DFT (4A)

- CTFS and DFT
- CTFT and DFT
- DTFT and DFT
- CTFT and DFT
-

Young Won Lim

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

CTFS with Complex Coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1, 2, \ldots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$A_0 = a_0$$

$$A_k = \frac{1}{2}(a_k - jb_k)$$

$$B_k = \frac{1}{2}(a_k + jb_k)$$

$$k = 1, 2, \ldots$$

$$x(t) = \sum_{k=0}^{\infty} \left(A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$A_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t}\, dt$$

$$k = 0, 1, 2, \ldots$$

$$B_k = \frac{1}{T} \int_0^T x(t) e^{+jk\omega_0 t}\, dt$$

$$k = 1, 2, \ldots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t}\, dt$$

$$k = -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} A_0 = a_0 & (k = 0) \\ A_k = \frac{1}{2}(a_k - jb_k) & (k > 0) \\ B_k = \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

CTFS and DTFS

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$ **CTFS**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$x(t) \approx \sum_{k=-M}^{+M} C_k \, e^{+jk\omega_0 t} \qquad N = 2M + 1$$

$$jk\omega_0 t \rightarrow k\left(\frac{2\pi}{T}\right) n \left(\frac{T}{N}\right) = \left(\frac{2\pi}{T}\right) n k$$

$$x[n] = \sum_{k=-M}^{+M} \gamma_k \, e^{+j\left(\frac{2\pi}{N}\right) nk}$$

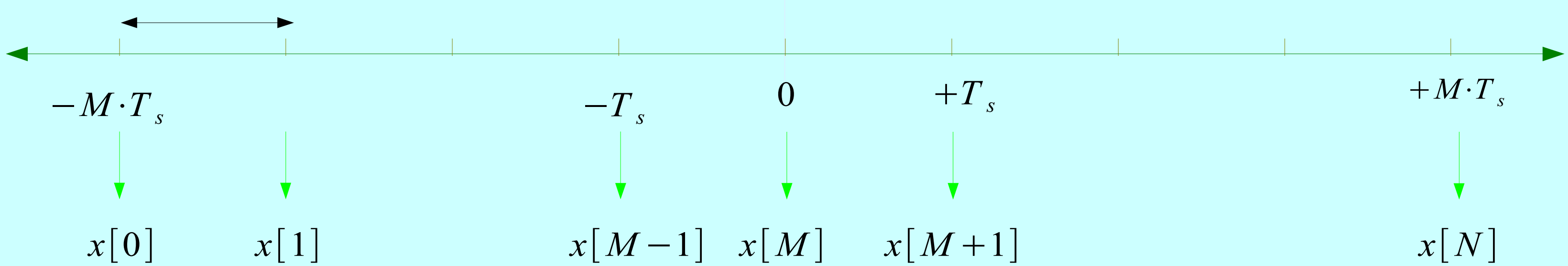
$$n = 0,\ 1,\ 2,\ \ldots,\ N-1,$$

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right) nk\,\omega_0 t}$$

$$k = -M,\ \ldots,\ 0,\ \ldots,\ +M$$

$$x_{FS}(t) = \sum_{k=-\infty}^{+\infty} \gamma_k \, e^{+jk\omega_0 t}$$ **DTFS**

$T_s = \dfrac{T}{N}$ **Time Samples**



$0 \le n < L$ $\qquad T_s \rightarrow 0$

CTFT and DFT

Continuous Time Fourier <u>Transform</u>

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Discrete Fourier <u>Transform</u>

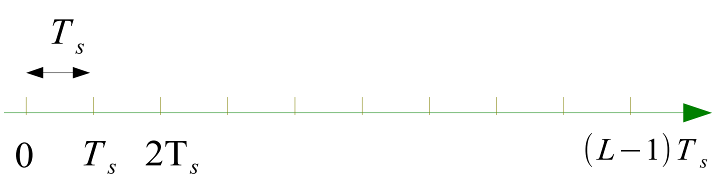
$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

From CTFT to DFT (1)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Time Samples

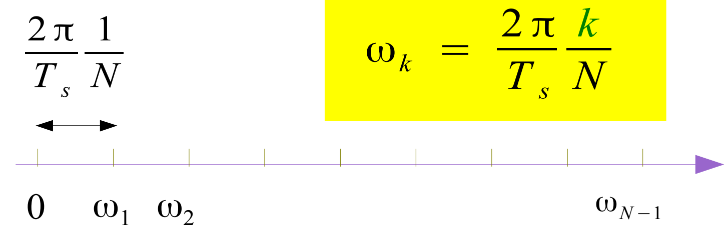


$$t \to nT_s \qquad dt \to T_s \qquad \int \to \sum$$

$$0 \le n < L \qquad\qquad T_s \to 0$$

Frequency Samples

$$\omega_k = \frac{2\pi}{T_s}\frac{k}{N}$$



$$\omega \to \omega_k \qquad d\omega \to \frac{2\pi}{T_s}\frac{1}{N} \qquad \int \to \sum$$

$$0 \le k < N \qquad\qquad 0 \le \omega_k < \frac{2\pi}{T_s}$$

From CTFT to DFT (2)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$t \to nT_s \qquad dt \to T_s \qquad \int \;\to\; \sum \qquad 0 \le n < L$$

$$\hat{X}(j\omega) = \sum_{n=-\infty}^{+\infty} x(nT_s)\, e^{-j\omega nT_s} \cdot T_s \quad \Leftrightarrow \quad x(nT_s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{X}(j\omega)\, e^{+j\omega nT_s}\, d\omega$$

$$\omega \to \omega_k \qquad d\omega \to \frac{2\pi}{T_s}\frac{1}{N} \qquad \int \;\to\; \sum \qquad 0 \le k < N$$

$$\omega_k = \frac{2\pi}{T_s}\frac{k}{N}$$

$$\hat{X}(j\omega_k) = T_s \sum_{n=0}^{L-1} x[n]\, e^{-j\omega_k nT_s} \quad \Leftrightarrow \quad x[n] = \frac{1}{2\pi} \sum_{k=0}^{N-1} \hat{X}(j\omega_k)\, e^{+j\omega_k nT_s}\; \frac{2\pi}{T_s}\frac{1}{N}$$

From CTFT to DFT (3)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Time Samples

$$t \to nT_s \qquad dt \to T_s \qquad \int \to \sum$$

Frequency Samples

$$\omega \to \omega_k \qquad d\omega \to \frac{2\pi}{T_s}\frac{1}{N} \qquad \int \to \sum$$

$$\hat{X}(j\omega_k) = T_s \sum_{n=0}^{L-1} x[n]\, e^{-j\omega_k n T_s} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi} \sum_{k=0}^{N-1} \hat{X}(j\omega_k)\, e^{+j\omega_k n T_s}\, \frac{2\pi}{T_s}\frac{1}{N}$$

$$\omega_k T_s \to \frac{2\pi}{N}k \quad \Rightarrow \quad \omega_k n T_s \to \frac{2\pi}{N}kn \qquad \omega_k = \frac{2\pi}{T_s}\frac{k}{N}$$

$$\frac{1}{T_s}\hat{X}(j\omega_k) = \sum_{n=0}^{L-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn} \quad \Longleftrightarrow \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{T_s}\hat{X}(j\omega_k)\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

From CTFT to DFT (4)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Time Samples

$$t \rightarrow n T_s \qquad dt \rightarrow T_s \qquad \int \rightarrow \sum$$

Frequency Samples

$$\omega \rightarrow \omega_k \qquad d\omega \rightarrow \frac{2\pi}{T_s}\frac{1}{N} \qquad \int \rightarrow \sum$$

$$\omega_k T_s \rightarrow \frac{2\pi}{N} k \quad \Rightarrow \quad \omega_k n T_s \rightarrow \frac{2\pi}{N} k n \qquad \omega_k = \frac{2\pi}{T_s}\frac{k}{N}$$

$$\frac{1}{T_s}\hat{X}(j\omega_k) = \sum_{n=0}^{L-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} \frac{1}{T_s}\hat{X}(j\omega_k)\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

From CTFT to DFT (5)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$\frac{1}{T_s}\hat{X}(j\omega_k) = \sum_{n=0}^{L-1} x[n]\, e^{-j(\frac{2\pi}{N})kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} \frac{1}{T_s}\hat{X}(j\omega_k)\, e^{+j(\frac{2\pi}{N})kn}$$

Discrete Fourier Transform

$$L = N$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

DTFT and CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s)\, e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$\Leftrightarrow$ $=$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

DTFT of a sampled signal

$$X(e^{j\hat{\omega}}) \Big|_{\hat{\omega} = \omega T_s} = X(e^{j\omega T_s}) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c\left(j\left(\omega - \frac{2\pi k}{T_s}\right)\right)$$

CTFT of a sampled signal

DTFT and DFT

DTFT of a sampled signal

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

$$\hat{\omega} = \omega T_s$$

$$\hat{\omega} \to \hat{\omega}_k \qquad 0 \le \hat{\omega}_k < 2\pi \qquad 0 \le k < N \qquad 0 \le n < L$$

$$X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n]\, e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = \left(\frac{2\pi}{N}\right) k$$

DFT of a sampled signal

$$X[k] = \quad X(e^{j(2\pi/N)k}) = \sum_{n=0}^{L-1} x[n]\, e^{-j(2\pi/N)k n}$$

DTFT *sampled in frequency*

DFT of a sampled signal

$$X[k] = X(e^{j(2\pi/N)k}) = \sum_{n=0}^{L-1} x[n]\, e^{-j(2\pi/N)kn}$$

DTFT *sampled in frequency*

$$X(e^{j\omega T_s})\Big|_{\omega = \frac{2\pi k}{N T_s}}$$

CTFT evaluated at $\omega = \dfrac{2\pi k}{N T_s}$

$$= \frac{1}{T_s} \sum_{l=-\infty}^{+\infty} X_c(j(\omega - l\omega_s))\Big|_{\omega = \frac{2\pi k}{N T_s}}$$

$$= \frac{1}{T_s} \sum_{l=-\infty}^{+\infty} X_c(j(\omega - l\frac{2\pi}{T_s}))\Big|_{\omega = \frac{2\pi k}{N T_s}}$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003