Spectra (1A)

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Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k} \right)$$

$$k = 1,\ 2,\ \ldots$$

$$g_k \cos(k\omega_0 t + \phi_k) = \underline{g_k \cos(\phi_k)} \cos(k\omega_0 t) - \underline{g_k \sin(\phi_k)} \sin(k\omega_0 t)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

Two-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T}\int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T}\int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1, 2, \ldots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t}\right)$$

$$A_0 = a_0$$

$$A_n = \frac{1}{2}(a_n - j b_n)$$

$$B_n = \frac{1}{2}(a_n + j b_n)$$

$$x(t) = \sum_{n=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} A_0 & (k = 0) \\ A_k & (k > 0) \\ B_k & (k < 0) \end{cases}$$

$$|C_k| = |A_k| = |B_k| \quad (k \neq 0)$$

$$Arg(C_k) = \begin{cases} +\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k}\right) & (k > 0) \\ -\phi_k = \tan^{-1}\left(+\frac{b_k}{a_k}\right) & (k < 0) \end{cases}$$

Single-Sided Spectrum

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,dt$$

$n = 1, 2, \ldots$

$$x(t) = g_0 + \sum_{n=1}^{\infty} g_n \cos(n\omega_0 t + \phi_n)$$

$$g_0 = a_0$$

$$g_n = \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

$n = 1, 2, \ldots$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$g_n \cos(n\omega_0 t + \phi_n) = g_n \cos(\phi_n)\cos(n\omega_0 t) - g_n \sin(\phi_n)\sin(n\omega_0 t)$$

Power Spectrum (1)

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k\, e^{+jk\omega_0 t}$$

$$c_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\,dt$$

$$k = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$c_k = \begin{cases} a_0 & (k=0) \\ \frac{1}{2}(a_k - jb_k) & (k>0) \\ \frac{1}{2}(a_k + jb_k) & (k<0) \end{cases}$$

Average Power $P = \frac{1}{T}\int_0^T |x(t)|^2\,dt$

$$P = |a_0|^2 + \frac{1}{2}\sum_{k=1}^{+\infty}\left(|a_k|^2 + |b_k|^2\right)$$

$$P = \frac{1}{T}\int_0^T |x(t)|^2\,dt$$

$$P = |c_0|^2 + \sum_{k=1}^{+\infty}\left(|c_k|^2 + |c_{-k}|^2\right)$$

Power Spectrum

$$P_k = \begin{cases} |a_0|^2 & (k=0) \\ \frac{1}{2}\left(|a_k|^2 + |b_k|^2\right) & (k>0) \end{cases}$$

Power Spectrum

$$P_k = \begin{cases} |c_0|^2 & (k=0) \\ \left(|c_k|^2 + |c_{-k}|^2\right) & (k>0) \end{cases}$$

Power Spectrum (2)

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k \, e^{+jk\omega_0 t}$$

Average Power $P = \frac{1}{T}\int_0^T |x(t)|^2 \, dt$

$$P = |a_0|^2 + \frac{1}{2}\sum_{k=1}^{+\infty}\left(|a_k|^2 + |b_k|^2\right)$$

Average Power $P = \frac{1}{T}\int_0^T |x(t)|^2 \, dt$

$$P = |c_0|^2 + \sum_{k=1}^{+\infty}\left(|c_k|^2 + |c_{-k}|^2\right)$$

$$|c_k|^2 + |c_{-k}|^2 = \tfrac{1}{2}\left(|a_k|^2 + |b_k|^2\right)$$

$$|c_k|^2 = \tfrac{1}{4}\left(|a_k|^2 + |b_k|^2\right)$$

$$|c_{-k}|^2 = \tfrac{1}{4}\left(|a_k|^2 + |b_k|^2\right)$$

$$c_k = \begin{cases} a_0 & (k = 0) \\ \tfrac{1}{2}(a_k - jb_k) & (k > 0) \\ \tfrac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

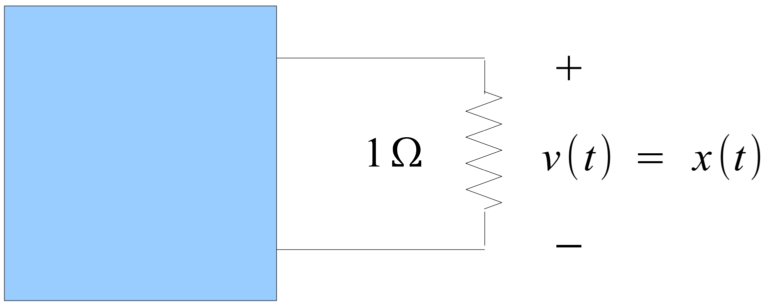
Power Spectrum

$$P_k = \begin{cases} |a_0|^2 & (k = 0) \\ \frac{1}{2}\left(|a_k|^2 + |b_k|^2\right) & (k > 0) \end{cases}$$

Power Spectrum

$$P_k = \begin{cases} |c_0|^2 & (k = 0) \\ \left(|c_k|^2 + |c_{-k}|^2\right) & (k > 0) \end{cases}$$

Continuous Periodic Signal



instantaneous power $x^2(t)$

average power $\frac{1}{T}\int_0^T x^2(t)\,dt$

T : *period*

$v(t) = x(t)$ Continuous Periodic

CTFS (Fourier Series)

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n\, e^{+jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$n = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

Parseval's Theorem

$$\frac{1}{T}\int_0^T x^2(t)\,dt = \sum_{n=-\infty}^{+\infty} |C_n|^2$$

average power

sum of power spectrum C_n

CTFS and CTFT

Continuous Time Fourier Series

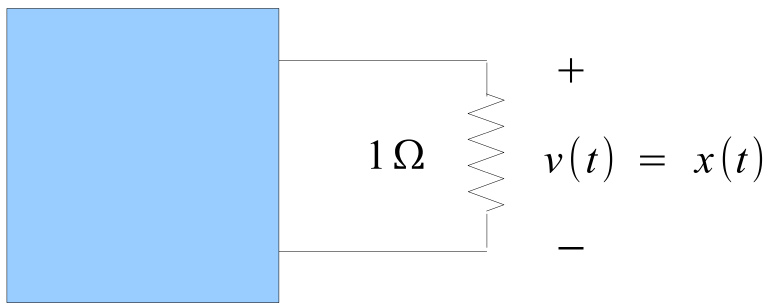
$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_n\, e^{+jn\omega_0 t}$$

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt \quad \Leftrightarrow \quad x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{+j2\pi f t}\, df$$

Continuous Aperiodic Signal



instantaneous power $x^2(t)$

total energy $\int_{-\infty}^{+\infty} x^2(t)\, dt$

$v(t) = x(t)$ Continuous Aperiodic

CTFT (Fourier Integral)

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{+j2\pi f t}\, d f$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, d t$$

Parseval's Theorem

$$\int_{-\infty}^{+\infty} x^2(t)\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

total energy

integral of energy spectral density $|X(f)|^2$

Average Power of Random Signals

A truncated sample function

$$x_T(t) = x(t) \qquad -\tfrac{T}{2} < t < +\tfrac{T}{2}$$
$$= 0 \qquad otherwise$$

Fourier Transform

$$X_T(\omega) = \int_{-\infty}^{+\infty} x_T(t)\, e^{-j\omega t}\, dt \iff x_T(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_T(\omega)\, e^{+j\omega t}\, d\omega$$

$$X_T(f) = \int_{-\infty}^{+\infty} x_T(t)\, e^{-j2\pi f t}\, dt \iff x_T(t) = \int_{-\infty}^{+\infty} X_T(f)\, e^{+j2\pi f t}\, df$$

Parseval's Therorem

$$\int_{-\infty}^{+\infty} |x_T(t)|^2\, dt = \int_{-\infty}^{+\infty} |X_T(f)|^2\, df$$

total energy

Average Power

$$\frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\, dt = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\, dt$$

total energy / T

Power Spectral Density of Random Signals

Average Power

$$\frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\,dt = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\,df \Rightarrow \int_{-\infty}^{+\infty} \hat{S}_{xx}(f)\,df$$

Raw Power Spectral Density

$$\frac{|X_T(f)|^2}{T} = \hat{S}_{xx}(f)$$

Power Spectral Density

$$\lim_{T\to\infty} \frac{E\left[|X_T(f)|^2\right]}{T} = S_{xx}(f)$$

$$\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\,dt = \lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\,df$$

➡ *not converge*

$$E\left[\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\,dt\right] = E\left[\lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\,df\right]$$

Random Signal

$$Var(x(t)) = \sigma_x^2 = \int_{-\infty}^{+\infty} \lim_{T\to\infty} \frac{E\left[|X_T(f)|^2\right]}{T}\,df \Rightarrow \int_{-\infty}^{+\infty} S_{xx}(f)\,df$$

Power and Power Density Spectra

Average Power

$$E[x^2(t)] = \frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\,dt$$

Average Power of N sample of x(t)

$$\frac{1}{N}\sum_{n=0}^{N-1} x_n^2 \approx \frac{1}{N\Delta t}\int_0^{N\Delta t} x^2(t)\,dt = \frac{1}{N^2}\sum_{m=0}^{N-1} |X_m|^2$$

$$\sum_{n=0}^{N-1} x_n^2 = \frac{1}{N}\sum_{m=0}^{N-1} |X_m|^2 \qquad m = 0, 1, \cdots, N-1$$

Periodogram:

$$P_{xx}(m) = \frac{1}{N}|X_m|^2$$

$$m = 0, 1, \cdots, N-1$$

Average Power

$$\sum_{n=0}^{N-1} x_n^2 = \frac{1}{N}\sum_{m=0}^{N-1} P_{xx}(m)$$

Total Energy

$$T\sum_{n=0}^{N-1} x_n^2 = T\sum_{m=0}^{N-1} P_{xx}(m)$$

$$\approx \int_0^{N\Delta t} x^2(t)\,dt$$

	Periodic Signals	Aperiodic Signals	Random Signals	
Frequency Spacing	$\Delta f = \frac{1}{N \Delta t}$	$\Delta f = \frac{1}{N \Delta t}$	$\sum S \Delta f = \frac{1}{N \Delta t} \sum S$	$\frac{1}{N \Delta t} \sum x^2 \Delta t$
Two Sided	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$	$S(k) = \frac{\Delta t}{N} \lvert X(k) \rvert^2$	$P = \sum_{k=0}^{N-1} S(k) \Delta f$
One Sided				
$k = 0, \ \frac{N}{2}$	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$	$S_1(k) = 2S(k)$	$P = \sum_{k=0}^{N/2} S_1(k) \Delta f$
$k = 1, \cdots, \frac{N}{2} - 1$	$\frac{2}{N} X(k)$	$\frac{2 \Delta t}{N} X(k)$	$S_1(k) = S(k)$	
Frequency Scale	$k \Delta f$	$k \Delta f$	$k \Delta f$	

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] K.H. Shin, J.K. Hammond, Fundamentals of Signal Processing for Sound and Vibration Engineers, Wiley, 2008