Observer based State Feedback and Estimation for Chaotic Systems: Application on Lorentz Attractor

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***Abstract*- Chaotic systems are widely used in various aspects of engineering. Controlling chaotic system have been demonstrated in various articles both analytically and by trial and error methodologies. One of the common constraints of the proposed successful schemes are that they require complete knowledge of the states of the system. In this article, we proposed a observer based state feedback and estimation scheme, which is well applicable to the cases where complete knowledge of states are not available. We have studied the feasibility of our system with application on Lorentz Attractor with on-off controller and Rössler Attractor.**

***Keywords*- Chaos, Observer based State Feedback, Transformation, Non-linear System, Duffing oscillator, Linearization.**

I. Introduction

Chaotic systems are deterministic systems, which are time response appears random [1]. A small difference in initial condition makes the future states to be widely diverging. This happens although the system is deterministic, no randomness is present in the system [2]. This property of chaotic system makes prediction of future behavior of the system very difficult.

Due to its irregularity and complexity, the presence of chaos is mostly avoided in many systems. If there is a chaotic element present in the system, one of the options is to redesign the whole system to exclude the particular element. But if this option is not available, it is mostly desirable to control the chaotic system to limit cycle or a steady state. The pioneers Ott, Grebogi, and Yorke (OGY) presented a method to convert chaotic systems, using small time dependent changes of the accessible parameters of the system to a system with limit cyclic orbits [3]. Another approach was reported by Hubler et. al. [4], [5] where the authors applied small perturbation to the forcing function. Similar approaches can be found in [6], [7]. In both of the cases the resulting states are no longer a solution of the original equation of the system. Since any chaotic system possesses an infinite numbers of unstable periodic orbits, application of small changes can settle the system to a large number of different orbit. This phenomenon is particularly interesting because the same change of parameters can only change the steady state slightly if the system is not chaotic but periodic or a stable system. This property of chaotic systems can be exploited to choose a particular orbit from a diverse range of orbits. What this method fails to address is that in many cases it is desirable to control the chaotic element to a steady state rather than a limit cycle. But these method can only be applicable when the target is a stable orbit. Also since small measurement errors in initial state exponentially diverges to widely different states in future outcomes, the application of these method becomes limited. To overcome measurement error, Hubinger et. al. Suggested local control of the controllable parameter in OGY method. But all these solutions were unable to steady the element to a constant steady state [8].

Although the discussed methods reduced chaotic systems to a limit cycle, they were unable to set the states to a stable point. This problem was addressed using classical feedback by Pyragas et. al. [9], [10]. The drawback of the system is either the gains of the feedback path has to be set experimentally, or through an extensive analysis of the system, which is not possible or feasible in all cases. Proposed by Kokotovic et. al., a recursive method named backstepping also a widely used and popular method for stabilization of chaotic systems [11], [12].

If the system can be represented by the following equation

$$\dot{x} = Ax + B\,\beta^{-1}(x)[u - \alpha(x)], \quad \ldots 1)$$

then it can be linearized via the state feedback

$$u = \alpha(x) + \beta(x)v$$

and techniques for linear systems can be applied to modify the system [13], [14]. Note that this procedure is an Exact Linearization of the original system unlike approximation by Jacobian Linearization. But the limitation of this procedure is that the system must be in a form of equation 1). To relax this restriction, Su invented a differential geometric method [14]. The only restriction that remains is that the system have to be diffeomorph [15] to equation 1), which in turn can be transformed to an unique controller canonical form using similarity transformation [16], [17]. Then we can simply use classical state feedback scheme and modify the system to meet our requirements (we can place the eigenvalues of the A matrix on the left half plane if we desire the system to be stable or we can place two of them in the imaginary axis if we want the chaotic system to follow some orbit).

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One of the common constraints in the successful schemes is that the states of the chaotic system must be visible to properly construct the linear system. There are very few methods presented which discusses about observer based state feedback for chaotic systems. Among them, Leu et. al. employed fuzzy-neural networks to approximate nonlinear system [18], [19]. In this article we have proposed a much simpler model to employ observer based state feedback for chaotic system.

This paper includes 5 sections. In Section II, we introduce a study of the mathematical background for the proposed method. Section III presents the proposed method. Section IV contains simulation results of the proposed method for some chaotic systems. We conclude the article in Section V.

II. Exact Linearization by diffeomorphic Transformation

Su relaxed the restriction of Exact Linearization by introducing the concept of $\Im$ equivalence [16].

Two systems $\dot{x}=f(x,u)$ and $\dot{y}=g(y,v)$ are said to be $\Im$ equivalent if there exists a C^∞ diffeomorphism

$$T:\mathbb{R}^{n+1}\rightarrow\mathbb{R}^{n+1}$$

with

$$T_1(x,u)=y_1,\dots T_n(x,u)=y_n \text{ and } T_{n+1}(x,u)=v,$$

such that

$$\dot{T}_i=g_i(T_1,\dots T_{n+1}) \quad \forall i.$$

Then using a little manipulation it can be shown that

$$\frac{\partial T_1}{\partial u}=0,\dots\frac{\partial T_n}{\partial u}=0,\frac{\partial T_{n+1}}{\partial u}\neq 0$$

from the fact that every linear controllable system is $\Im$ equivalent to

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\\ \vdots\\ \dot{y}_{n+1}\end{pmatrix}=\begin{pmatrix}y_2\\ \vdots\\ y_{n+1}\\ 0\end{pmatrix}+\begin{pmatrix}0\\0\\ \vdots\\1\end{pmatrix}v \qquad \dots 1)$$

corresponding T transformation for a system $\dot{x}=f(x)+g(x).\phi(x,u)$

where $\phi(x,u)$ is a scalar, $\phi(0,0)=0, f(0)=0$ and $\frac{\partial\phi}{\partial u}\neq 0$ is given by

$$T_{i+1}=\sum_{j=1}^{n}\frac{\partial T_i}{\partial x_j}\left(f_j(x)+g_j(x)\cdot\phi(x,u)\right),\forall i\neq n+1$$

since $\frac{\partial T_1}{\partial u}=0,\dots\frac{\partial T_n}{\partial u}=0$

$$T_{i+1}=\sum_{j=1}^{n}\frac{\partial T_i}{\partial x_j}f_j(x),\ \forall i=1,\dots,n-1$$

and

$$T_{n+1}=\sum_{j=1}^{n}\frac{\partial T_n}{\partial x_j}\left(f_j(x)+g_j(x)\cdot\phi(x,u)\right)$$

which can be written in terms of product space [20] as

$$T_{i+1}=\langle dT_i,f\rangle,\ \forall i=1,\dots,n-1$$
$$T_{n+1}=\langle dT_n,f\rangle+\langle dT_n,g\rangle\phi \qquad \dots 2)$$

III. State Feedback Model

With the help of equation 1, equation 2 can be rewritten as

$$v=T_{n+1}(x,u)$$

since the transformation is diffeomorphic

$$u=T^{-1}{}_{n+1}(x,v)$$

With this knowledge, we can design the observer-based state feedback model as proposed in figure 1.

To validate this model, we have tested it over two chaotic systems, Lorentz Attractor [21], [22], [23].

A. *Lorentz Attractor*

The equations for Lorentz attractor are given by

$$\frac{dx}{dt}=p(y-x)$$
$$\frac{dy}{dt}=-xz-y$$
$$\frac{dz}{dt}=xy-z-R$$

In this equation $R=R_0+u$ is the Rayleigh number, u is the controlled forcing function and p is Prandtl number. For $R_0=28$, this system demonstrates chaotic behavior with 3 unstable equilibrium, $(\sqrt{R_0-1},\sqrt{R_0-1},-1),(-\sqrt{R_0-1},-\sqrt{R_0-1},-1)$, $(0,0,R_0)$ [24]. We shift the coordinate to $(x-\sqrt{R_0-1},y-\sqrt{R_0-1},z+1)\rightarrow(x_1,x_2,x_3)$ to define the equilibrium point $(\sqrt{R_0-1},\sqrt{R_0-1},-1)\rightarrow(0,0,0)$. With the change in coordinates, the state equation becomes

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-px_1+px_2\\ x_1-x_2-\sqrt{R_0-1}\,x_3-x_1x_3\\ \sqrt{R_0-1}(x_1+x_2)-x_3+x_1x_2\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}u$$

Corresponding $\Im$ equivalent can be obtained as [25]

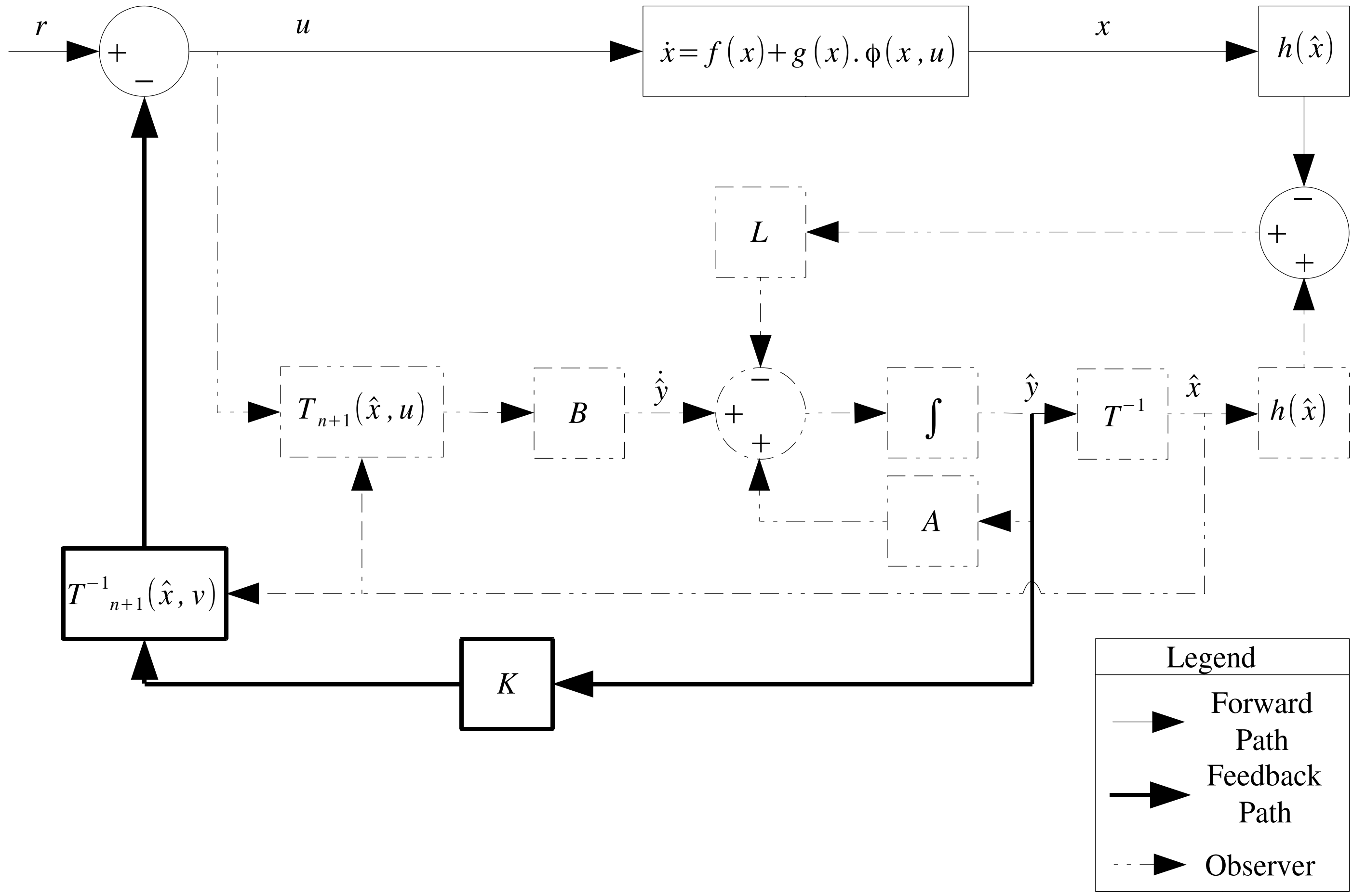

Figure 1. Proposed Observer-based State Feedback Model

$$T_1 = x_1$$

$$T_2 = -px_1 + px_2$$

$$T_3 = (p^2+p)(x_1-x_2) - p\sqrt{R_0-1}\,x_3 - px_1x_3$$

and

$$T_4 = v = \langle dT_{3,} f\rangle + \langle dT_{3,} g\rangle u = q(x) + s(x)u$$

$$\begin{aligned}\langle dT_{3,} f\rangle &= q(x)\\ &= (p^2+p-px_3)(-px_1+px_2)\\ &\quad -(p^2+p)(x_1-x_2-\sqrt{R_0-1}-x_1x_3)x_2\\ &\quad -p(\sqrt{R_0-1}+x_1)[\sqrt{R_0-1}(x_1+x_2)-x_3+x_1x_2]\end{aligned}$$

$$\langle dT_{3,} g\rangle = s(x) = p(x_1+\sqrt{R_0-1})$$

The inverse transformation can be obtained as

$$T^{-1}{}_1 : x_1 = y_2$$

$$T^{-1}{}_2 : x_2 = \frac{y_3}{p} + y_2$$

$$T^{-1}{}_3 : x_3 = \frac{-y_4 - (1+\frac{1}{p})\,y_3}{\sqrt{R_0-1} + y_2}$$

$$T^{-1}{}_4 : u = \frac{v-q(x)}{s(x)}$$

IV. Simulation Results

For the simulation, we choose the poles of the observer to be at $(-1,-2,-3,-4)$. Corresponding K matrix was found to be $K = [24\ \ 50\ \ 35\ \ 10]$. Theoretically this choice would change the chaotic system to a stable system and force the state of the system to the unstable equilibrium at origin. Simulation was performed using various initial states of the chaotic attractor and all the observation were in congruency with the anticipated result. Only one of the states x_1 were made observable during the experiment. Experiments with different observability were performed, but due to similarity of outcomes, the observation are omitted. Figure 2, Figure 3 and Figure 4 represent response of the Lorentz attractor with different initial conditions and both controlled and uncontrolled responses.

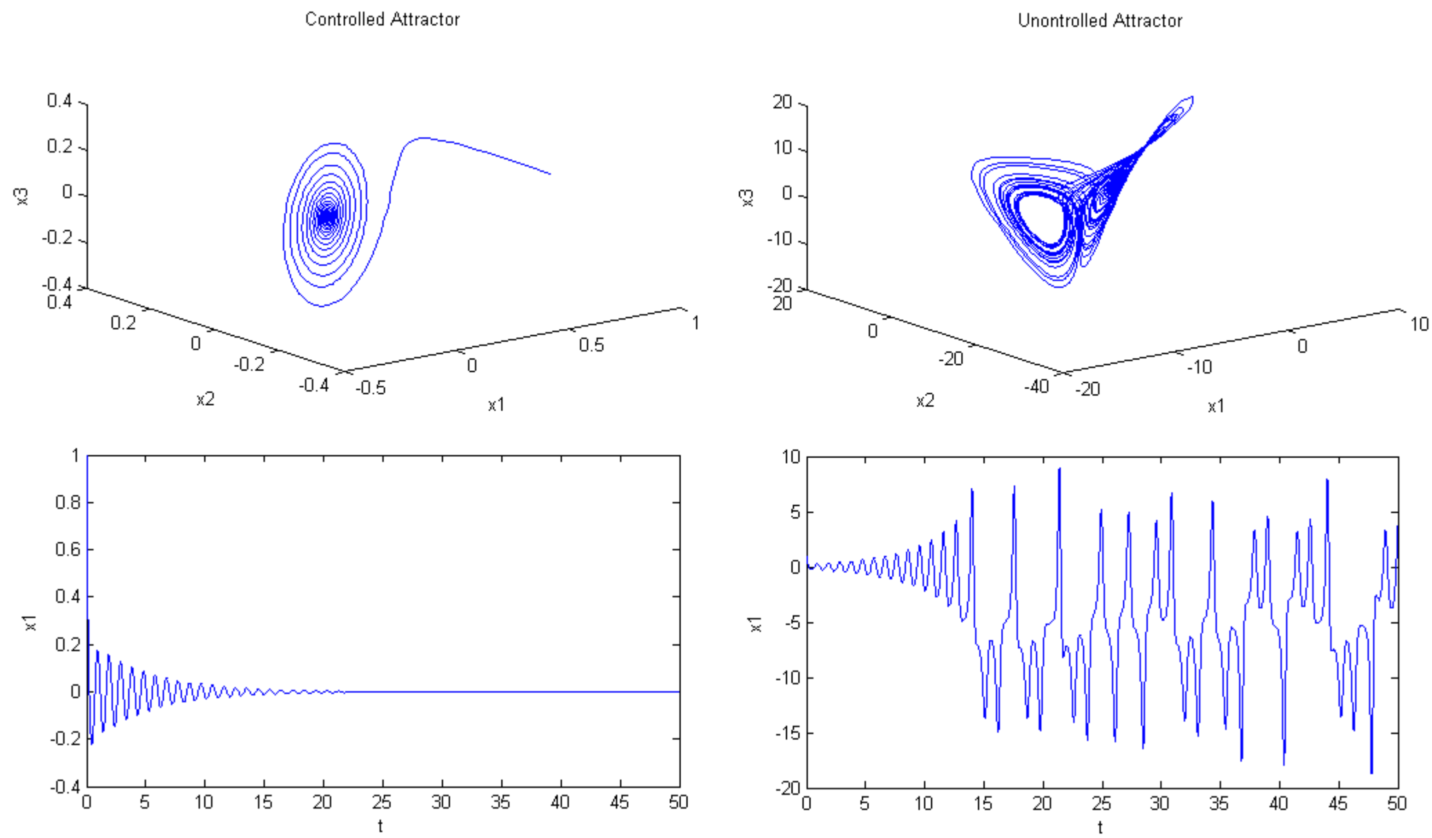

Fig 2. Response of Lorrentz Attractor with initial state of $[1\ \ 0\ \ 0]$

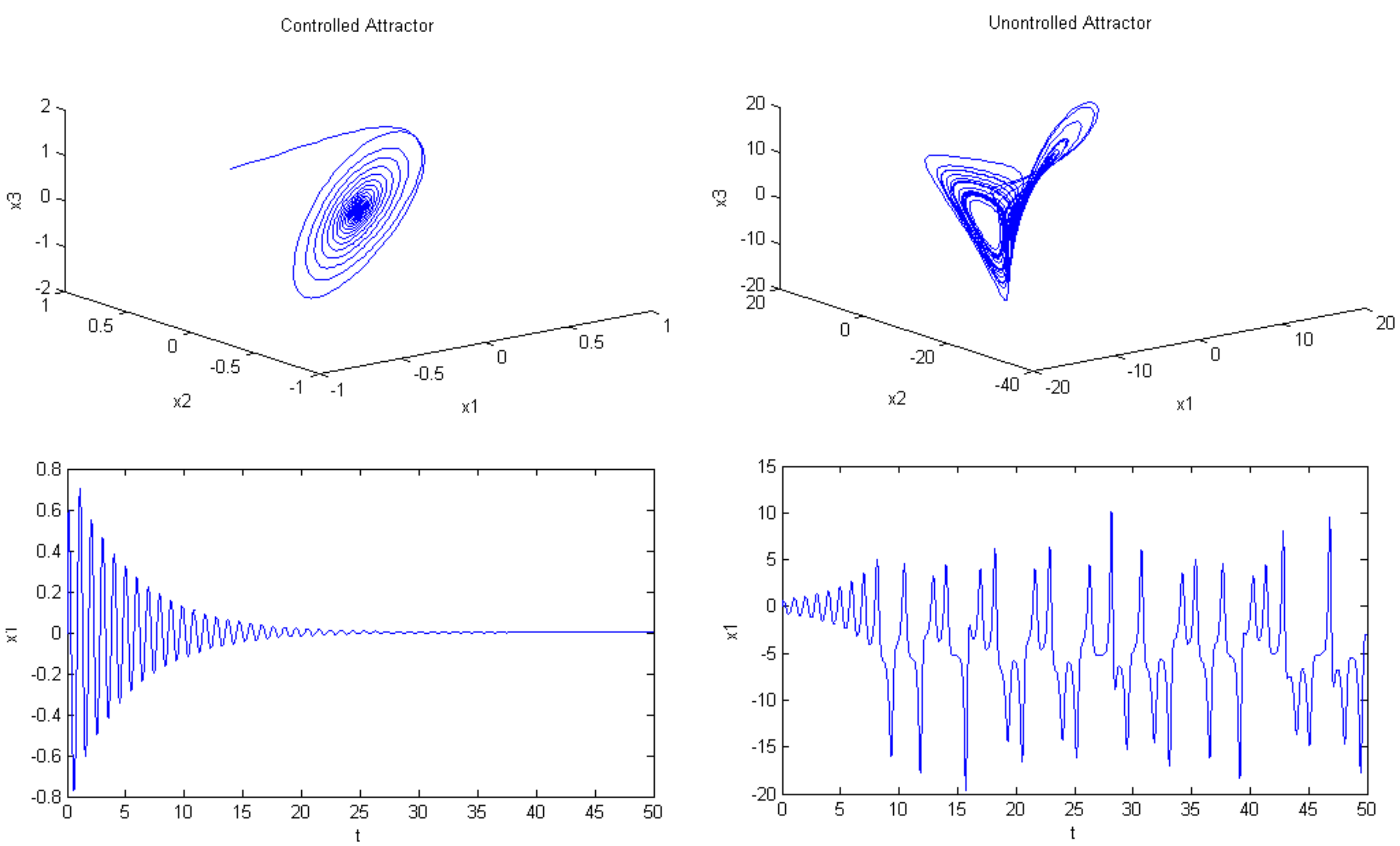

Fig 3. Response of Lorrentz Attractor with initial state of $[0\ \ 1\ \ 0]$

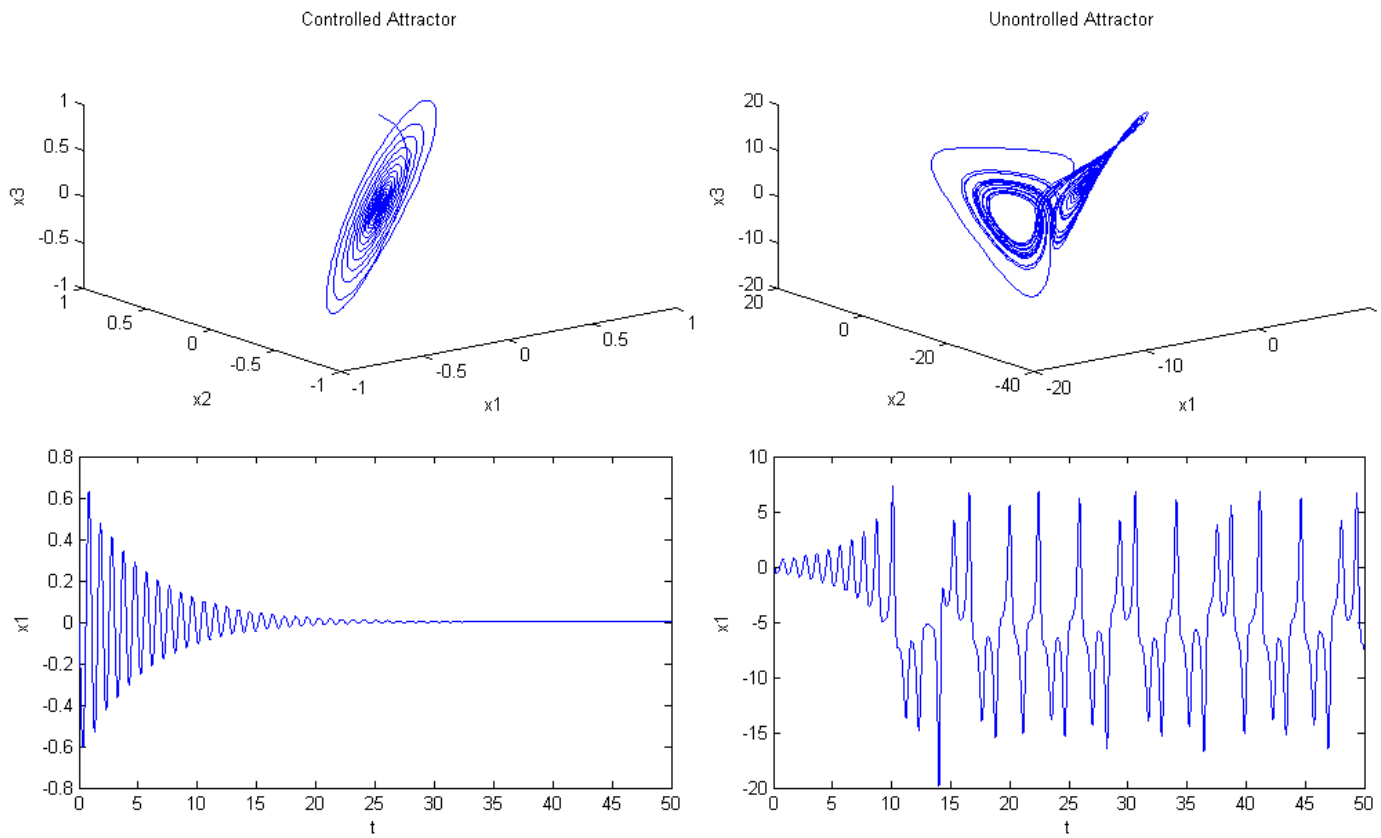

Fig 4. Response of Lorrentz Attractor with initial state of $[0\ \ 0\ \ 1]$

V. Conclusion

In this article, we present a novel approach for stabilization of chaotic system when the states of the system are observable. We proved the feasibility of our approach by studying the behavior of Lorentz attractor with different initial conditions under the proposed scheme. With careful modification the approach can also be applied to obtain a limit cyclic orbit from the chaotic system. Also the same methodologies are applicable for other non-linear systems without loss of generalization.

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