# Vector Functions (1A)

- Vector Functions
- Motion
- Curvature

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# Vector Valued Functions

Set of points $\qquad (x,\ y,\ z)$

Parametric functions
$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases} \implies \quad (f(t),\ g(t),\ h(t))$$

$$\text{parameter } t$$

Vector Valued Function

Position Vector

a given $t$ $\implies$ A point in 3-d space

$$r(t) = \langle f(t),\ g(t),\ h(t) \rangle$$
$$= f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

# Vector Valued Functions (1)

Vector Valued Function     $r(t) = \langle f(t),\ g(t),\ h(t) \rangle$

$$= f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$



domain

range

$t$

$t_0$

$z$

$r(t_0)$

$y$

$x$

a given  $t_0$     ➡     A position vector
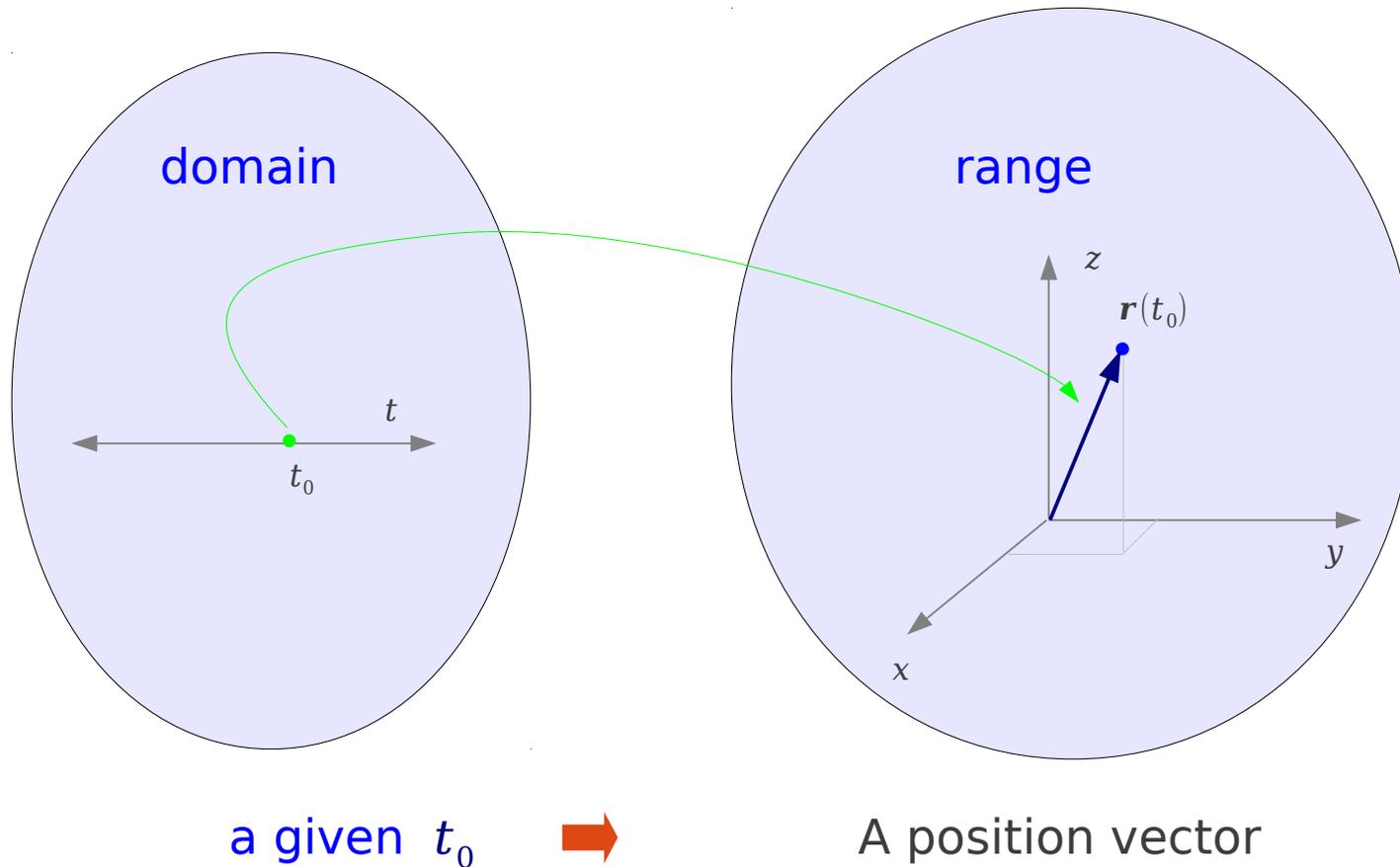
# Vector Valued Functions (2)

Vector Field $(x, \ y, \ z)$ $\boldsymbol{F}(x, \ y, \ z)$

domain

$z$

$(x_0, \ y_0, \ z_0)$

$y$

$x$

range

$z$

$\boldsymbol{F}(x, \ y, \ z)$

$y$

$x$
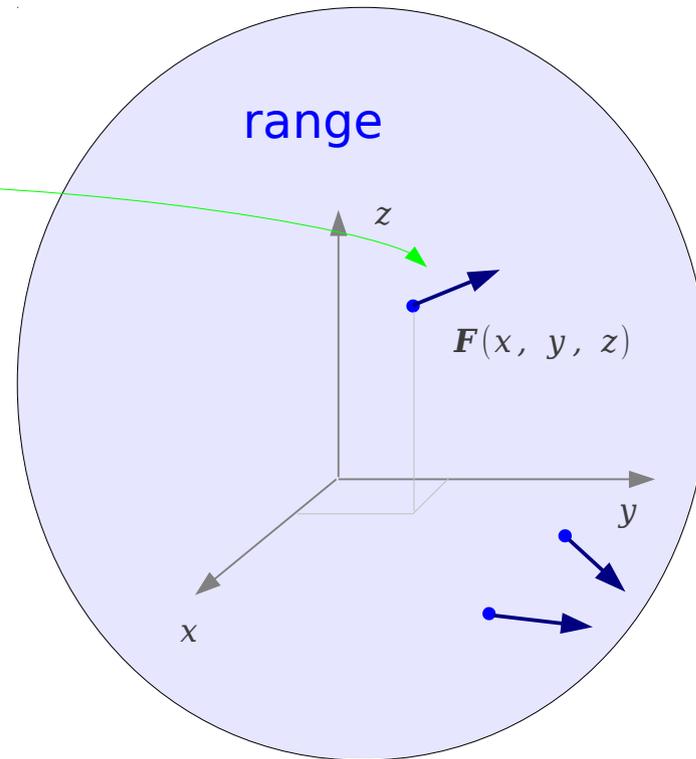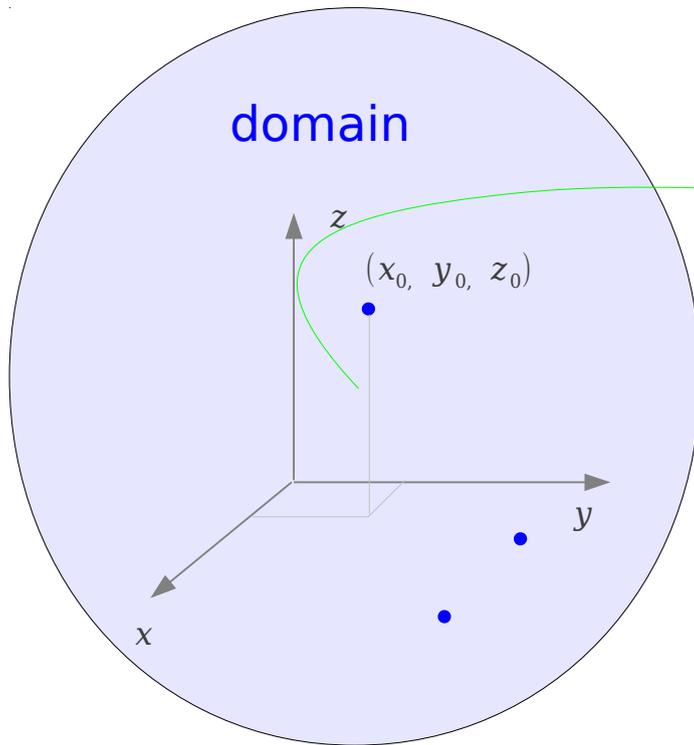
a given point in a 3-d space

$(x_0, \ y_0, \ z_0)$

A vector

$(u_0, \ v_0, \ w_0)$

# Limit of a Vector Function

Vector Valued Function $\qquad r(t) = \langle f(t),\ g(t),\ h(t) \rangle$

Limit of a Vector Valued Function

$$\lim_{t \to a} r(t) = \left\langle \lim_{t \to a} f(t),\ \lim_{t \to a} g(t),\ \lim_{t \to a} h(t) \right\rangle$$

Limit of a Vector Valued Function

$r(a)$ is defined

$\displaystyle\lim_{t \to a} r(t)$ exists

$r(a) = \displaystyle\lim_{t \to a} r(t)$

# Derivative of a Vector Function

Vector Valued Function $\qquad \boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$

Derivative of a Vector Valued Function

$$\boldsymbol{r}'(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t}$$

$\boldsymbol{r}(t) \qquad$ Position Vector $\qquad \boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$

$\boldsymbol{r}'(t) \qquad$ Tangent Vector $\qquad \boldsymbol{r}'(t) = \langle f'(t),\ g'(t),\ h'(t) \rangle$

# Arc Length (1)

Vector Valued Function

$$\boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$$

Derivative of a Vector Valued Function

$$\boldsymbol{r}'(t) = \langle f'(t),\ g'(t),\ h'(t) \rangle$$

$(f(t_0+1),\ g(t_0+1),\ h(t_0+1))$

$(f(t_0),\ g(t_0),\ h(t_0))$

$(f(t_0),\ g(t_0),\ h(t_0))$

$z$

$t$

$t_0$  $t_0+1$

$y$

$x$
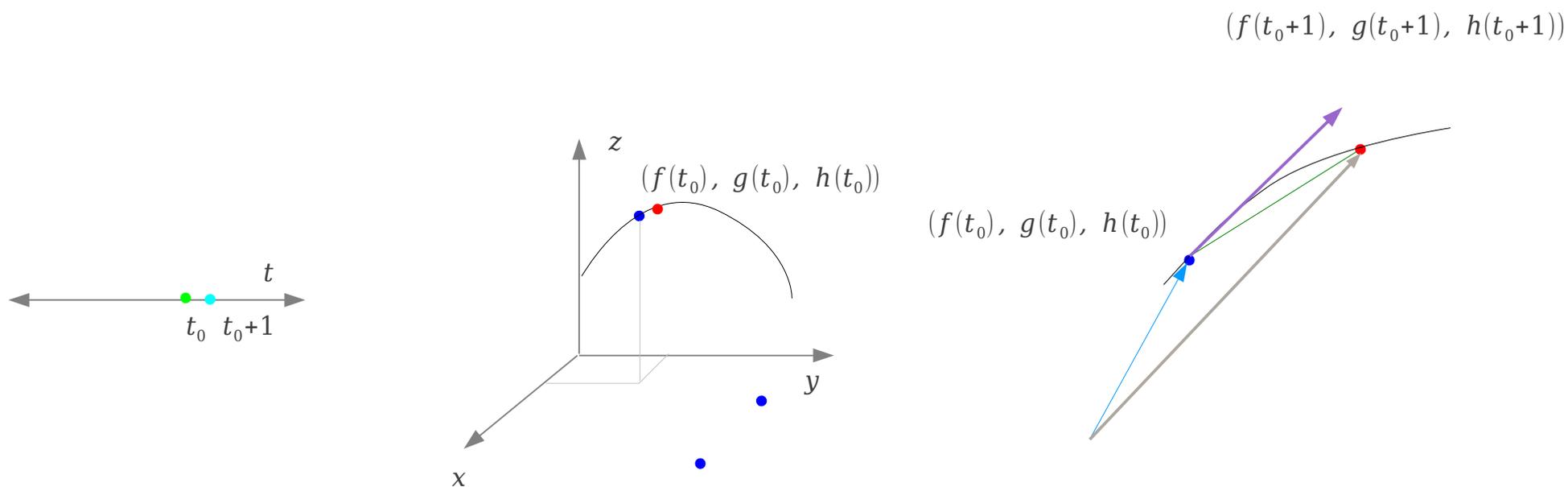
# Arc Length (2)

Vector Valued Function

$$\boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$$

Derivative of a Vector Valued Function

$$\boldsymbol{r}'(t) = \langle f'(t),\ g'(t),\ h'(t) \rangle$$

$$\boldsymbol{r}'(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t}$$

$t$

$t_0 \quad t_0 + \Delta t$

$(f(t_0),\ g(t_0),\ h(t_0))$

$z$

$h(t_0 + \Delta t) - g(t_0)$

$g(t_0 + \Delta t) - g(t_0)$

$f(t_0 + \Delta t) - f(t_0)$

$y$

$(f(t_0 + \Delta t),\ g(t_0 + \Delta t),\ h(t_0 + \Delta t))$

$x$
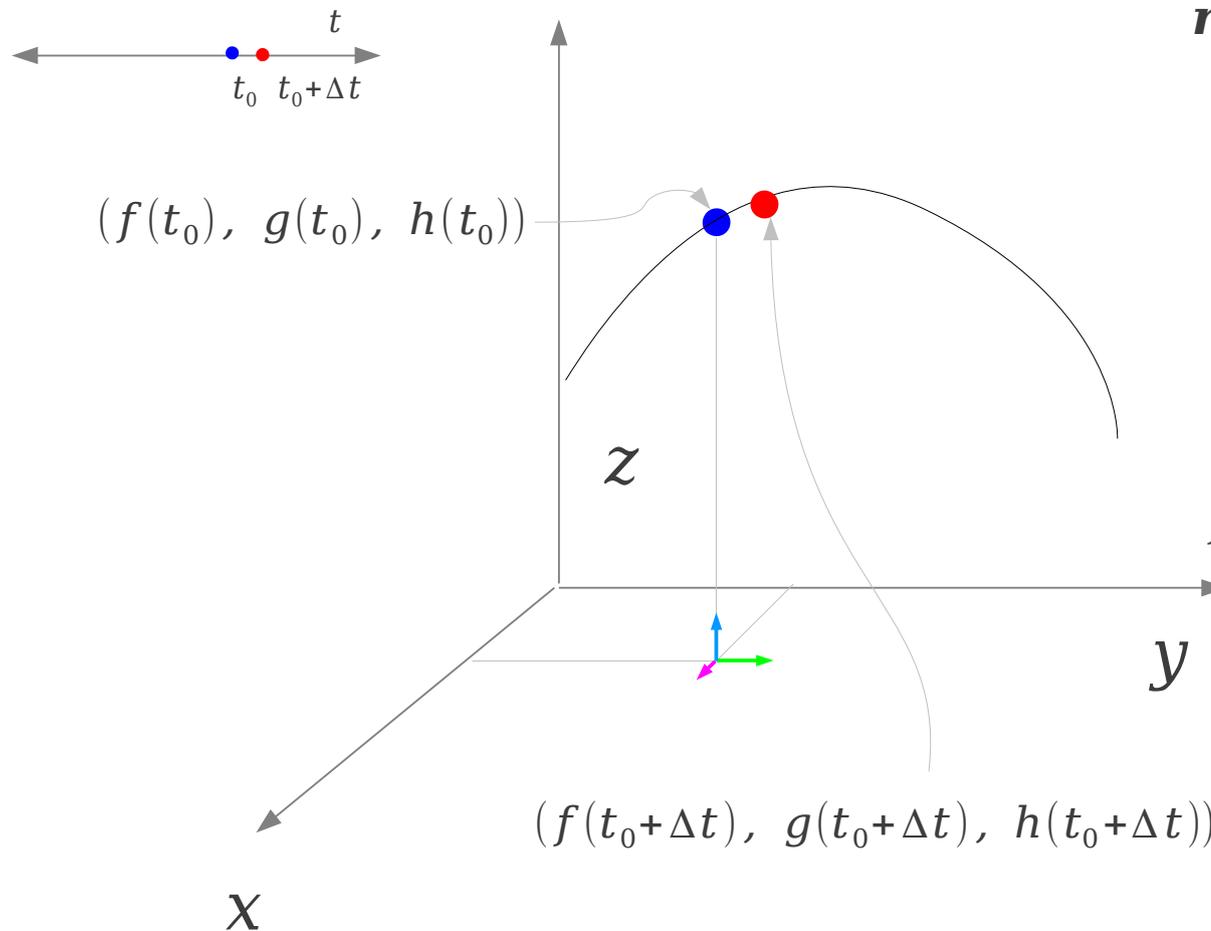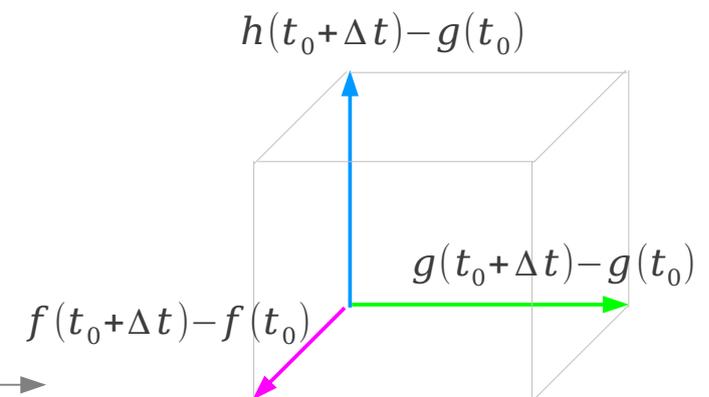
# Arc Length (3)

Vector Valued Function

$$\boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$$

Derivative of a Vector Valued Function

$$\boldsymbol{r}'(t) = \langle f'(t),\ g'(t),\ h'(t) \rangle$$

$$\boldsymbol{r}'(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t}$$

$$\frac{h(t_0 + \Delta t) - h(t_0)}{\Delta t_0}$$

$$\frac{g(t_0 + \Delta t) - g(t_0)}{\Delta t_0}$$

$$\frac{d h}{d t}$$

$$\frac{d g}{d t}$$

$$\frac{d f}{d t}$$
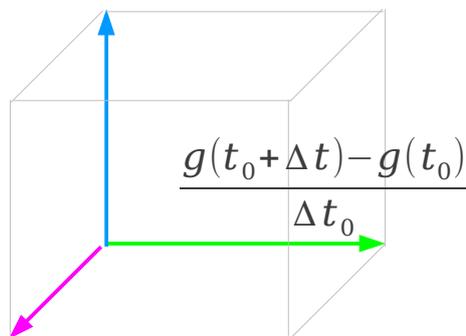
$$\frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t_0}$$
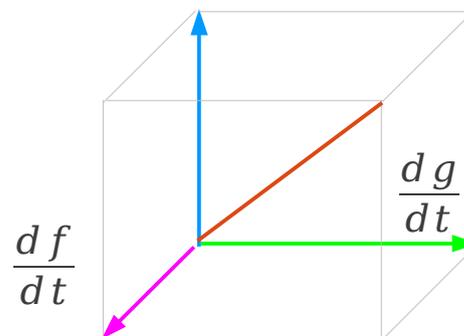
$$\sqrt{[f'(t)]^2 + [g'(t)]^2 + [h(t)]^2} = \|\boldsymbol{r}'(t)\|$$

# Arc Length (4)

Vector Valued Function

$$\boldsymbol{r}(t) = \langle f(t), \ g(t), \ h(t) \rangle$$

Derivative of a Vector Valued Function

$$\boldsymbol{r}'(t) = \langle f'(t), \ g'(t), \ h'(t) \rangle$$

$$\boldsymbol{r}'(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t}$$

$t$

$t_0 \quad t_0 + \Delta t$

$z$

$y$

$(f(t_0 + \Delta t), \ g(t_0 + \Delta t), \ h(t_0 + \Delta t))$

$x$

$(f(t_0), \ g(t_0), \ h(t_0))$
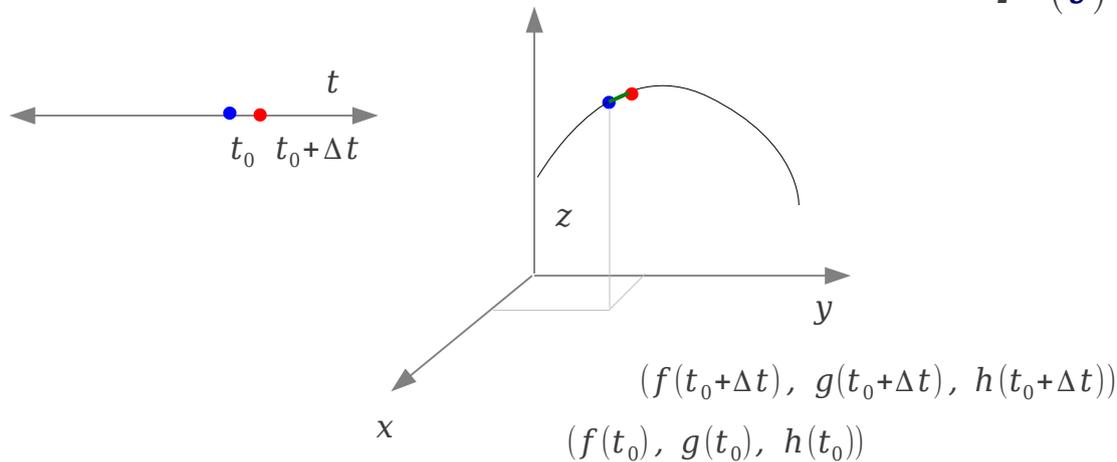
Arc Length

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h(t)]^2} \ dt$$

$$= \int_a^b \|\boldsymbol{r}'(t)\| \ dt$$

# Composite Function



$$\theta = \omega t$$

$$s_0 = u(t_0)$$

$$\cos(\omega t)$$

range

$$s_0 = u(t_0)$$

$$r(s_0)$$
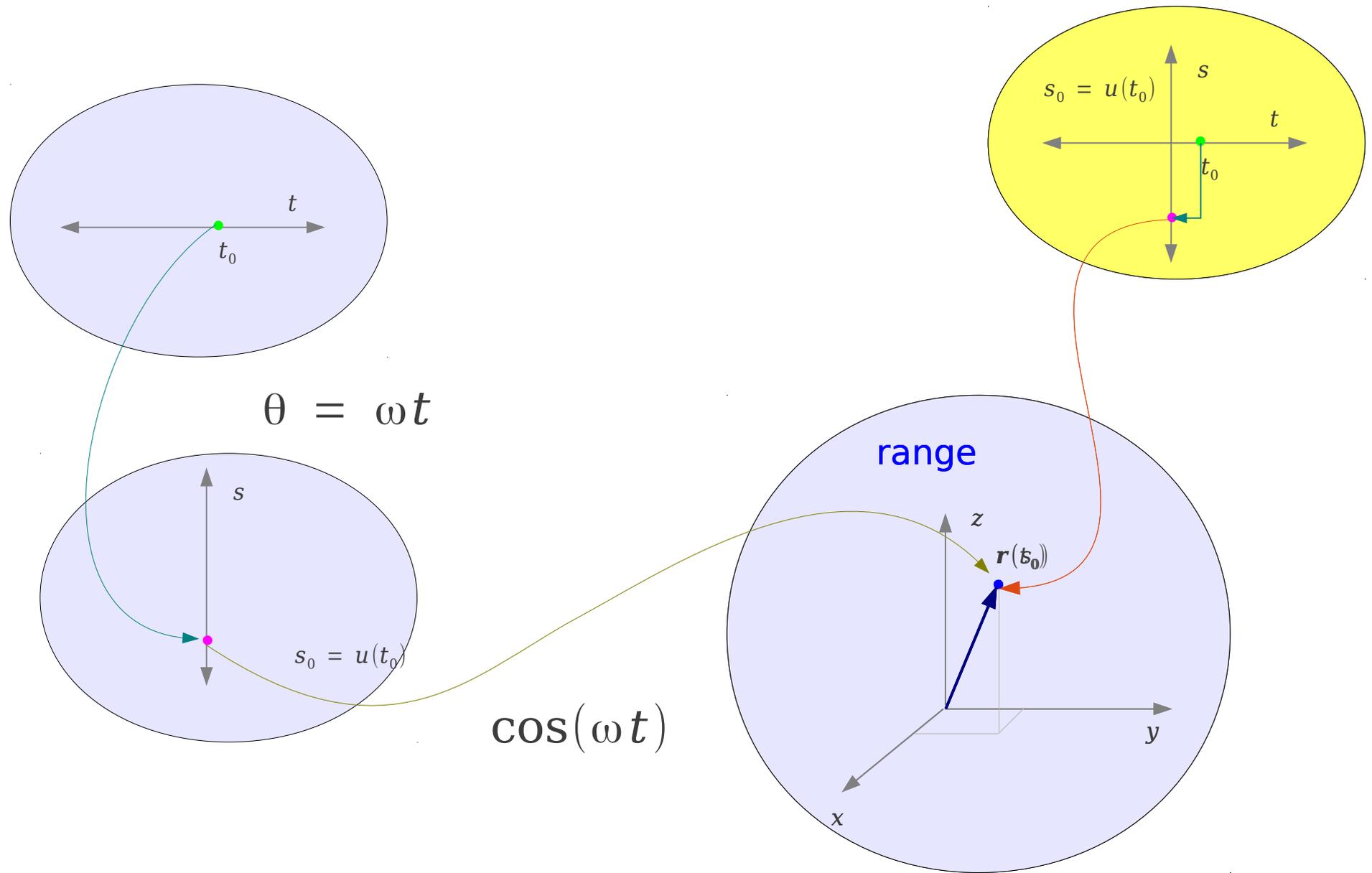
# Chain Rule of a Vector Function (1)

Derivative of a Vector Valued Function

$$s = u(t)$$

$$\frac{d\,s}{d\,t} = \frac{d\,u(t)}{d\,t} \quad \Longrightarrow \quad u'(t)$$

$$\frac{d\boldsymbol{r}}{d\,t} = \frac{d\boldsymbol{r}}{d\,s}\frac{d\,s}{d\,t} = \boldsymbol{r}'(s)\,u'(t)$$

$s_0 = u(t_0)$

$s$

$t$

$t_0$

range

$z$

$\boldsymbol{r}(s_0)$

$y$

$x$
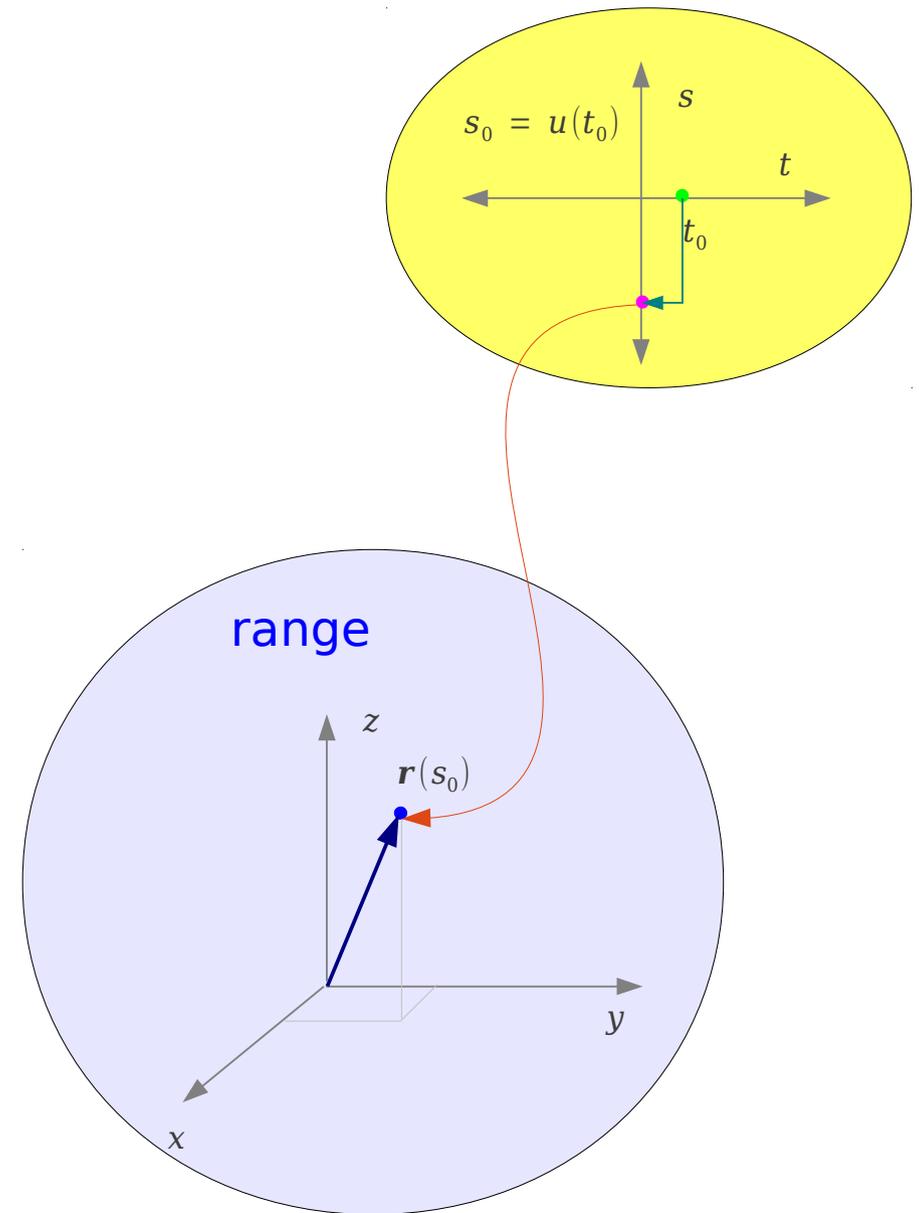
# Chain Rule of a Vector Function (2)

**Vector Valued Function**

$$\boldsymbol{r}(s) = \langle f(s),\ g(s),\ h(s) \rangle$$

**Scalar Function**

$$s = u(t)$$

$$\boldsymbol{r}(u(t)) = \langle f(u(t)),\ g(u(t)),\ h(u(t)) \rangle$$

**Derivative of a Vector Valued Function**

$$s = u(t)$$

$$\frac{ds}{dt} = \frac{du(t)}{dt} \quad \Longrightarrow \quad u'(t)$$

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} = \boldsymbol{r}'(s)u'(t)$$

# Integration of a Vector Function

Vector Valued Function

$$\boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$$

$$= f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

Limit of a Vector Valued Function

$$\int \boldsymbol{r}(t)\, dt = \left\langle \int f(t)\, dt,\ \int g(t)\, dt,\ \int h(t)\, dt \right\rangle$$

$$= \int f(t)\, dt\, \boldsymbol{i} + \int g(t)\, dt\, \boldsymbol{j} + \int h(t)\, dt\, \boldsymbol{k}$$

# Displacement, Velocity, Acceleration

**Displacement**

$$\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

**Velocity**

$$\boldsymbol{v}(t) = \boldsymbol{r}'(t) = f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}$$

**Acceleration**

$$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{r}''(t) = f''(t)\boldsymbol{i} + g''(t)\boldsymbol{j} + h''(t)\boldsymbol{k}$$

**Speed**

$$\|\boldsymbol{v}(t)\| = \left\|\frac{\boldsymbol{r}(t)}{dt}\right\| = \|f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}\|$$

$$= \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

# Unit Tangent of a Vector Function (1)

Displacement

$$\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

Velocity

$$\boldsymbol{v}(t) = \boldsymbol{r}'(t) = f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}$$

Unit Tangent

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}$$

$$\boldsymbol{r}'(t) \quad \Longrightarrow \quad \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt}$$

Arc length
$s$

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h(t)]^2}\, dt = \int_a^b \|\boldsymbol{r}'(t)\|\, dt$$

$$\frac{ds}{dt} = \|\boldsymbol{r}'(t)\|$$

# Unit Tangent of a Vector Function (2)

Displacement $\qquad r(t) = f(t)i + g(t)j + h(t)k$

Velocity $\qquad v(t) = r'(t) = f'(t)i + g'(t)j + h'(t)k$

Unit Tangent

velocity · speed
· direction

direction $\quad T(t) = \dfrac{r'(t)}{\|r'(t)\|}$ $\qquad \dfrac{dr}{ds} = \dfrac{\dfrac{dr}{dt}}{\dfrac{ds}{dt}} = \dfrac{r'(t)}{\|r'(t)\|} = T(t)$

speed

$r'(t)$ ➡ $\dfrac{dr}{dt} = \dfrac{dr}{ds}\dfrac{ds}{dt}$ ⬅ $\dfrac{ds}{dt} = \|r'(t)\|$

# Curvature of a Vector Function (1)

Vector Valued Function
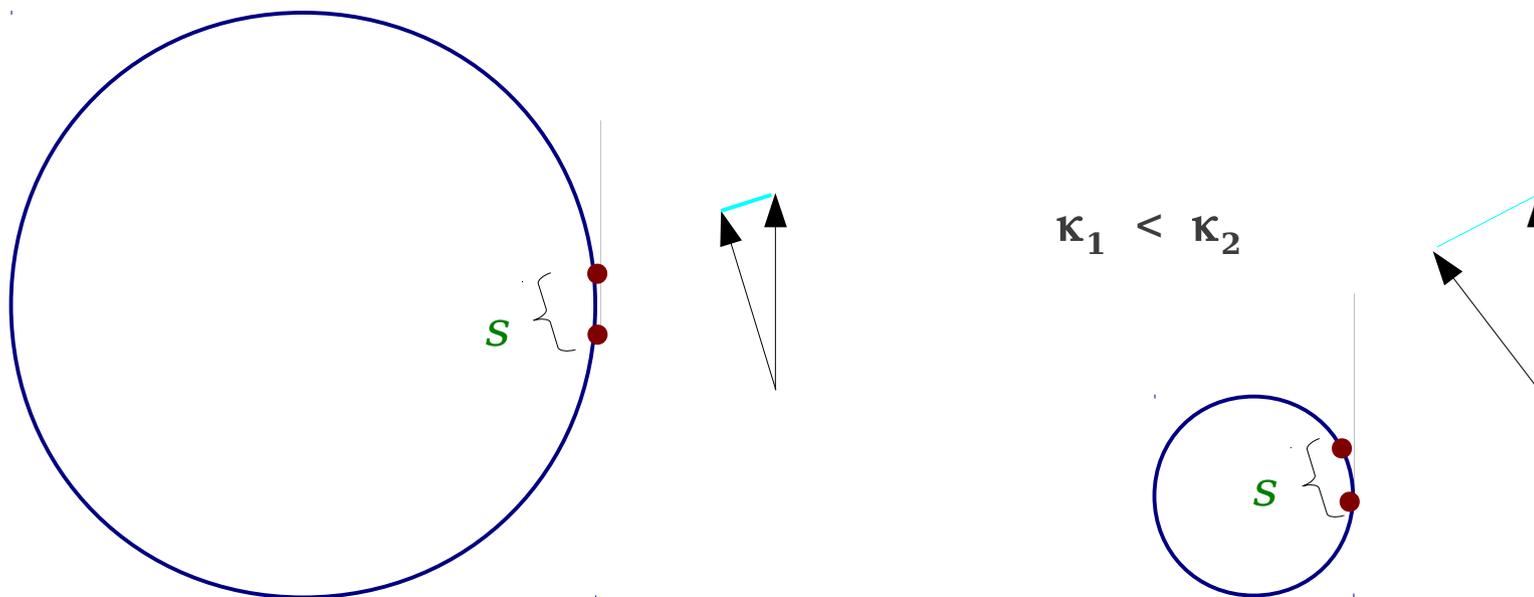
$$\boldsymbol{r}(t) \;=\; f(t)\boldsymbol{i} \;+\; g(t)\boldsymbol{j} \;+\; h(t)\boldsymbol{k}$$

Unit Tangent

$$\boldsymbol{T}(t) \;=\; \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} \;=\; \frac{d\boldsymbol{r}}{ds}$$

Curvature

$$\kappa \;=\; \left\|\frac{d\boldsymbol{T}}{ds}\right\|$$

$s$

$\kappa_1 \;<\; \kappa_2$

$s$

# Curvature of a Vector Function (2)

Vector Valued Function

$$\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

Unit Tangent

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \frac{d\boldsymbol{r}}{ds}$$

Curvature

$$\kappa = \left\| \frac{d\boldsymbol{T}}{ds} \right\|$$

Arc length    $s$

$$\frac{d\boldsymbol{T}}{dt} = \frac{d\boldsymbol{T}}{ds}\frac{ds}{dt} \qquad\qquad \frac{ds}{dt} = \|\boldsymbol{r}'(t)\|$$

$$\frac{d\boldsymbol{T}}{ds} = \frac{\dfrac{d\boldsymbol{T}}{dt}}{\dfrac{ds}{dt}} = \frac{\boldsymbol{T}'(t)}{\|\boldsymbol{r}'(t)\|} = \kappa(t)$$

# Line Equations (2)

# References

[1]  http://en.wikipedia.org/
[2]  http://planetmath.org/
[3]  M.L. Boas, "Mathematical Methods in the Physical Sciences"
[4]  D.G. Zill, "Advanced Engineering Mathematics"