Complex Functions (1A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Derivatives

the complex function f is defined in a neighborhood of a point z_0

Derivative of f at z_0

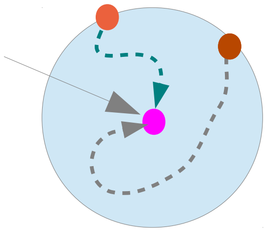
$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided that this limit exists

f is said to be **differentiable** at z_0

Δz can approach zero
from any convenient direction

z_0



Analyticity

f **differentiable** at z_0

f **differentiable** at every point in some *neighborhood* of z_0

⇒

the complex function f is **analytic** at a point z_0

A complex function can be differentiable *at a point* z_0 but differentiable *nowhere else*

$$f(z) = z\bar{z} = |z|^2 = x^2 + y^2$$

$$\lim_{\Delta z \to 0} \frac{(x+\Delta x)^2 + (y+\Delta y)^2 - x^2 - y^2}{\Delta x + i\Delta y}$$

A function that is **analytic** at **every** point z

:analytic function

$$f(z) = z^2$$

Analytic Functions

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

$$\Delta f = f(z + \Delta z) - f(z)$$

$$\Delta z = \Delta x + i \Delta y$$

$f(z)$: **analytic** in a region $\Longleftrightarrow$ $f(z)$ has a (unique) derivative at every point of the region

$f(z)$: **analytic** at a point $z = a$ $\Longleftrightarrow$ $f(z)$ has a (unique) derivative at every point of some small circle about $z = a$

Singular Point

Regular point of $f(z)$ ⟺ a point at which $f(z)$ is analytic

Singular point of $f(z)$ ⟺ a point at which $f(z)$ is not analytic

Isolated Singular point of $f(z)$ ⟺ a point at which $f(z)$ is analytic
everywhere else
inside some small circle
about the singular point

Cauchy-Riemann Condition (1)

$$f(z) = u(x,y) + i v(x,y)$$

: **differentiable** at a point

$$z = x + iy$$

$$\frac{\partial u}{\partial x} = + \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = - \frac{\partial u}{\partial y}$$

Necessary Condition for Analyticity

$$f(z) = u(x,y) + i v(x,y)$$

$$\frac{\partial}{\partial x} \qquad \frac{\partial}{\partial y}$$

$$f(z) = u(x,y) + i v(x,y)$$

$$\frac{\partial}{\partial x} \qquad \frac{\partial}{\partial y}$$

Cauchy-Riemann Condition (2)

$f(z) = u(x,y) + iv(x,y)$: **differentiable** at a point $z = x + iy$

⇒ $f'(z)$ exists

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \qquad \Delta z = \Delta x + i\Delta y$$

$$= \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta z}$$

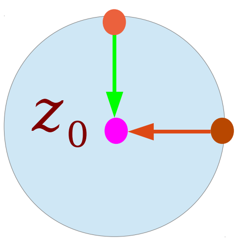
horizontal approach ⇒ $\Delta x \to 0$

$\Delta z \to 0$ $\qquad \boxed{\Delta y = 0}$

vertical approach ⇒ $\Delta y \to 0$

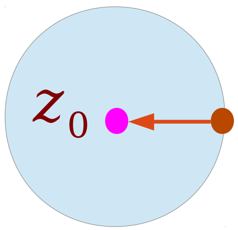
$\Delta z \to 0$ $\qquad \boxed{\Delta x = 0}$

must have the same $f'(z)$



Cauchy-Riemann Condition (3)

horizontal approach $\Delta z \to 0 \Rightarrow \Delta x \to 0 \quad \Delta y = 0$

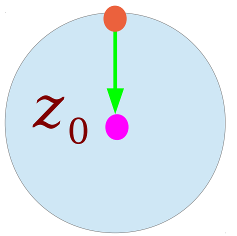


$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y) - u(x,y) - i v(x,y)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad = \frac{\partial f}{\partial x}$$

vertical approach $\Delta z \to 0 \Rightarrow \Delta y \to 0 \quad \Delta x = 0$



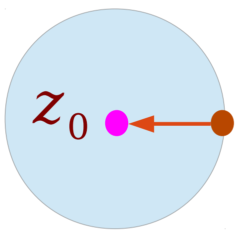
$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y) - u(x,y) - i v(x,y)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i \Delta y} + i \frac{v(x, y+\Delta y) - v(x,y)}{i \Delta y}$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad = -i \frac{\partial f}{\partial y}$$

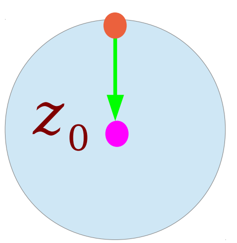
Cauchy-Riemann Condition (4)

horizontal approach $\Delta z \to 0 \Rightarrow \Delta x \to 0 \quad \boxed{\Delta y = 0}$



$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad = \frac{\partial f}{\partial x}$$

vertical approach $\Delta z \to 0 \Rightarrow \Delta y \to 0 \quad \boxed{\Delta x = 0}$



$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad = -i\frac{\partial f}{\partial y}$$

$$\Rightarrow \quad f'(z) = \boxed{\frac{\partial u}{\partial x}} + \boxed{i\frac{\partial v}{\partial x}} = \boxed{-i\frac{\partial u}{\partial y}} + \boxed{\frac{\partial v}{\partial y}} \quad \Rightarrow \quad \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \qquad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

To Be Analytic (1)

$f(z) = u(x,y) + iv(x,y)$: **analytic** in a domain D

$\Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$f(z) = u(x,y) + iv(x,y)$: **analytic** in a domain D

$\Leftarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$u(x,y),\ v(x,y)$: **continuous** on in a domain D

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$: **continuous** on in a domain D

To Be Analytic (2)

if the real functions and v(x,y) are continuous

and have continuous first order partial derivatives

in a neighborhood of z,

and if u(x,y) and v(x,y) satisfy

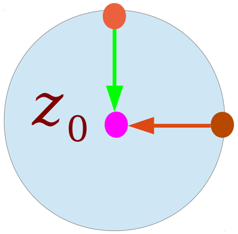
the Cauchy-Riemann equations at the point z,

then the complex function f(z) = u(x,y) + iv(x,y)

is **differentiable** at z

and f'(z) is as belows.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$



Derivatives

$f(z) = u(x,y) + iv(x,y)$: **analytic** in a region R

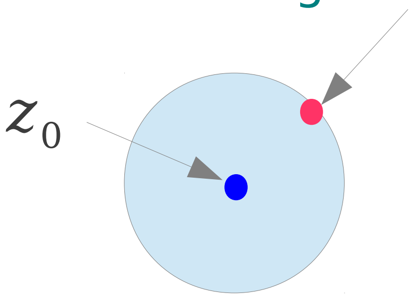
derivatives of all orders at points inside region

$$f'(z_0),\ f''(z_0),\ f^{(3)}(z_0),\ f^{(4)}(z_0),\ f^{(5)}(z_0),\ \cdots$$

Taylor series expansion about any point z_0 inside the region

The power series converges inside the circle about z_0

This circle extends to the nearest singular point



Laplace Equation

$f(z) = u(x, y) + i v(x, y)$: **analytic** in a region R

$\Rightarrow$ $u(x, y)$, $v(x, y)$ satisfy Laplace's equation in the region

harmonic functions

$u(x, y)$, $v(x, y)$ satisfy Laplace's equation in simply connected region

$\Rightarrow$ Real / imaginary part of an analytic function $f(z)$

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”