$$u_t + u\nabla u - \epsilon \nabla^2 u = 0$$

θ-implicit integration $\quad \theta = 1 \rightarrow$ implicit $\quad \theta = 0 \rightarrow$ explicit

$$\frac{u^k - u^{k-1}}{\Delta t} + \theta\left[u^k \nabla u^k - \epsilon \nabla^2 u^k\right] + (1-\theta)\left[u^{k-1}\nabla u^{k-1} - \epsilon \nabla^2 u^{k-1}\right] = 0$$

u^{k-1} is a known coefficient, move to rhs

$$\frac{u^k}{\Delta t} + \theta\left[u^k \nabla u^k - \epsilon \nabla^2 u^k\right] = \underbrace{\frac{u^{k-1}}{\Delta t} - (1-\theta)\left[u^{k-1}\nabla u^{k-1} - \epsilon \nabla^2 u^{k-1}\right]}_{f_{k-1}}$$

$u^k \nabla u^k$ is non-linear, linearize

$$u^k_{p+1} = u^k_p + \delta u^k_p \simeq u^k$$

where p is iteration index, δu^k_p is assumed small, u^k_p is a coefficient

$$\frac{u^k_p + \delta u^k_p}{\Delta t} + \theta\left[(u^k_p + \delta u^k_p)\nabla(u^k_p + \delta u^k_p) - \epsilon \nabla^2(u^k_p + \delta u^k_p)\right] = f_{k-1}$$

∇, and ∇^2 are linear operators, drop higher order terms

$$\frac{\delta u^k_p}{\Delta t} + \theta\left[u^k_p \nabla u^k_p + u^k_p \nabla \underline{\delta u^k_p} + \underline{\delta u^k_p}\nabla u^k_p + \cancelto{H.O.T}{\delta u^k_p \nabla \delta u^k_p} - \epsilon \nabla^2 u^k_p - \epsilon \nabla^2 \underline{\delta u^k_p}\right] = f_{k-1} - \frac{u^k_p}{\Delta t}$$

u^k_p is a known coefficient

$$\frac{\delta u^k_p}{\Delta t} + \theta\left[u^k_p \nabla \delta u^k_p + \delta u^k_p \nabla u^k_p - \epsilon \nabla^2 \delta u^k_p\right] = f_{k-1} \underbrace{- \frac{u^k_p}{\Delta t} - \theta\left[u^k_p \nabla u^k_p - \epsilon \nabla^2 u^k_p\right]}_{f_p}$$

$$\frac{\delta u^k_p}{\Delta t} + \theta\left[u^k_p \nabla \delta u^k_p + \delta u^k_p \nabla u^k_p - \epsilon \nabla^2 \delta u^k_p\right] = f_{k-1} + f_p$$

The PDE has been linearized, now put into variational form

$$\int_\Omega \left(\frac{\delta u^k_p}{\Delta t} + \theta\left[u^k_p \nabla \delta u^k_p + \delta u^k_p \nabla u^k_p - \epsilon \nabla^2 \delta u^k_p\right]\right) \cdot v \, d\Omega = \int_\Omega (f_{k-1} + f_p)\, v \, d\Omega$$

Need to integrate second-order derivatives $\quad \int_\Omega u v' \, d\Omega = (uv|_\Gamma - \int u' v \, d\Omega$

$$-\int_\Omega \epsilon \nabla^2 \delta u^k_p \cdot v \, d\Omega = -(\epsilon \nabla \delta u^k_p \cdot v|_\Gamma + \int_\Omega \epsilon \nabla \delta u^k_p \cdot \nabla v \, d\Omega$$

$$-\int_\Omega \epsilon \nabla^2 u^k_p \cdot v \, d\Omega = -(\epsilon \nabla u^k_p \, v|_\Gamma + \int_\Omega \epsilon \nabla u^k_p \cdot \nabla v \, d\Omega$$

$$\int_\Omega \left(\frac{\delta u^k_p}{\Delta t} v + \theta u^k_p \nabla \delta u^k_p v + \theta \delta u^k_p \nabla u^k_p v + \theta \epsilon \nabla \delta u^k_p \cdot \nabla v\right) d\Omega - \theta(\epsilon \nabla \delta u^k_p \, v|_\Gamma =$$

$$\int_\Omega \left(\frac{u^{k-1}}{\Delta t} v - (1-\theta) u^{k-1} \nabla u^{k-1} v - (1-\theta)\epsilon \nabla u^{k-1} \cdot \nabla v\right) d\Omega + (1-\theta)(\epsilon \nabla u^{k-1} v|_\Gamma$$

$$\int_\Omega \left(-\frac{u^k_p}{\Delta t} v - \theta u^k_p \nabla u^k_p v - \theta \epsilon \nabla u^k_p \cdot \nabla v\right) d\Omega + \theta(\epsilon \nabla u^k_p \, v|_\Gamma$$