

# Matrices (3A)

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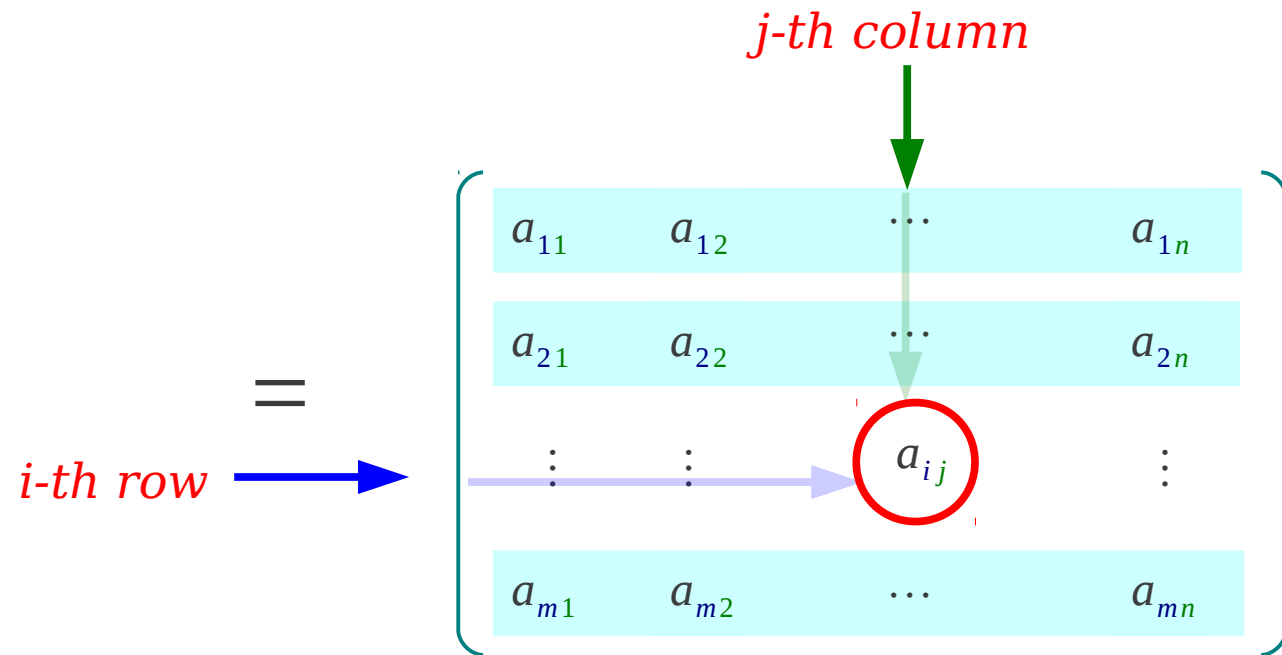
# A General $m \times n$ Matrix

$$A = [a_{ij}]_{m \times n} = [a_{ij}]$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

# A Element of a Matrix

$$A = [a_{ij}]_{m \times n} = [a_{ij}]$$



$$(A)_{m \times n} = a_{ij}$$

# Matrix Multiplication (1)

1st row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

2nd row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

m-th row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\mathbf{A} = [a_{ij}]_{m \times n}$$

$$\mathbf{x} = [x_{ij}]_{n \times 1}$$

$$\mathbf{b} = [b_{ij}]_{m \times 1}$$

$$\mathbf{A} \quad \mathbf{x} = \mathbf{b}$$

$$(m) \times n \quad n \times (1) \quad (m \times 1)$$

# Matrix Multiplication (2)

1st row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

↓ 1st column

$$\begin{aligned}
 \mathbf{A} &= [a_{ij}]_{m \times n} \\
 \mathbf{B} &= [b_{ij}]_{n \times 2} \\
 \mathbf{C} &= [c_{ij}]_{m \times 2}
 \end{aligned}$$

2nd row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

↓ 1st column

$$\mathbf{A} \mathbf{B} = \mathbf{C}$$

$$(m \times n) (n \times 2) = (m \times 2)$$

# Matrix Multiplication (3)

1st row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \\ \vdots \\ \vdots \\ b_{n1} \\ b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

2nd column

$$A = [a_{ij}]_{m \times n}$$

$$B = [b_{ij}]_{n \times 2}$$

$$C = [c_{ij}]_{m \times 2}$$

2nd row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \\ \vdots \\ \vdots \\ b_{n1} \\ b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

2nd column

$$A B = C$$

$$(m \times n) (n \times 2) = (m \times 2)$$

# Matrix Multiplication (4)

*m-th row* →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

*1st column*

$$A = [a_{ij}]_{m \times n}$$

$$B = [b_{ij}]_{n \times 2}$$

$$C = [c_{ij}]_{m \times 2}$$

*m-th row* →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

*2nd column*

$$A \quad B = C$$

$$(m \times n) \quad (n \times 2) \quad (m \times 2)$$



# Matrix Multiplication (4)

*m-th row* →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

*1st column*

$$A = [a_{ij}]_{m \times n}$$

$$B = [b_{ij}]_{n \times 2}$$

$$C = [c_{ij}]_{m \times 2}$$

*m-th row* →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

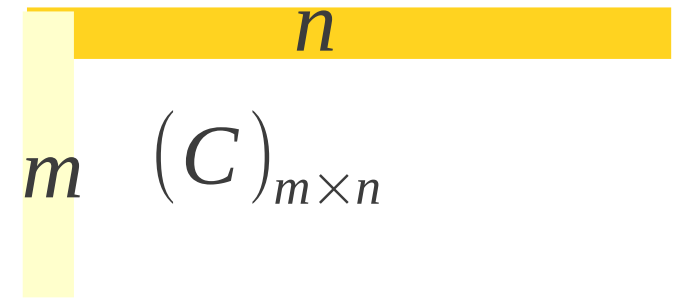
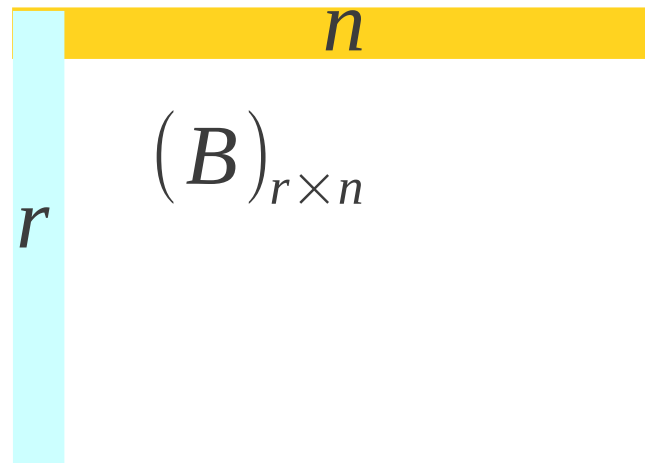
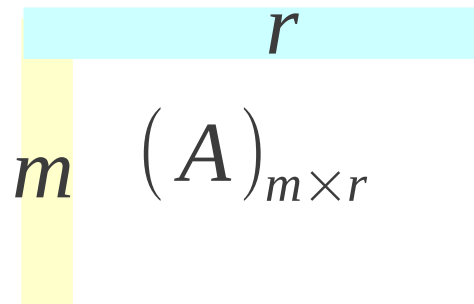
*2nd column*

$$A B = C$$

$$(m \times n) (n \times 2) = (m \times 2)$$

# Multiplication of Matrices (1)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \boxed{c_{ii}} & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rn} \end{pmatrix}$$



# Multiplication of Matrices (2)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \boxed{c_{ij}} & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rn} \end{pmatrix}$$

$$(AB)_{ij} = c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} \cdots + a_{ir}b_{rj}$$

$$= \sum_{k=1}^r a_{ik}b_{kj}$$

# Partitioned Matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

$$A = [a_{ij}]_{m \times n}$$

$$B = [b_{ij}]_{n \times 2}$$

$$C = [c_{ij}]_{m \times 2}$$

$$\begin{matrix} B \\ n \times 2 \end{matrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \begin{matrix} n \times 1 & n \times 1 \end{matrix}$$

$$\begin{matrix} AB \\ m \times 2 \end{matrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} \begin{matrix} m \times 1 & m \times 1 \end{matrix}$$

$$\begin{matrix} m \times n & n \times 2 & m \times n & n \times 1 & m \times n & n \times 1 \end{matrix}$$

# Partitioned Matrix

$$\begin{array}{l}
 \mathbf{a}_1 \rightarrow \\
 \mathbf{a}_2 \rightarrow \\
 \mathbf{a}_m \rightarrow
 \end{array}
 \begin{pmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 a_{21} & a_{22} & \cdots & a_{2n} \\
 \vdots & \vdots & & \vdots \\
 a_{m1} & a_{m2} & \cdots & a_{mn}
 \end{pmatrix}
 \begin{pmatrix}
 b_{11} & b_{12} \\
 b_{21} & b_{22} \\
 \vdots & \vdots \\
 b_{n1} & b_{n2}
 \end{pmatrix}
 =
 \begin{pmatrix}
 c_{11} & c_{12} \\
 c_{21} & c_{22} \\
 \vdots & \vdots \\
 c_{m1} & c_{m2}
 \end{pmatrix}$$

$$\mathbf{A} = [a_{ij}]_{m \times n}$$

$$\mathbf{B} = [b_{ij}]_{n \times 2}$$

$$\mathbf{C} = [c_{ij}]_{m \times 2}$$

$$\begin{array}{l}
 \mathbf{A} \\
 m \times n
 \end{array}
 =
 \begin{bmatrix}
 \mathbf{a}_1 \\
 \mathbf{a}_2 \\
 \vdots \\
 \mathbf{a}_m
 \end{bmatrix}
 \begin{array}{l}
 1 \times n \\
 1 \times n \\
 \\
 1 \times n
 \end{array}
 \quad
 \begin{array}{l}
 \mathbf{A} \mathbf{B} \\
 m \times 2 \\
 m \times n \quad n \times 2
 \end{array}
 =
 \begin{bmatrix}
 \mathbf{a}_1 \\
 \mathbf{a}_2 \\
 \vdots \\
 \mathbf{a}_m
 \end{bmatrix}
 \begin{array}{l}
 1 \times 2 \quad 1 \times n \quad n \times 2 \\
 1 \times 2 \quad 1 \times n \quad n \times 2 \\
 \\
 1 \times 2 \quad 1 \times n \quad n \times 2
 \end{array}$$

# Determinant

## Determinant of order 2

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

## Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ & b_2 & b_3 \\ & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} & a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \quad \begin{bmatrix} & & a_3 \\ b_1 & b_2 & \\ c_1 & c_2 & \end{bmatrix}$$

# Determinant

## Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ & b_2 & b_3 \\ & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} & a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \quad \begin{bmatrix} & & a_3 \\ b_1 & b_2 & \\ c_1 & c_2 & \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = + a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

# Determinant

## Determinant of order 3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
 = a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\
 - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\
 + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$



# Minor

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The **minor** of entry  $a_{ij}$   $M_{ij}$

The determinant of the submatrix  
that remains after **deleting**  $i$ -th row and  $j$ -th column

# Minor

The **minor** of entry  $a_{ij}$

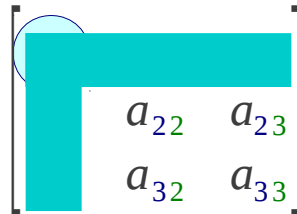
$$M_{ij}$$

The determinant of the submatrix  
that remains after **deleting**  $i$ -th row and  $j$ -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$n \times n$$


$$\begin{bmatrix} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$(n-1) \times (n-1)$$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

# Cofactor

The **cofactor** of entry  $a_{ij}$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

The **minor** of entry  $a_{ij}$

$$M_{ij}$$

The determinant of the submatrix that remains after **deleting**  $i$ -th row and  $j$ -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$(n-1) \times (n-1)$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Cofactor

$$C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

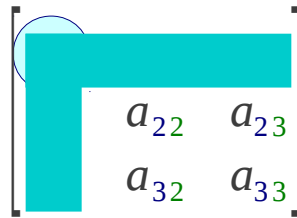
$$\begin{bmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{bmatrix}$$

$$\begin{bmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{bmatrix} = \begin{bmatrix} (-1)^2 & (-1)^3 & (-1)^4 \\ (-1)^3 & (-1)^4 & (-1)^5 \\ (-1)^4 & (-1)^5 & (-1)^6 \end{bmatrix}$$

# Minor Example (1)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$


$$\begin{bmatrix} & & & \\ & a_{22} & a_{23} & \\ & a_{32} & a_{33} & \end{bmatrix}$$

Sub-matrix

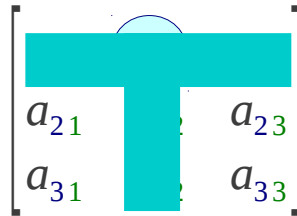
$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{11} = +M_{11}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

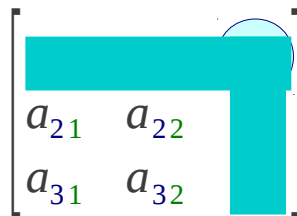

$$\begin{bmatrix} & & & \\ a_{21} & & a_{23} & \\ a_{31} & & a_{33} & \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{12} = -M_{12}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$


$$\begin{bmatrix} & & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & & \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

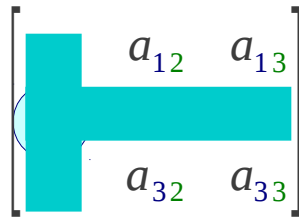
$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{13} = +M_{13}$$

# Minor Example (2)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$


$$\begin{bmatrix} & a_{12} & a_{13} \\ & & \\ a_{32} & & a_{33} \end{bmatrix}$$

Sub-matrix

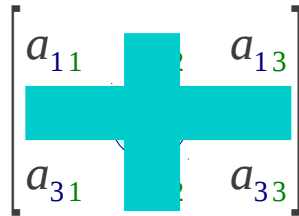
$$\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

Minor

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{21} = -M_{21}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

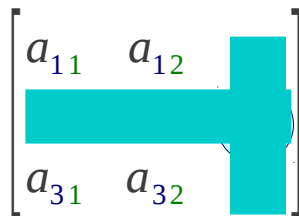

$$\begin{bmatrix} a_{11} & & a_{13} \\ & & \\ a_{31} & & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{22} = +M_{22}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$


$$\begin{bmatrix} a_{11} & a_{12} & \\ & & \\ a_{31} & a_{32} & \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{23} = -M_{23}$$

# Minor Example (2)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

Minor / Cofactor

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$C_{31} = +M_{31}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$C_{32} = -M_{32}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$C_{33} = +M_{33}$$

# Determinant

The **determinant** of an  $n \times n$  matrix  $\mathbf{A}$   $\det(\mathbf{A})$

Cofactor expansion along the  $i$ -th row

Cofactor expansion along the  $j$ -th column

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \det(\mathbf{A}) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \end{aligned}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \det(\mathbf{A}) &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \end{aligned}$$

# Adjoint

The **cofactor** of entry  $a_{ij}$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

The **minor** of entry  $a_{ij}$

$$M_{ij}$$

The determinant of the submatrix  
that remains after **deleting**  $i$ -th row and  $j$ -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

$$a_{11} \Leftrightarrow C_{11}$$

$$a_{12} \Leftrightarrow C_{12}$$

$$a_{13} \Leftrightarrow C_{13}$$

$$a_{21} \Leftrightarrow C_{21}$$

$$a_{22} \Leftrightarrow C_{22}$$

$$a_{23} \Leftrightarrow C_{23}$$

$$a_{31} \Leftrightarrow C_{31}$$

$$a_{32} \Leftrightarrow C_{32}$$

$$a_{33} \Leftrightarrow C_{33}$$

**Matrix of Cofactors**

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose



**Adjoint**

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$



# Inverse Matrix

The **cofactor** of entry  $a_{ij}$   $C_{ij} = (-1)^{i+j} M_{ij}$

The **minor** of entry  $a_{ij}$   $M_{ij}$

The determinant of the submatrix  
that remains after **deleting**  $i$ -th row and  $j$ -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose



Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \text{adj}(\mathbf{A}) = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

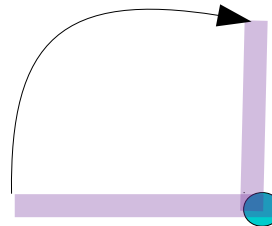
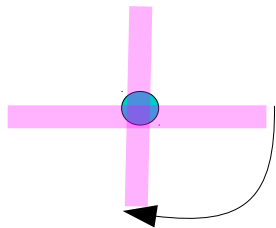
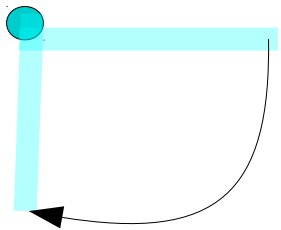
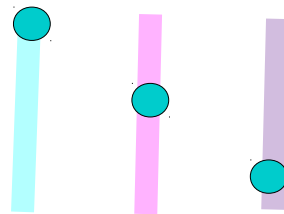
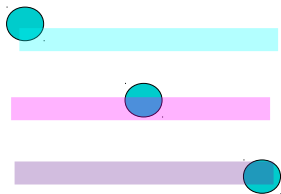
# Matrix Transpose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$[a_{ij}]$

$[a_{ji}]$



# Cofactor Expansion and Determinant

$\mathbf{A}$   $n \times n$

zero row

zero col

has

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



$$\det(\mathbf{A}) = 0$$

$$\det(\mathbf{A}^T) = \det(\mathbf{A}) = 0$$

# Properties of Determinants

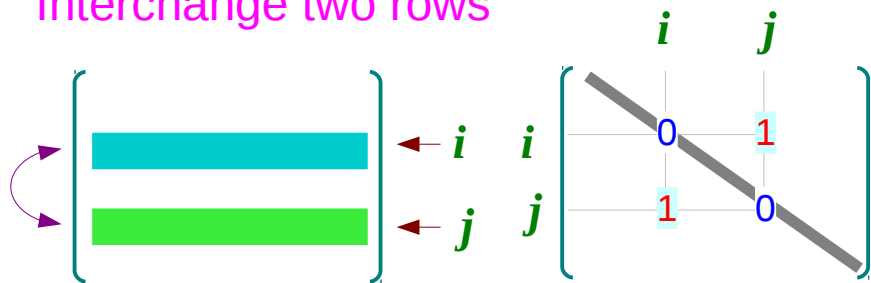
$$\det(\mathbf{A}) = 0$$

$$\det(k\mathbf{A}) = k^n \det(\mathbf{A})$$

$$\det(\mathbf{A}+\mathbf{B}) \neq \det(\mathbf{A})+\det(\mathbf{B})$$

# Elementary Matrix and Determinant

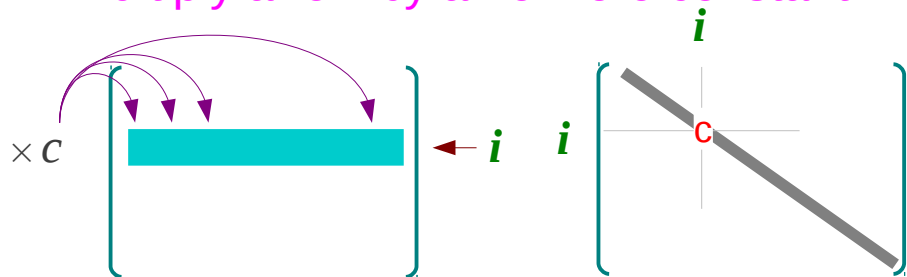
Interchange two rows



$$\det(\mathbf{B}) = -\det(\mathbf{A})$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

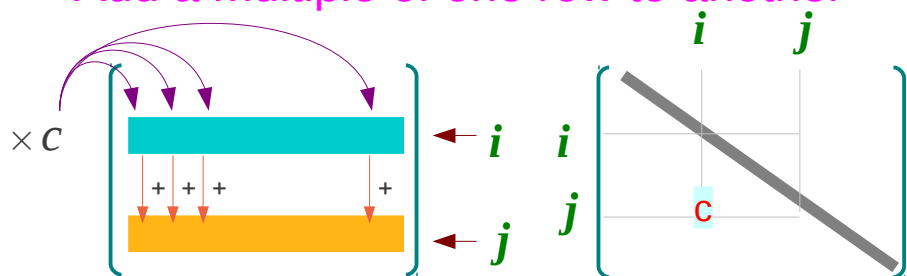
Multiply a row by a nonzero constant



$$\det(\mathbf{B}) = c \det(\mathbf{A})$$

$$\begin{vmatrix} c a_{11} & c a_{12} & c a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = c \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Add a multiple of one row to another



$$\det(\mathbf{B}) = \det(\mathbf{A})$$

$$\begin{vmatrix} a_{11} + c a_{21} & a_{12} + c a_{22} & a_{13} + c a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

## References

- [1] <http://en.wikipedia.org/>
- [2] Anton & Busby, "Contemporary Linear Algebra"
- [3] Anton & Rorres, "Elementary Linear Algebra"