# Complex Functions (1A)

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# Analytic Functions

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

$$\Delta f = f(z+\Delta z) - f(z)$$

$$\Delta z = \Delta x + i \Delta y$$

$f(z)$ : **analytic** in a region ⟺

$f(z)$ has a (unique) derivative at every point of the region

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

$f(z)$ : **analytic** at a point $z = a$ ⟺

$f(z)$ has a (unique) derivative at every point of some small circle about $z = a$

# Singular Point

**Regular** point of $f(z)$ ⬌

a point at which $f(z)$ is analytic

**Singular** point of $f(z)$ ⬌

a point at which $f(z)$ is not analytic

**Isolated Singular** point of $f(z)$ ⬌

a point at which $f(z)$ is analytic everywhere else inside some small circle about the singular point

# Cauchy-Riemann Condition

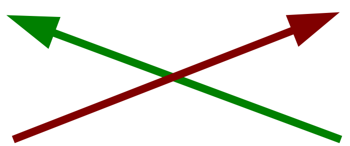
$f(z) = u(x,y) + i v(x,y)$ : **analytic** in a region

⇒ in that region $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \qquad \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

$f(z) = u(x,y) + i v(x,y)$

$\dfrac{\partial}{\partial x} \qquad \dfrac{\partial}{\partial y}$

$f(z) = u(x,y) + i v(x,y)$



$\dfrac{\partial}{\partial x} \qquad \dfrac{\partial}{\partial y}$

# Analytic

$$f(z) = u(x,y) + iv(x,y)$$

$u(x,y),\ v(x,y),\ \dfrac{\partial u}{\partial x},\ \dfrac{\partial u}{\partial y},\ \dfrac{\partial v}{\partial x},\ \dfrac{\partial v}{\partial y}$ : continuous

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Rightarrow \quad f(z) = u(x,y) + iv(x,y)$$

: **analytic** at all points inside a region

not necessarily on the boundary

# Derivatives

$f(z) = u(x,y) + iv(x,y)$ : **analytic** in a region R

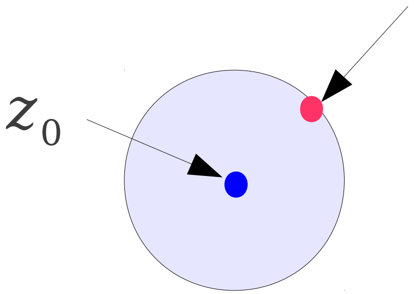
➡ derivatives of all orders at points inside region

$$f'(z_0),\ f''(z_0),\ f^{(3)}(z_0),\ f^{(4)}(z_0),\ f^{(5)}(z_0),\ \cdots$$

➡ Taylor series expansion about any point $z_0$ inside the region

The power series converges inside the circle about $z_0$

This circle extends to the nearest singular point



# Laplace Equation

$f(z) = u(x,y) + iv(x,y)$ : **analytic** in a region R

⇒ $u(x,y)$, $v(x,y)$ satisfy Laplace's equation in the region

harmonic functions

$u(x,y)$, $v(x,y)$ satisfy Laplace's equation in simply connected region

⇒ Real / imaginary part of an analytic function $f(z)$

# Cauchy's Theorem

$f(z)$ : **analytic** on and inside C ➡ $\oint_{around\,C} f(z)\,dz = 0$

simple closed curve

a continuously turning tangent

except possibly at a finite number of points

allow a finite number of corners (not smooth)

# Cauchy's Integral Formula

$f(z)$ : **analytic** on and inside simple close curve C

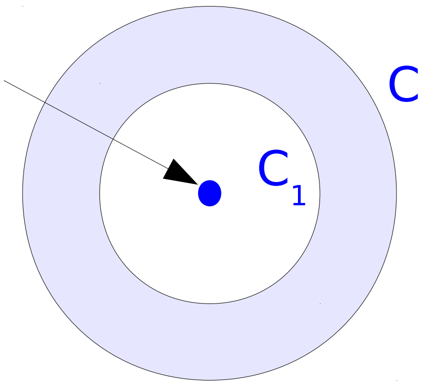
$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$

the value of $f(z)$

at a point $z = a$ inside C

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw$$

# Laurent's Theorem




$f(z)$ : **analytic** in the region R

between circles $C_1$, $C_2$

centered at $z_0$

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

Principal part

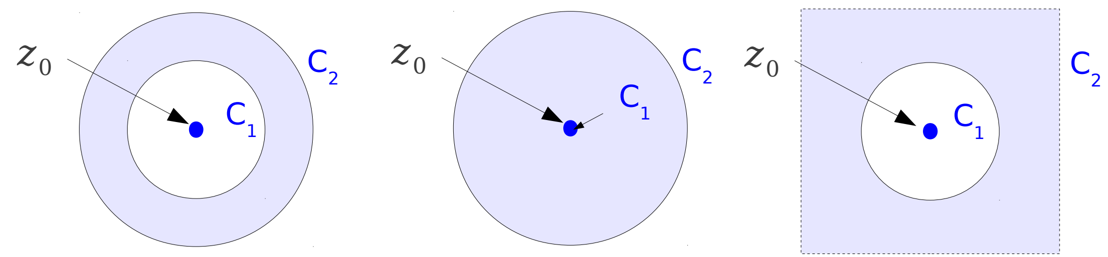
: **convergent** in the region R

# Laurent's Theorem - Coefficients

$f(z)$ : **analytic** in the region R

between circles $C_1$, $C_2$

centered at $z_0$




$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

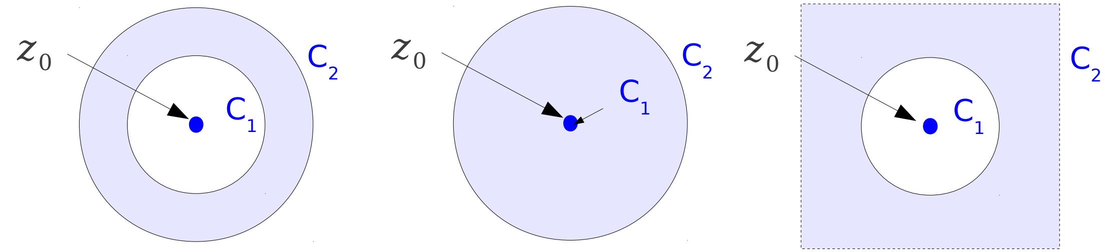
: **convergent** in the region R

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}}$$

# Laurent's Theorem – Some Points

$f(z)$ : **analytic** in the region R
between circles $C_1$, $C_2$
centered at $z_0$




$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

*regular point* $z_0$

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n}$$

*pole of order n* $z_0$

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{(z-z_0)}$$

*simple pole* $z_0$

$b_1$ *residue of* $f(z)$

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \cdots$$

*essential singularity* $z_0$



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# Residue Theorem (1)

$f(z)$ : **analytic** on and inside C ⇒ $\oint_{around\, C} f(z)\, dz = 0$

$f(z)$ : **analytic** on and inside C except $z_0$ ⇒

$z_0$ Isolated singular point

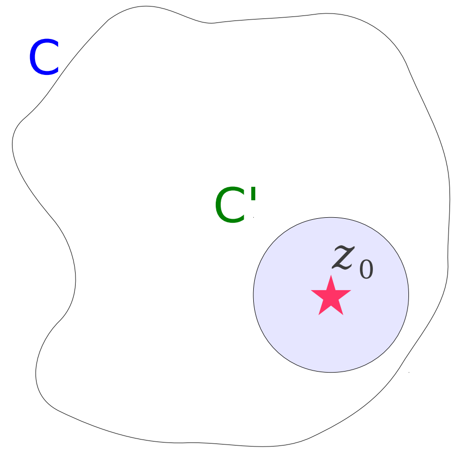
$$\oint_C f(z)dz = 2\pi i \cdot \sum \textit{the residues of } f(z) \textit{ inside } C$$

The integral around C is in the counterclockwise direction

# Residue Theorem (2)

$f(z)$ : **analytic** on and inside C except $z_0$

$z_0$ Isolated singular point




$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$
$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \cdots$$

$$\oint_C a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \, dz = 0$$

**analytic** on and inside C

along C' $z = z_0 + \rho e^{i\theta}$

$$\oint_{C'} \frac{b_1}{(z-z_0)} dz = b_1 \int_0^{2\pi} \frac{\rho i e^{i\theta} d\theta}{\rho e^{i\theta}} = b_1 \int_0^{2\pi} i \, d\theta = 2\pi i b_1$$

$$\oint_{C'} \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \cdots \, dz = 0$$

$$\int_0^{2\pi} e^{ik\theta} d\theta = \left[\frac{e^{ik\theta}}{ik}\right]_0^{2\pi} = 0$$

# Finding Residues (1)

$$\oint_C f(z)dz = 2\pi i \cdot \sum \textit{the residues of } f(z) \textit{ inside } C$$

The integral around C is in the counterclockwise direction

Methods of Finding Residues

**Laurent Series:** $b_1$ $1/(z-z_0)$

**Simple Pole :** $f(z)\cdot(z-z_0)$

**Multiple Pole :** $f(z)\cdot(z-z_0)^m$

# Finding Residues (2)

$$\oint_C f(z)dz = 2\pi i \cdot \sum \textit{the residues of } f(z) \textit{ inside } C$$

The integral around C is in the counterclockwise direction

**Laurent Series:** $b_1$ $1/(z-z_0)$

**Simple Pole :** $f(z)\cdot(z-z_0)$

**Multiple Pole :** $f(z)\cdot(z-z_0)^m$

$$R(z_0) = \lim_{z\to z_0}(z-z_0)f(z)$$

$$R(z_0) = \frac{g(z_0)}{h'(z_0)} \quad \Leftarrow \quad f(z) = \frac{g(z)}{h(z)} \qquad g(z_0) \neq 0, \quad h'(z_0) \neq 0, \quad h(z_0) = 0$$

$$R(z_0) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)$$

## References


[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”