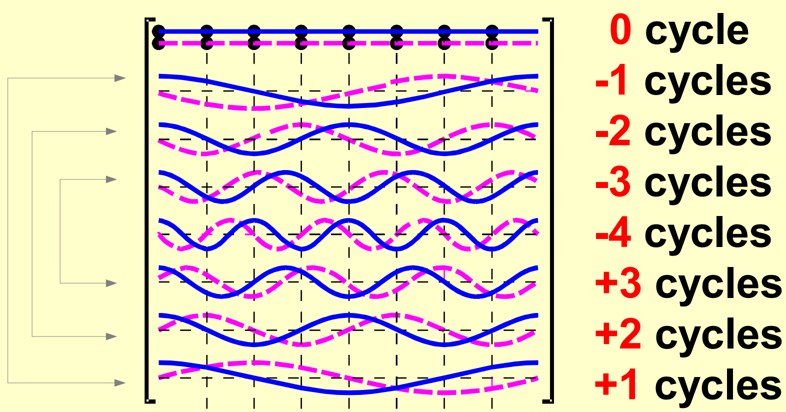


X[5] measures "-3 cycle" component in x

$$\langle \boldsymbol{r}_5^H, \boldsymbol{x} \rangle = \boldsymbol{r}_5 \cdot \boldsymbol{x} \leq \|\boldsymbol{r}_5^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_5^H$

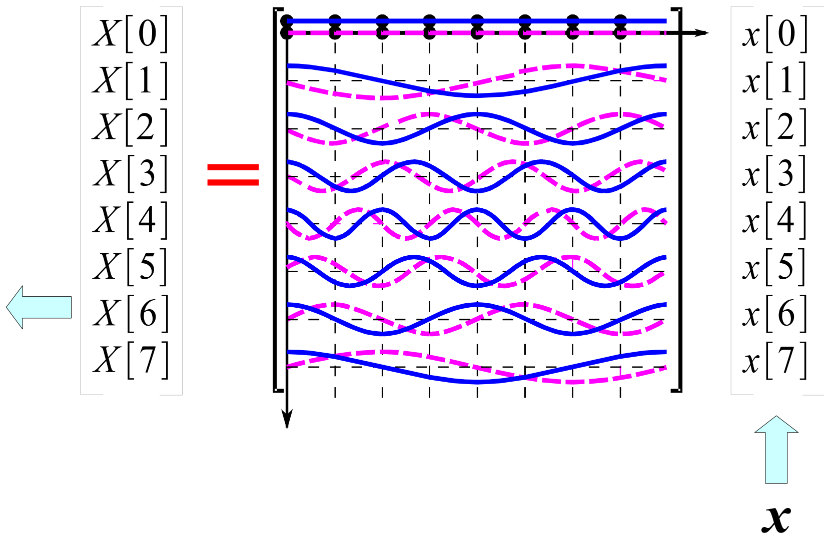
*When **x** looks like this, **X**[5] is max.*



$$Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

N=8 DFT : Inner Product X[6]

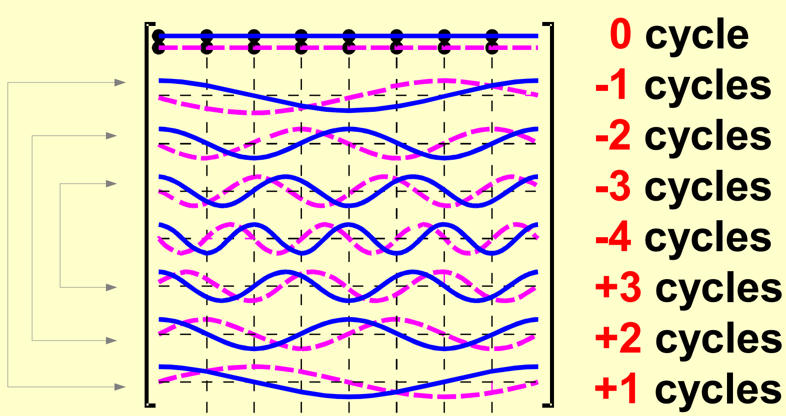


X[6] measures "-2 cycle" component in x

$$\langle \boldsymbol{r}_6^H, \boldsymbol{x} \rangle = \boldsymbol{r}_6 \cdot \boldsymbol{x} \;\le\; \|\boldsymbol{r}_6^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_6^H$

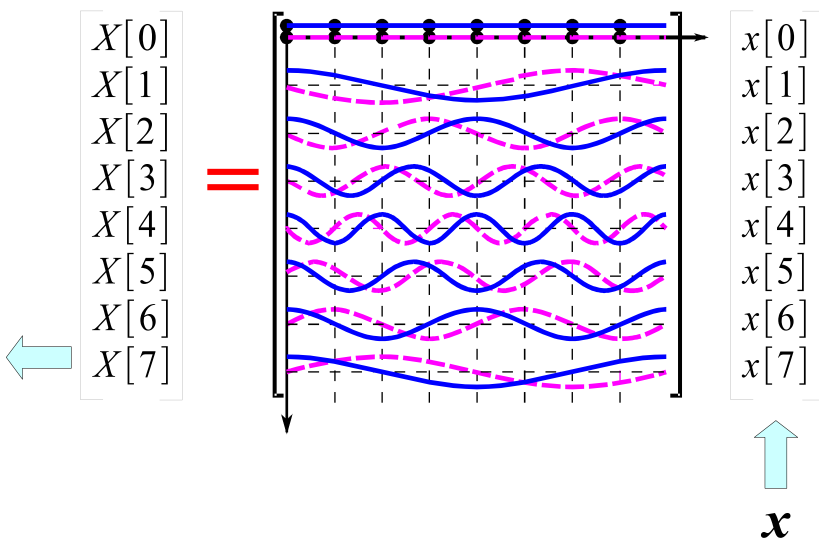
*When **x** looks like this, **X**[6] is max.*



$$Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

N=8 DFT : Inner Product X[7]

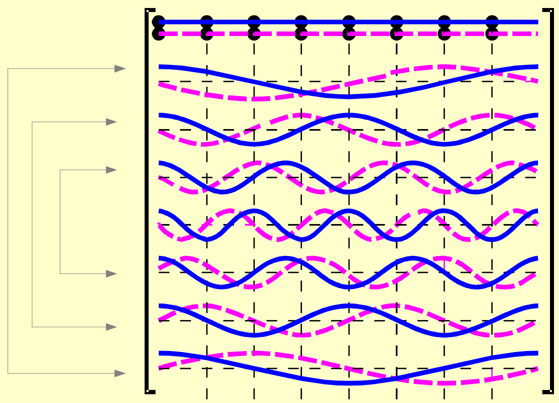


X[7] measures "-1 cycle" component in x

$$\langle \boldsymbol{r}_7^H, \boldsymbol{x} \rangle = \boldsymbol{r}_7 \cdot \boldsymbol{x} \;\le\; \|\boldsymbol{r}_7^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_7^H$

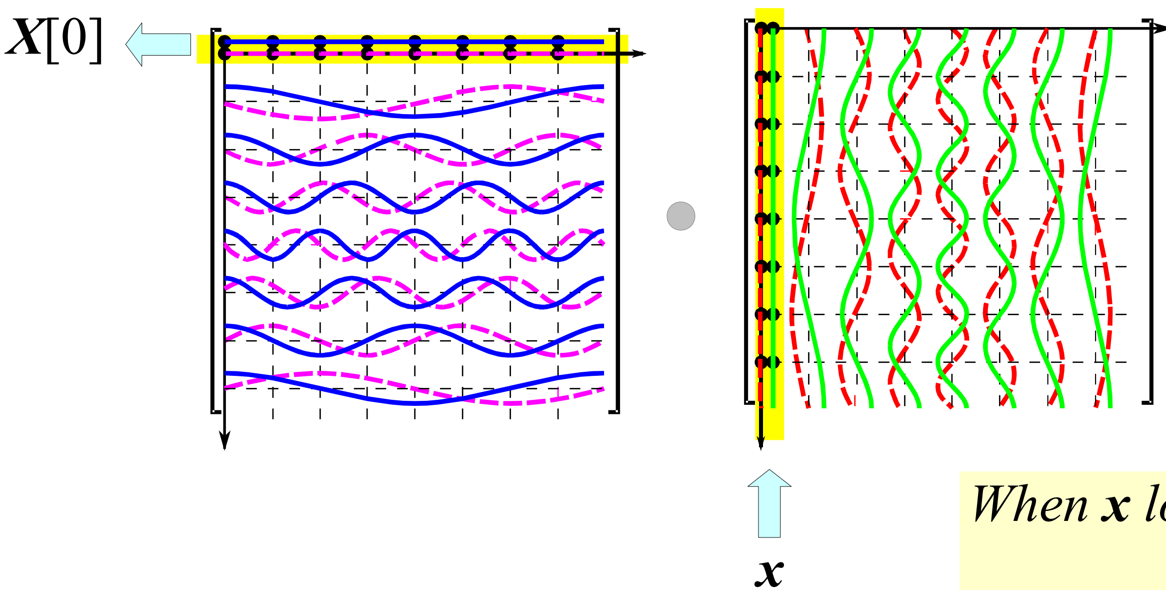
*When **x** looks like this, **X**[7] is max.*



$$Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{A} \cdot \boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U} \cdot \boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

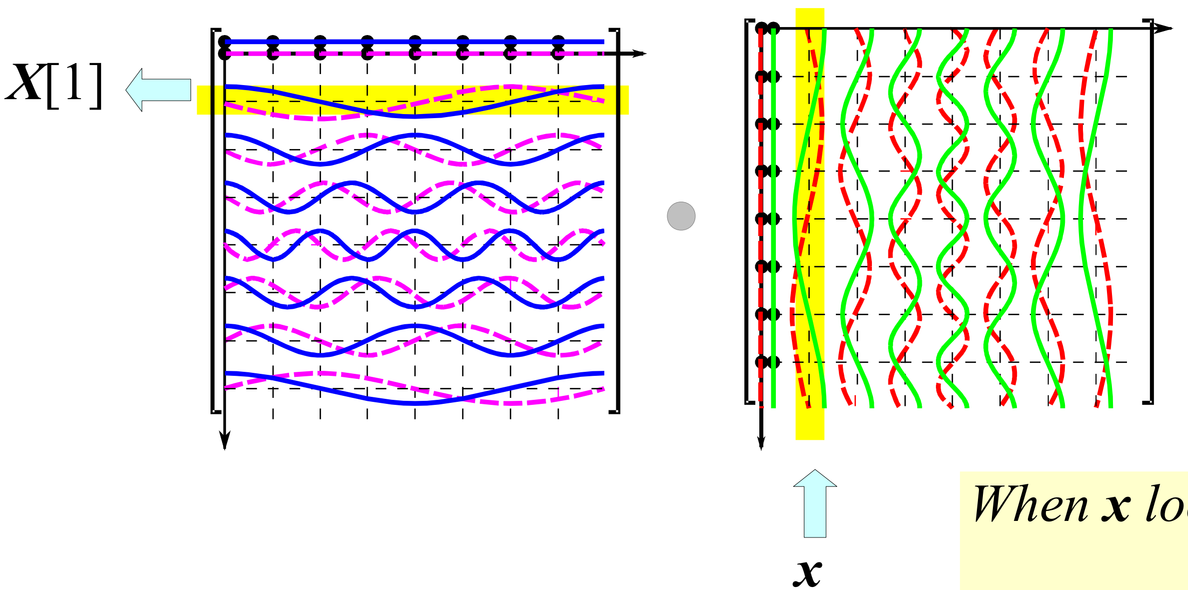
*When **x** looks like this, **X**[0] is max. (=**N**)*

$\boldsymbol{X}[k] = 0 \quad for \;\; k \neq 0$

$$\boldsymbol{X}[0] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[1] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{A}\cdot\boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U}\cdot\boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

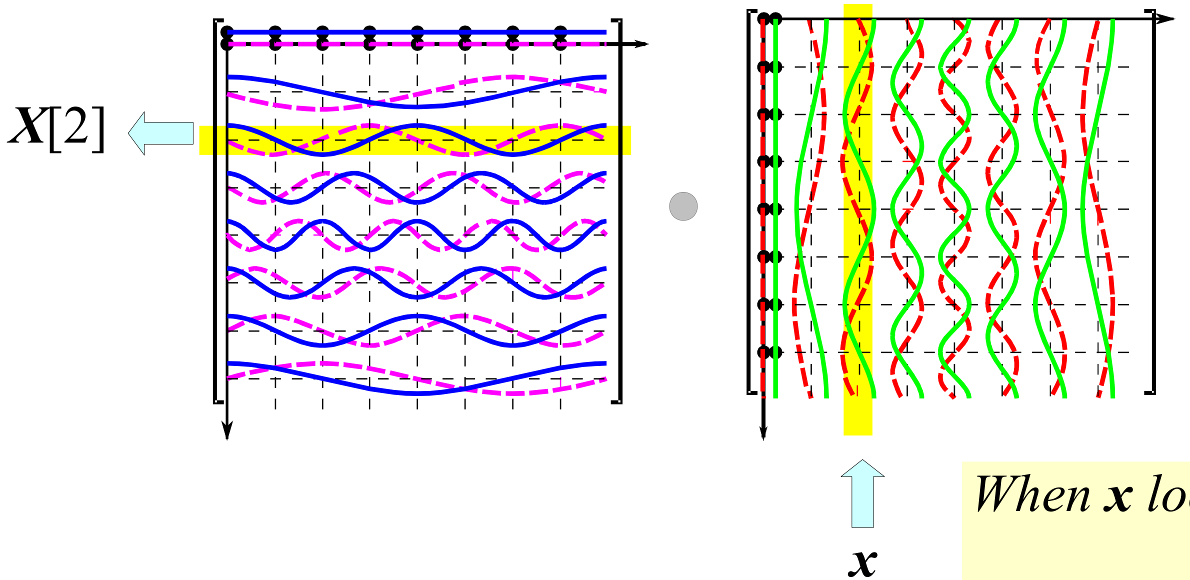
*When **x** looks like this, **X**[1] is max.* (=***N***)

$\boldsymbol{X}[k] = 0 \quad \text{for } k \neq 1$

$$X[1] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 7} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[2] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{A}\cdot\boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U}\cdot\boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

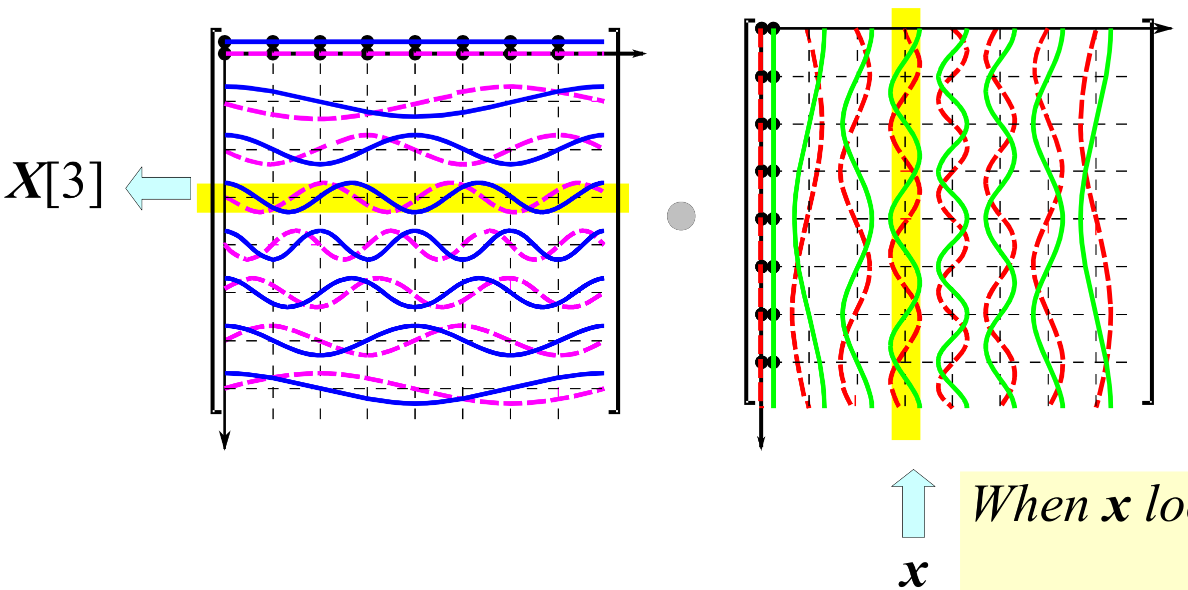
When $\boldsymbol{x}$ *looks like this,* $\boldsymbol{X}[2]$ *is max.* $(=\boldsymbol{N})$

$\boldsymbol{X}[k] = 0$ *for* $k \neq 2$

$$X[2] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 6} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[3] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$



$$A \cdot B = A \cdot A^H = N\, I$$

$$U \cdot U^H = I$$

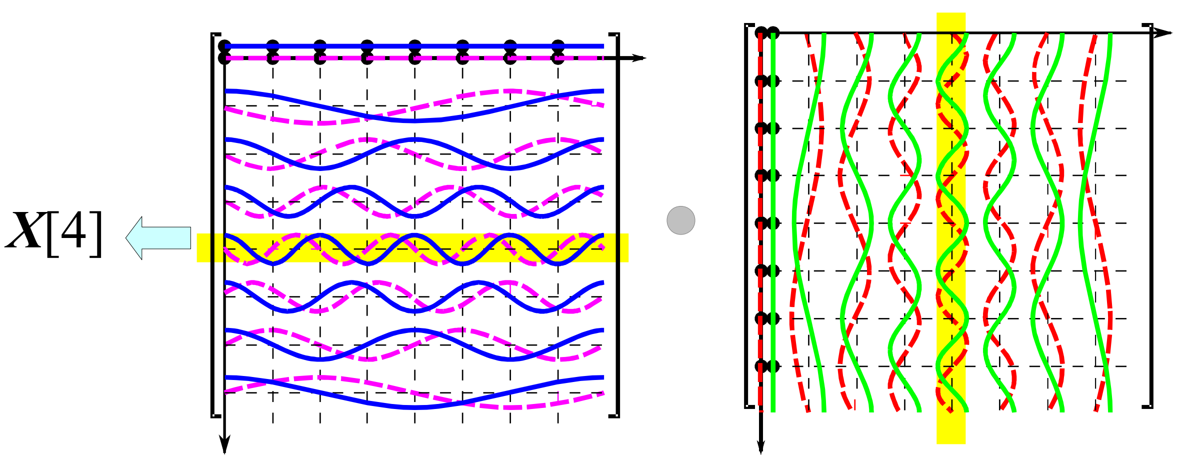
Unitary Matrix

*When **x** looks like this, **X**[3] is max. (=**N**)*

$\boldsymbol{X}[k] = 0 \quad \text{for } k \neq 3$

$$X[3] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 5} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$



$$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{A}\cdot\boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U}\cdot\boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

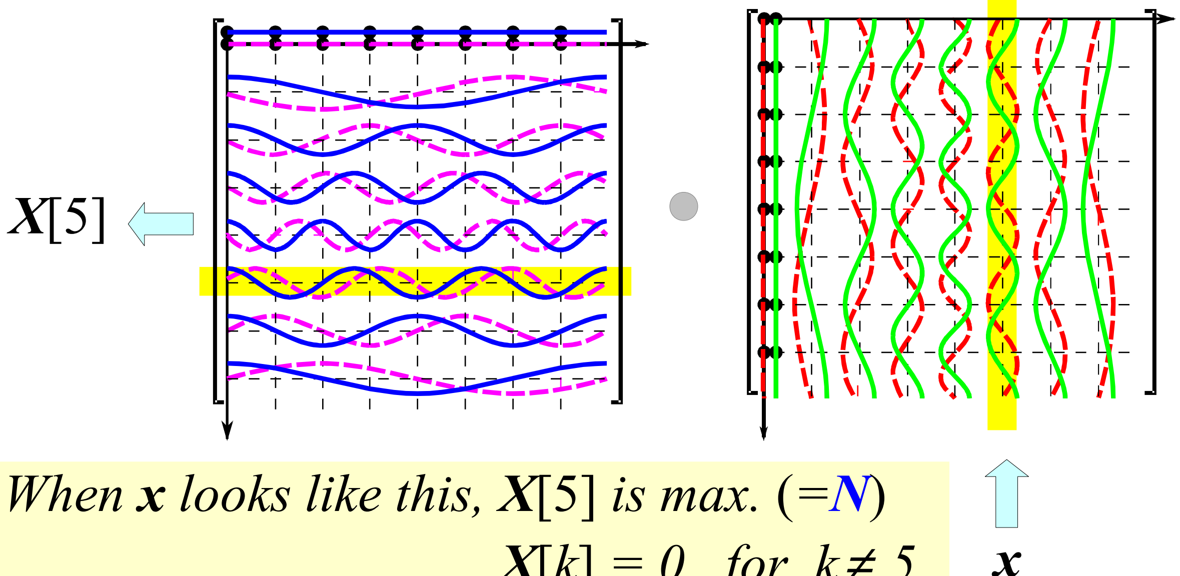
*When **x** looks like this, **X**[4] is max. (=**N**)*
***X**[k] = 0 for k ≠ 4*

x

$$X[4] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[5] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$



$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{A} \cdot \boldsymbol{A}^H = N\,\boldsymbol{I}$

$\boldsymbol{U} \cdot \boldsymbol{U}^H = \boldsymbol{I}$

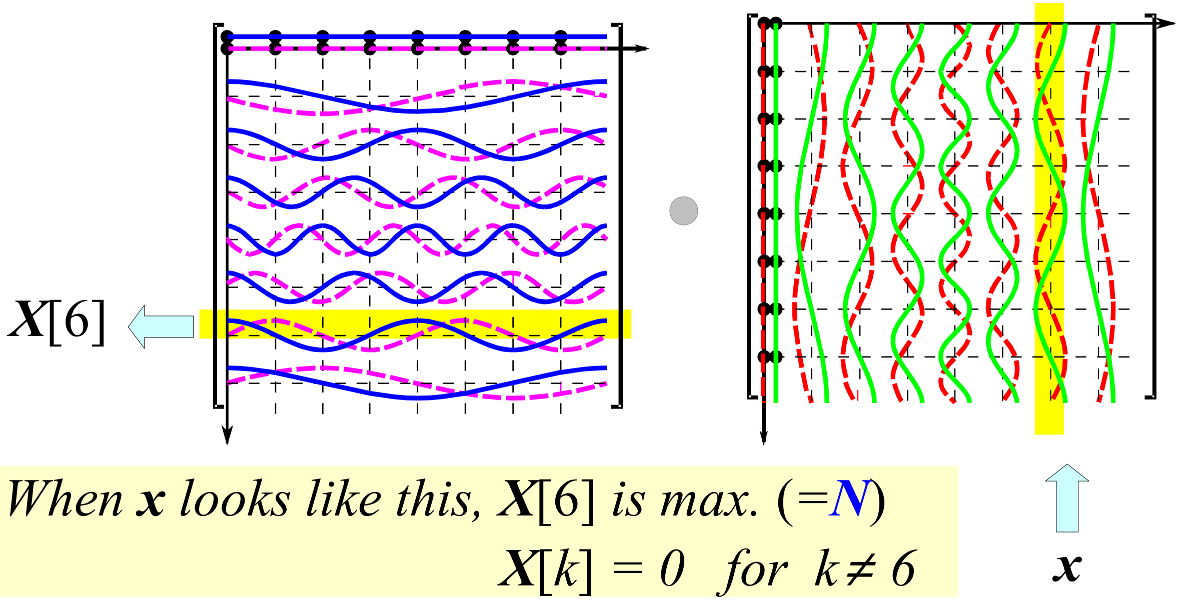
Unitary Matrix

*When **x** looks like this, **X**[5] is max.* (=***N***)

$\boldsymbol{X}[k] = 0 \quad for \;\; k \neq 5$

$$\boldsymbol{X}[5] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 3} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{A}\cdot\boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U}\cdot\boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

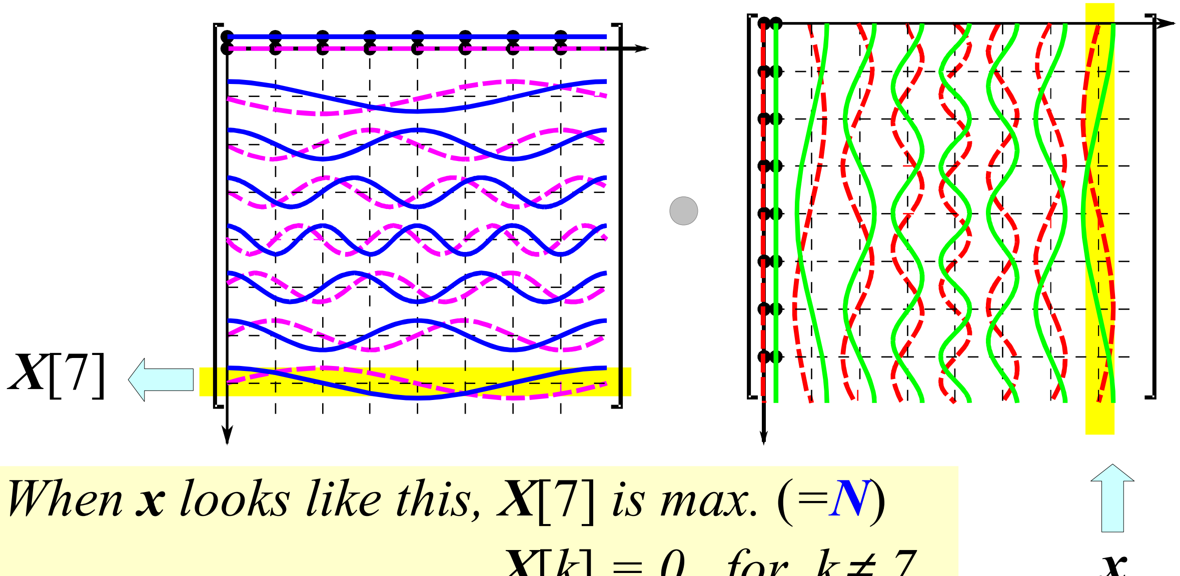
When ***x*** *looks like this,* ***X***[6] *is max. (=****N****)*

X[*k*] = 0 *for* *k* ≠ 6

$$X[6] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[7] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N\, I$$

$$U \cdot U^H = I$$

Unitary Matrix

*When **x** looks like this, **X**[7] is max. (=**N**)*

$X[k] = 0$ for $k \neq 7$

$$X[7] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 1} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A "graphical interpretation" of the DFT and FFT, by Steve Mann