Given:

The binomial series expansion

$$(x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k \qquad (1)$$

And

$$\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!} \qquad (2)$$

Find:

Use equations (1) and (2) to show that $(x+y)^{-\frac{1}{2}} = \sum_{i=0}^{\infty} \alpha_i x^i$ (3)

Where $\alpha_i = \dfrac{1\cdot 3\cdot \ldots \cdot (2i-1)}{2\cdot 4\cdot \ldots \cdot (2i)}$ (4)

Solution:

Using the general solution (1) and equation (3) we can find the constants

$$\begin{aligned} \alpha &= 1 \\ b &= -x \\ r &= -\frac{1}{2} \\ k &= i \end{aligned}$$

Substitute these into the general series expansion (1)

$$(1-x)^{-\frac{1}{2}} = \sum_{i=0}^{\infty} \binom{-\frac{1}{2}}{i} \left(1^{-\frac{1}{2}-i}\right)\left(x^i\right)$$

substituting in $\binom{-\frac{1}{2}}{i} = \dfrac{-\frac{1}{2}(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-i+1)}{i!}$

gives $(1-x)^{-\frac{1}{2}} = \sum_{i=0}^{\infty} \dfrac{\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\ldots\left(\frac{1}{2}+i-1\right)}{(1)(2)(3)(4)\ldots(i)}\left(x^i\right)$

multiply by $\frac{2}{2}$ and $(1-x)^{-\frac{1}{2}} = \sum_{i=0}^{\infty} \dfrac{(1)(2+1)\ldots(2i-1)}{(2)(4)\ldots(2i)}\left(x^i\right)$

to get $(1-x)^{-\frac{1}{2}} = \sum_{i=0}^{\infty} \dfrac{(1)(3)\ldots(2i-1)}{(2)(4)\ldots(2i)}\left(x^i\right) = \sum_{i=0}^{\infty} \alpha_i x^i$ recall: $\alpha_i = \dfrac{1\cdot 3\cdot \ldots \cdot (2i-1)}{2\cdot 4\cdot \ldots \cdot (2i)}$