# Spectra (1A)

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# Single-Sided Spectrum

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right)$$

$$x(t) = g_0 + \sum_{n=1}^{\infty} g_n \cos(n\omega_0 t + \phi_n)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t)\, dt$$

$n = 1, 2, \ldots$

$$g_0 = a_0$$

$$g_n = \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = \tan^{-1}\left( -\frac{b_n}{a_n} \right)$$

$n = 1, 2, \ldots$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$g_n \cos(n\omega_0 t + \phi_n) = g_n \cos(\phi_n)\cos(n\omega_0 t) - g_n \sin(\phi_n)\sin(n\omega_0 t)$$

# Two-Sided Spectrum

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t)\, dt$$

$$n = 1, 2, \ldots$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} \left( A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t} \right)$$

$$A_0 = a_0$$

$$A_n = \frac{1}{2} (a_n - j b_n)$$

$$B_n = \frac{1}{2} (a_n + j b_n)$$

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{+jn\omega_0 t}$$

$$C_n = \frac{1}{T} \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$n = -2, -1, 0, +1, +2, \ldots$$

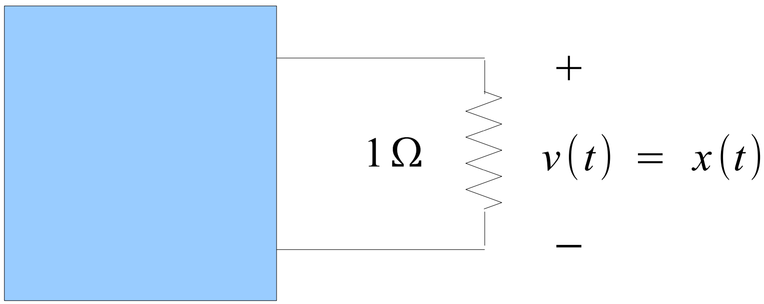
$$C_n = \begin{cases} A_0 & (n = 0) \\ A_n & (n > 0) \\ B_n & (n < 0) \end{cases}$$

$$|C_n| = \frac{A_n}{2} \quad (n \neq 0)$$

$$Arg(C_n) = \begin{cases} +\phi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right) & (n > 0) \\ -\phi_n = \tan^{-1}\left(+\frac{b_n}{a_n}\right) & (n < 0) \end{cases}$$

# Periodogram

# Constant Periodic Signal




instantaneous power $x^2(t)$

average power $\frac{1}{T}\int_0^T x^2(t)\,dt$

$T$ : *period*

$v(t) = x(t)$ Continuous Periodic

CTFS (Fourier Series)

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n\, e^{+jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$n = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

Parseval's Theorem

$$\frac{1}{T}\int_0^T x^2(t)\,dt = \sum_{n=-\infty}^{+\infty} |C_n|^2$$

average power

sum of

power spectrum $C_n$

# CTFS and CTFT

**Continuous Time Fourier Series**

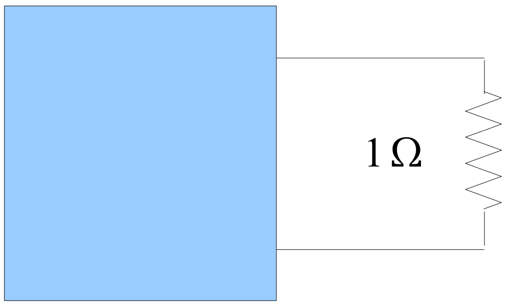
$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_n\, e^{+jn\omega_0 t}$$

**Continuous Time Fourier Transform**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt \quad \Leftrightarrow \quad x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{+j2\pi f t}\, df$$

# Constant Aperiodic Signal




$+$

$v(t) = x(t)$

$-$

instantaneous power $x^2(t)$

total energy $\int_0^T x^2(t)\, dt$

$T \; : \; period$

$v(t) = x(t)$ Continuous Aperiodic

CTFT (Fourier Integral)

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{+j2\pi f t} d f$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t} d t$$

Parseval's Theorem

$$\int_0^T x^2(t)\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

total energy

integral of energy spectral density $|X(f)|^2$

# Periodogram

## A truncated sample function

$$x_T(t) = x(t) \qquad -T/2 < t < +T/2$$

$$= 0 \qquad otherwise$$

## Fourier Transform

$$X_T(\omega) = \int_{-\infty}^{+\infty} x_T(t)\, e^{-j\omega t}\, dt \qquad\qquad x_T(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_T(\omega)\, e^{+j\omega t}\, d\omega$$

$$X_T(f) = \int_{-\infty}^{+\infty} x_T(t)\, e^{-j2\pi f t}\, dt \qquad\qquad x_T(t) = \int_{-\infty}^{+\infty} X_T(f)\, e^{+j2\pi ft}\, df$$

## Parseval's Therorem

$$\int_{-\infty}^{+\infty} |x_T(t)|^2\, dt = \int_{-\infty}^{+\infty} |X_T(f)|^2\, df$$

## Average Power

$$\frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\, dt \qquad = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\, dt$$

# Average Power

## A truncated sample function

$$x_T(t) = x(t) \qquad -T/2 < t < +T/2$$

$$= 0 \qquad otherwise$$

## Fourier Transform

$$X_T(\omega) = \int_{-\infty}^{+\infty} x_T(t)\, e^{-j\omega t}\, dt \qquad\qquad x_T(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_T(\omega)\, e^{+j\omega t}\, d\omega$$

$$X_T(f) = \int_{-\infty}^{+\infty} x_T(t)\, e^{-j2\pi f t}\, dt \qquad\qquad x_T(t) = \int_{-\infty}^{+\infty} X_T(f)\, e^{+j2\pi ft}\, df$$

## Parseval's Therorem

$$\int_{-\infty}^{+\infty} |x_T(t)|^2\, dt = \int_{-\infty}^{+\infty} |X_T(f)|^2\, df$$

## Average Power

$$\frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\, dt \qquad = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\, dt$$

# Raw Power Spectral Density

## Average Power

$$\frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\,dt \quad = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\,df \quad \Rightarrow \int_{-\infty}^{+\infty} \hat{S}_{xx}(f)\,df$$

## Raw Power Spectral Density

$$\frac{|X_T(f)|^2}{T} = \hat{S}_{xx}(f)$$

$$\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\,dt \quad = \lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\,df$$

$$E\left[\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\,dt\right] \quad = E\left[\lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\,df\right]$$

# Power Spectral Density

## Average Power

$$\frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\,dt \quad = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\,df \quad \Rightarrow \int_{-\infty}^{+\infty} \hat{S}_{xx}(f)\,df$$

## Raw Power Spectral Density

$$\frac{|X_T(f)|^2}{T} = \hat{S}_{xx}(f)$$

$$E\left[\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\,dt\right] \quad = E\left[\lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\,df\right]$$

$$Var(x(t)) = \sigma_x^2 \quad = \lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{E\left[|X_T(f)|^2\right]}{T}\,df \quad \Rightarrow \int_{-\infty}^{+\infty} \hat{S}_{xx}(f)\,df$$

## Power Spectral Density

$$\lim_{T\to\infty} \frac{E\left[|X_T(f)|^2\right]}{T} = S_{xx}(f)$$

# Power and Power Density Spectra

Average Power

$$E[x^2(t)] = \frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\, dt$$

Average Power of N sample of x(t)

$$\frac{1}{N}\sum_{n=0}^{N-1} x_n^2 \approx \frac{1}{N\,\Delta t}\int_{0}^{N\,\Delta t} x^2(t)\, dt = \frac{1}{N^2}\sum_{m=0}^{N-1} |X_m|^2$$

$$\sum_{n=0}^{N-1} x_n^2 = \frac{1}{N}\sum_{m=0}^{N-1} |X_m|^2 \qquad m = 0,\ 1,\ \cdots,\ N-1$$

Periodogram:

$$P_{xx}(m) = \frac{1}{N}|X_m|^2$$

$$m = 0,\ 1,\ \cdots,\ N-1$$

Average Power

$$\sum_{n=0}^{N-1} x_n^2 = \frac{1}{N}\sum_{m=0}^{N-1} P_{xx}(m)$$

Total Energy

$$T\sum_{n=0}^{N-1} x_n^2 = T\sum_{m=0}^{N-1} P_{xx}(m)$$

$$\approx \int_{0}^{N\,\Delta t} x^2(t)\, dt$$

## References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] K.H. Shin, J.K. Hammond, Fundamentals of Signal Processing for Sound and Vibration Engineers, Wiley, 2008