

# General Vector Space (3A)

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# Vector Space

$V$ : non-empty set of objects

defined operations:

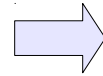
addition

$$\mathbf{u} + \mathbf{v}$$

scalar multiplication

$$k \mathbf{u}$$

if the following axioms are satisfied  
for all object  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$  and all scalar  $k$ ,  $m$



$V$ : vector space

objects in  $V$ : vectors

1. if  $\mathbf{u}$  and  $\mathbf{v}$  are objects in  $V$ , then  $\mathbf{u} + \mathbf{v}$  is in  $V$
2.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3.  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4.  $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$  (zero vector)
5.  $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + (\mathbf{u}) = \mathbf{0}$
6. if  $k$  is any scalar and  $\mathbf{u}$  is objects in  $V$ , then  $k\mathbf{u}$  is in  $V$
7.  $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8.  $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9.  $k(m\mathbf{u}) = (km)\mathbf{u}$
10.  $1(\mathbf{u}) = \mathbf{u}$

# Test for a Vector Space

1. Identify the set  $V$  of objects
2. Identify the addition and scalar multiplication on  $V$
3. Verify  $u + v$  is in  $V$  and  $ku$  is in  $V$   
**closure** under **addition** and **scalar multiplication**
4. Confirm other axioms.

1. if  $u$  and  $v$  are objects in  $V$ , then  $u + v$  is in  $V$
2.  $u + v = v + u$
3.  $u + (v + w) = (u + v) + w$
4.  $0 + u = u + 0 = u$  (zero vector)
5.  $u + (-u) = (-u) + (u) = 0$
6. if  $k$  is any scalar and  $u$  is objects in  $V$ , then  $ku$  is in  $V$
7.  $k(u + v) = ku + kv$
8.  $(k + m)u = ku + mu$
9.  $k(mu) = (km)u$
10.  $1(u) = u$

# Subspace

a subset  $W$  of a vector space  $V$

If the subset  $W$  is itself a vector space  $\Rightarrow$  the subset  $W$  is a **subspace** of  $V$

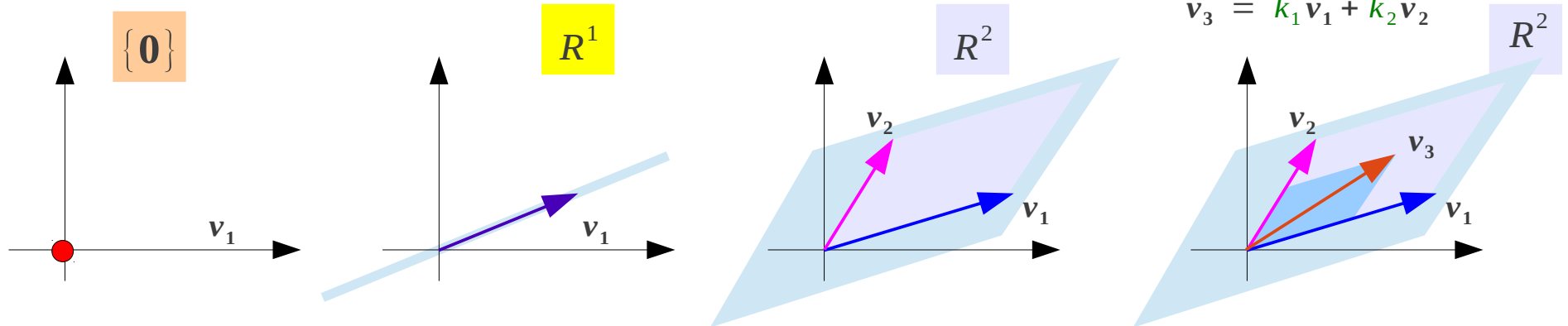
1. if  $u$  and  $v$  are objects in  $W$ , then  $u + v$  is in  $W$
2.  $u + v = v + u$
3.  $u + (v + w) = (u + v) + w$
4.  $0 + u = u + 0 = u$  (zero vector)
5.  $u + (-u) = (-u) + (u) = 0$
6. if  $k$  is any scalar and  $u$  is objects in  $W$ , then  $ku$  is in  $W$
7.  $k(u + v) = ku + kv$
8.  $(k + m)u = ku + mu$
9.  $k(mu) = (km)u$
10.  $1(u) = u$

# Subspace Example (1)

In vector space  $R^2$

|                                      |                   |             |                       |
|--------------------------------------|-------------------|-------------|-----------------------|
| any <b>one</b> vector                | (linearly indep.) | spans $R^1$ | line <u>through 0</u> |
| any <b>two</b> non-collinear vectors | (linearly indep.) | spans $R^2$ | plane                 |
| any <b>three or more</b> vectors     | (linearly dep.)   | spans $R^2$ | plane                 |

Subspaces of  $R^2$



# Subspace Example (2)

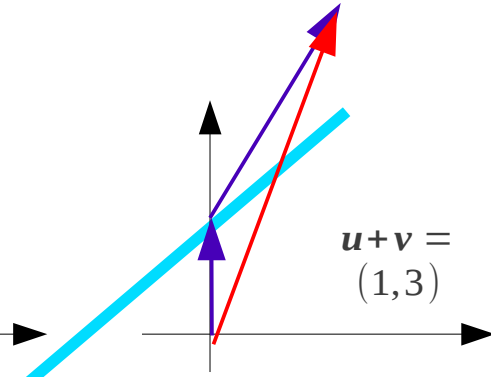
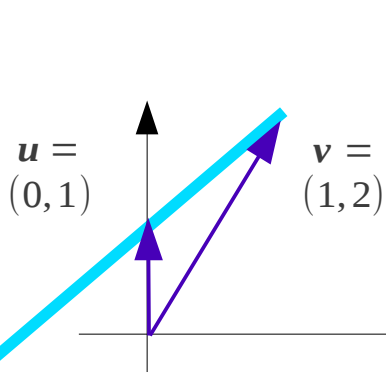
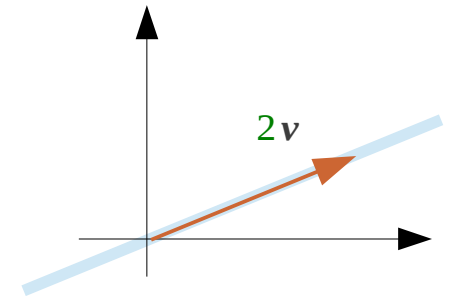
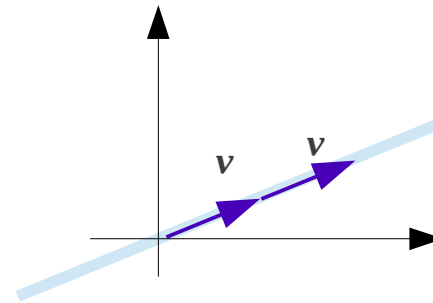
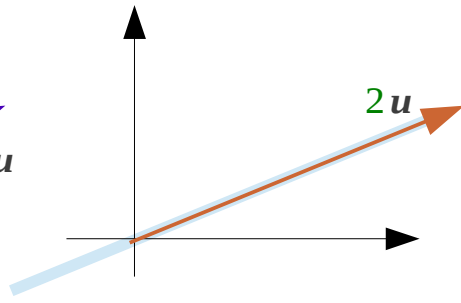
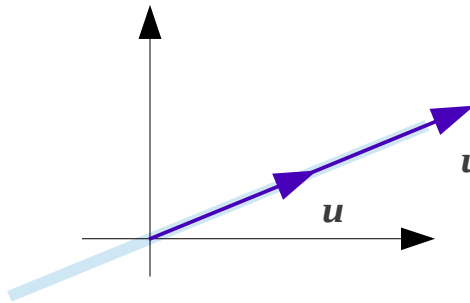
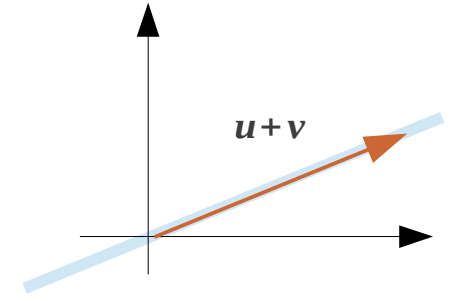
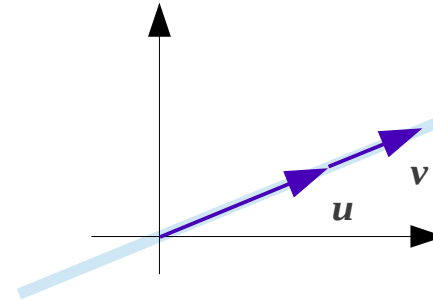
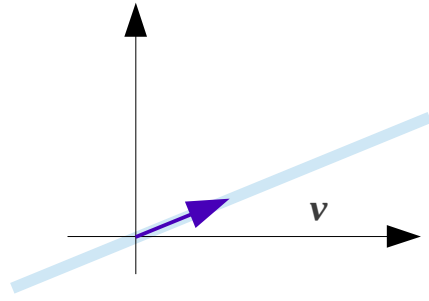
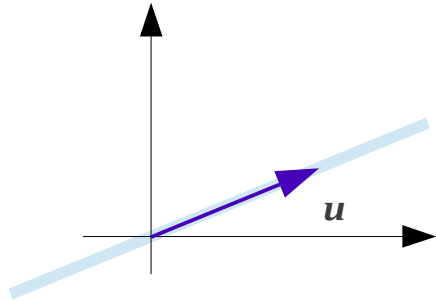
In vector space  $\mathbb{R}^2$

any one vector

(linearly indep.)

spans  $\mathbb{R}^1$

line through 0



~~vector space~~

# Subspace Example (3)

In vector space  $R^3$

|                                                         |                   |              |       |                        |
|---------------------------------------------------------|-------------------|--------------|-------|------------------------|
| any <b>one</b> vector                                   | (linearly indep.) | <b>spans</b> | $R^1$ | line <u>through 0</u>  |
| any <b>two</b> non-collinear vectors                    | (linearly indep.) | <b>spans</b> | $R^2$ | plane <u>through 0</u> |
| any <b>three</b> vectors<br>non-collinear, non-coplanar | (linearly indep.) | <b>spans</b> | $R^3$ | 3-dim space            |
| any <b>four or more</b> vectors                         | (linearly dep.)   | <b>spans</b> | $R^3$ | 3-dim space            |

**Subspaces of**  $R^2$

|         |                       |                        |             |
|---------|-----------------------|------------------------|-------------|
| $\{0\}$ | $R^1$                 | $R^2$                  | $R^3$       |
|         | line <u>through 0</u> | plane <u>through 0</u> | 3-dim space |



# Row & Column Spaces

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$\begin{aligned} \mathbf{r}_1 &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix} \\ \mathbf{r}_2 &= \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix} \\ &\vdots \\ \mathbf{r}_m &= \begin{pmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{aligned}$$

$\leftarrow \hspace{10em} \rightarrow$   
 $n$

$\mathbf{r}_i \in R^n$

**ROW Space**      subspace of  $R^n$   
 $= \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m\}$

**COLUMN Space**      subspace of  $R^m$   
 $= \text{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$

$\mathbf{c}_1 \quad \mathbf{c}_2 \quad \cdots \quad \mathbf{c}_n \quad \mathbf{c}_i \in R^m$

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \quad \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \quad \cdots \quad \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$\updownarrow$   
 $m$

# Row Space

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$r_i \in R^n$$

$$r_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix}$$

$$r_2 = \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}$$

$$r_m = \begin{pmatrix} \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$


$$n$$

**ROW Space**      **subspace of**  $R^n$

$$= \text{span} \{ r_1, r_2, \dots, r_m \}$$

$$k_1 r_1 + k_2 r_2 + \cdots + k_m r_m$$

$$= k_1 \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix}$$

$$+ k_2 \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}$$

$$+ k_m \begin{pmatrix} \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

# Column Spaces

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

**COLUMN Space**    subspace of  $R^m$   
 $= \text{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$

$c_i \in R^m$      $\mathbf{c}_1$      $\mathbf{c}_2$      $\mathbf{c}_n$

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \quad \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \quad \cdots \quad \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$k_1 \mathbf{c}_1 + k_2 \mathbf{c}_2 + \cdots + k_n \mathbf{c}_n$$

$$= k_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + k_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \quad \cdots \quad + k_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

# Null Space

$$\begin{matrix} & \xleftarrow{n} & & & & & \\ & & & & & & \\ \begin{matrix} \uparrow \\ m \\ \downarrow \end{matrix} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} & \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} & \begin{matrix} \uparrow \\ n \\ \downarrow \end{matrix} & = & \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{matrix} \uparrow \\ m \\ \downarrow \end{matrix}
 \end{matrix}$$

**NULL Space**

subspace of  $R^n$

solution space

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$Ax = x_1 c_1 + x_2 c_2 + \cdots + x_n c_n = 0$$

$$Ax = 0$$

$$Ax = x_1 c_1 + x_2 c_2 + \cdots + x_n c_n = b$$

$$Ax = b$$



# Solution Space of $\mathbf{Ax}=\mathbf{b}$ (1)

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

$$0 = 1$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$1 \cdot x_1 + 3 \cdot x_3 = -1$$

$$1 \cdot x_2 - 4 \cdot x_3 = 2$$

$$\left( \begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$1 \cdot x_1 - 5 \cdot x_2 + 1 \cdot x_3 = 4$$

Solve for a leading variable

$$x_1 = -1 - 3 \cdot x_3$$

$$x_1 = 4 + 5 \cdot x_2 - 1 \cdot x_3$$

$$x_2 = 2 + 4 \cdot x_3$$

Treat a free variable as a parameter

$$x_3 = t$$

$$x_2 = s \quad x_3 = t$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$$\begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

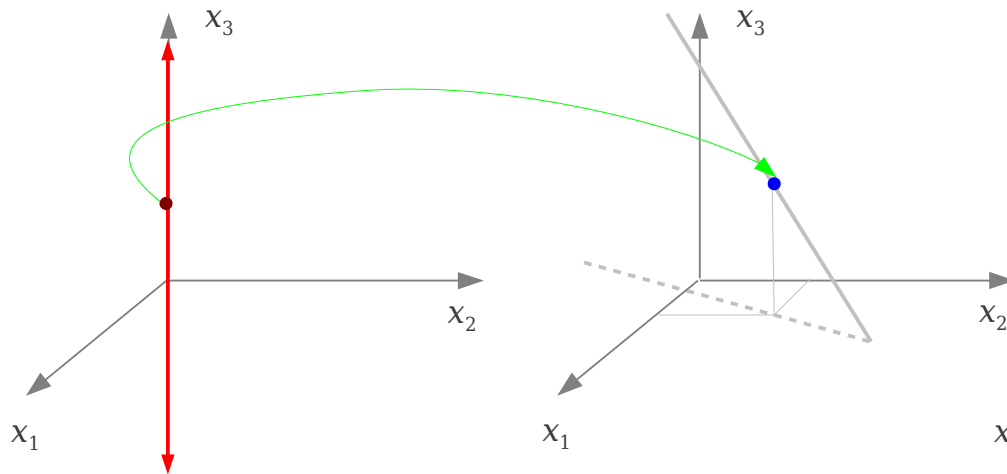
# Solution Space of $\mathbf{Ax}=\mathbf{b}$ (2)

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \quad \leftarrow \text{free variable}$$

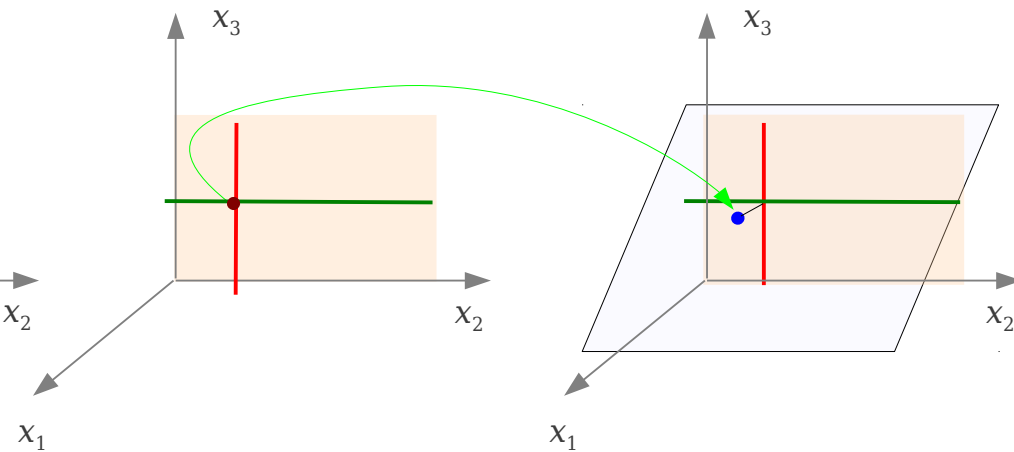
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases} \quad \leftarrow \text{free variable}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$



*infinitely many solutions*



*infinitely many solutions*

# Solution Space of $Ax=b$ (3)

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$$\begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

General Solution of



$$Ax = b$$

Particular Solution of

$$Ax = b$$

General Solution of

$$Ax = 0$$

Particular Solution of

$$Ax = b$$

General Solution of

$$Ax = 0$$



# Solution Space of $Ax=b$ (3)

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$$\begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

General  
Solution of  
 $Ax = b$



Particular  
Solution of  
 $Ax = b$

General  
Solution of  
 $Ax = 0$

Particular  
Solution of  
 $Ax = b$

General  
Solution of  
 $Ax = 0$

# Linear System & Inner Product (1)

## Linear Equations

### Corresponding Homogeneous Equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

$$\mathbf{a} = (a_1, a_2, \cdots, a_n)$$

$$\mathbf{x} = (x_1, x_2, \cdots, x_n)$$

normal vector

$$\mathbf{a} \cdot \mathbf{x} = b$$

$$\mathbf{a} \cdot \mathbf{x} = 0$$

each **solution** vector  $\mathbf{x}$  of a **homogeneous** equation  
**orthogonal** to the coefficient vector  $\mathbf{a}$

## Homogeneous Linear System

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = 0$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = 0$$

... ..

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = 0$$

$$\mathbf{r}_1 \cdot \mathbf{x} = 0$$

$$\mathbf{r}_2 \cdot \mathbf{x} = 0$$

...

$$\mathbf{r}_m \cdot \mathbf{x} = 0$$

# Linear System & Inner Product (2)

## Homogeneous Linear System

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$$\mathbf{r}_1 \cdot \mathbf{x} = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$

$$\mathbf{r}_2 \cdot \mathbf{x} = 0$$

... ..

...

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

$$\mathbf{r}_m \cdot \mathbf{x} = 0$$

each **solution** vector  $\mathbf{x}$  of a **homogeneous** equation  
**orthogonal** to the row vector  $\mathbf{r}_i$  of the coefficient matrix

Homogeneous Linear System  $\mathbf{A} \cdot \mathbf{x} = 0$   $\mathbf{A} : m \times n$

**solution set** consists of all vectors in  $R^n$   
that are **orthogonal** to every row vector of  $\mathbf{A}$

# Linear System & Inner Product (3)

Non-Homogeneous Linear System

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

$$\mathbf{A} : m \times n$$

Homogeneous Linear System

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$$

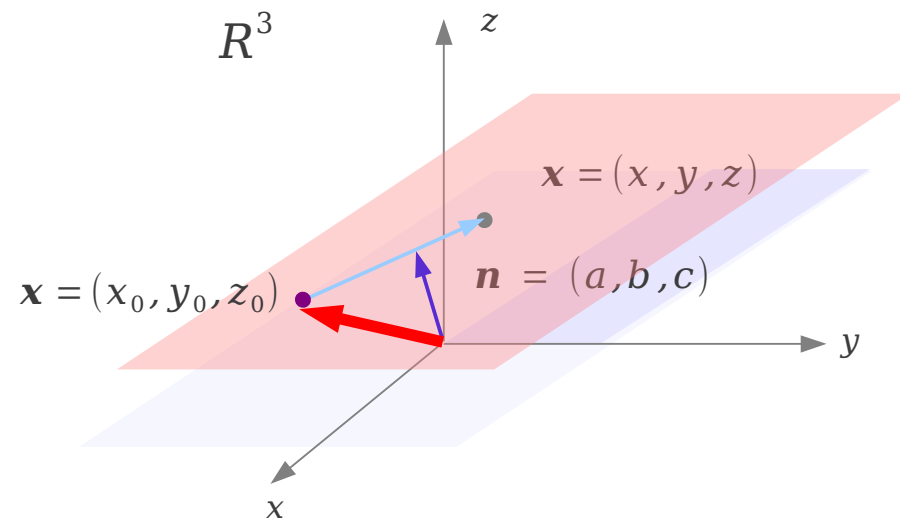
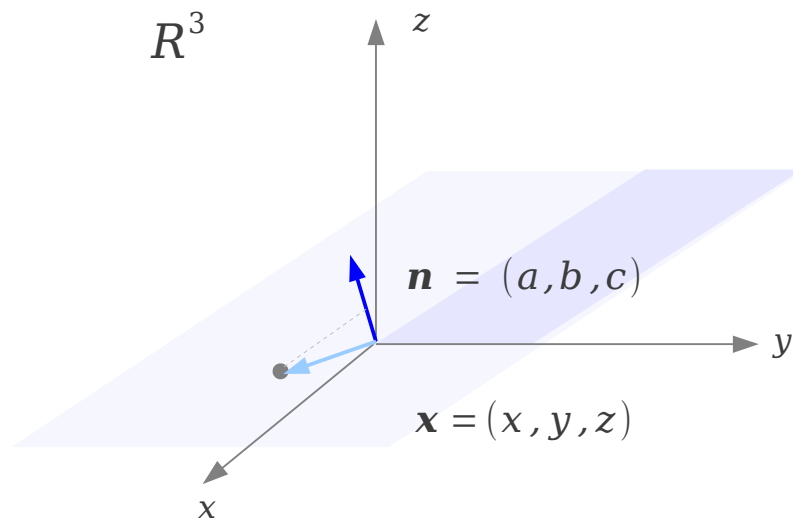
a particular solution

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

solution set consists of all vectors in  $R^n$  that are **orthogonal** to every row vector of  $\mathbf{A}$

+

a particular solution  $\mathbf{x}_0$   $\mathbf{A} \cdot \mathbf{x}_0 = \mathbf{b}$



# Linear System & Inner Product (4)

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left. \begin{array}{l} 2 \\ 3 \\ 1 \end{array} \right\} \begin{cases} \mathbf{r}_1 \cdot \mathbf{x} = 0 \\ \mathbf{r}_2 \cdot \mathbf{x} = 0 \\ \text{a line through the origin } R^1 \end{cases}$$

$$\left. \begin{array}{l} 1 \\ 3 \\ 2 \end{array} \right\} \begin{cases} \mathbf{r}_1 \cdot \mathbf{x} = 0 \\ \text{a plane through the origin } R^2 \end{cases}$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$$\begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

# Consistent Linear System $\mathbf{Ax}=\mathbf{b}$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

$\mathbf{Ax} = \mathbf{b}$  consistent  $\longleftrightarrow$

$$x_1 \mathbf{c}_1 + x_2 \mathbf{c}_2 + \cdots + x_n \mathbf{c}_n = \mathbf{b}$$

expressed in linear combination  
of column vectors

$\longleftrightarrow$   $\mathbf{b}$  is in the column space of  $\mathbf{A}$

$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

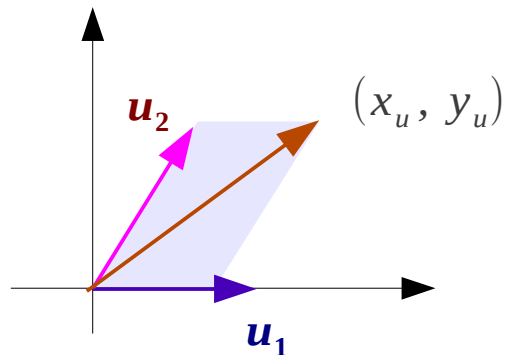
$$\mathbf{Ax} = x_1 \mathbf{c}_1 + x_2 \mathbf{c}_2 + \cdots + x_n \mathbf{c}_n = \mathbf{b}$$

# Dimension

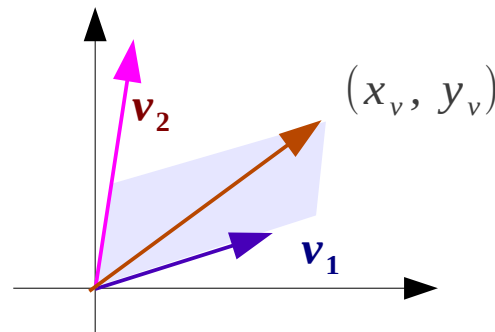
In a **finite-dimensional** vector space  $R^n$   ~~$R^\infty$~~   
all bases  $\rightarrow$  the **same number** of vectors  $n$

many bases but the same number of basis vectors

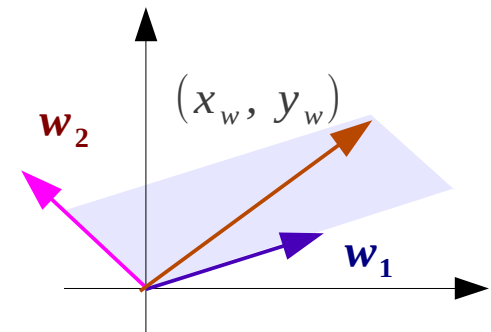
basis  $\{u_1, u_2\}$   $R^2$



basis  $\{v_1, v_2\}$   $R^2$



basis  $\{w_1, w_2\}$   $R^2$



The **dimension** of a **finite-dimensional** vector space  $V$

$\dim(V)$



the **number** of vectors in a **basis**

# Dimension of a Basis (1)

In vector space  $R^2$

|       |                               |                              |                        |                       |
|-------|-------------------------------|------------------------------|------------------------|-----------------------|
|       | any one vector                | (linearly indep.)            | <del>spans</del> $R^2$ | line <u>through 0</u> |
| basis | any two non-collinear vectors | (linearly indep.)            | spans $R^2$            | plane ←               |
|       | any three or more vectors     | <del>(linearly indep.)</del> | <del>spans</del> $R^2$ | plane                 |

In vector space  $R^3$

|       |                                                  |                              |                        |                        |
|-------|--------------------------------------------------|------------------------------|------------------------|------------------------|
|       | any one vector                                   | (linearly indep.)            | <del>spans</del> $R^3$ | line <u>through 0</u>  |
|       | any two non-collinear vectors                    | (linearly indep.)            | <del>spans</del> $R^3$ | plane <u>through 0</u> |
| basis | any three vectors<br>non-collinear, non-coplanar | (linearly indep.)            | spans $R^3$            | 3-dim space ←          |
|       | any four or more vectors                         | <del>(linearly indep.)</del> | <del>spans</del> $R^3$ | 3-dim space            |



# Dimension of a Basis (2)

In vector space  $R^n$

any  $n-1$  vectors

(linearly indep.)?

~~spans~~

~~$R^n$~~

line through 0

basis

$n$  vectors of a basis

(linearly indep.)

spans

$R^n$

plane

any  $n+1$  vectors

~~(linearly indep.)~~

spans?

$R^n$

plane

a finite-dimensional vector space  $V$

a basis  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$

- { a set of more than  $n$  vectors  $\rightarrow$  ~~(linearly indep.)~~
- { a set of less than  $n$  vectors  $\rightarrow$  ~~spans  $V$~~

$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  non-empty finite set of vectors in  $V$

$S$  is a basis



- {  $S$  linearly independent
- {  $S$  spans  $V$

# Plus / Minus Theorem

$S$  a nonempty set of vectors in a vector space  $V$

$S$  : linear independent

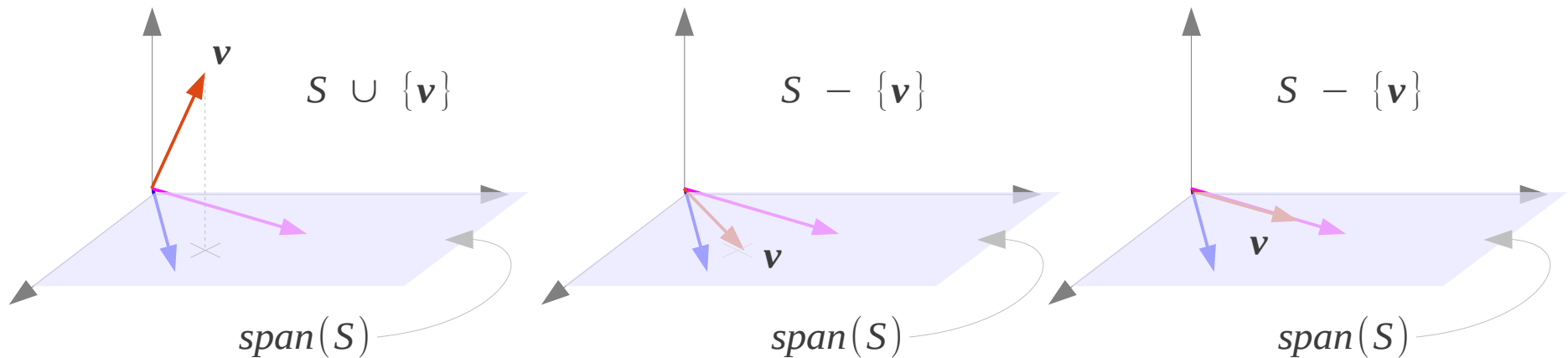
$\mathbf{v}$  a vector in  $V$  but outside of  $\text{span}(S)$

$\Rightarrow S \cup \{\mathbf{v}\}$  : linear independent

$\mathbf{v}, \mathbf{u}_i \in S$  linear combination

$\mathbf{v} = k_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 + \cdots + k_n \mathbf{u}_n$

$\Rightarrow \text{span}(S) = \text{span}(S - \{\mathbf{v}\})$



## References

- [1] <http://en.wikipedia.org/>
- [2] Anton, et al., Elementary Linear Algebra, 10<sup>th</sup> ed, Wiley, 2011
- [3] Anton, et al., Contemporary Linear Algebra,