Signals and Spectra (1A)

Young Won Lim
9/12/12

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Energy and Power

Instantaneous Power

Real signal

$$p(t) = x^2(t)$$

Energy dissipated
during $(-T/2, \; +T/2)$

$$E_x^T = \int_{-T/2}^{+T/2} x^2(t)\, dt$$

Affects the performance
of a communication system

Average power dissipated
during $(-T/2, \; +T/2)$

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

The rate at which energy is dissipated

Determines the voltage

Energy and Power Signals (1)

Energy dissipated during $(-T/2,\ +T/2)$

$$E_x^T = \int_{-T/2}^{+T/2} x^2(t)\,dt$$

Average power dissipated during $(-T/2,\ +T/2)$

$$P_x^T = \frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)\,dt$$

Energy Signal

Nonzero but finite energy

For all time

$$0 < E_x < +\infty$$

$$E_x = \lim_{T\to+\infty}\int_{-T/2}^{+T/2} x^2(t)\,dt$$

$$= \int_{-\infty}^{+\infty} x^2(t)\,dt$$

Power Signal

Nonzero but finite power

For all time

$$0 < P_x < +\infty$$

$$P_x = \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)\,dt$$

Energy and Power Signals (2)

Energy Signal

$$0 < E_x < +\infty$$

$$E_x = \lim_{T \to +\infty} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

$$= \int_{-\infty}^{+\infty} x^2(t)\, dt$$

$$P_x = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

$$= \lim_{T \to +\infty} \frac{B}{T} \to 0$$

Non-periodic signals
Deterministic signals

Power Signal

$$0 < P_x < +\infty$$

$$P_x = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

$$E_x = \lim_{T \to +\infty} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

$$= \lim_{T \to +\infty} B \cdot T \to +\infty$$

Periodic signals
Random signals

Energy and Power Spectral Densities (1)

Energy Spectral Density

$$
\begin{aligned}
E_x &= \int_{-\infty}^{+\infty} x^2(t)\,dt \\
&= \int_{-\infty}^{+\infty} |X(f)|^2\,df \\
&= \int_{-\infty}^{+\infty} \Psi(f)\,df \\
&= 2\int_{0}^{+\infty} \Psi(f)\,df
\end{aligned}
$$

$$\Psi(f) = |X(f)|^2$$

Power Spectral Density

$$
\begin{aligned}
P_x &= \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x^2(t)\,dt \\
&= \sum_{n=-\infty}^{+\infty} |c_n|^2 \\
&= \int_{-\infty}^{+\infty} G_x(f)\,df \\
&= 2\int_{0}^{+\infty} G_x(f)\,df
\end{aligned}
$$

$$G_x(f) = \sum_{n=-\infty}^{+\infty} |c_n|^2\,\delta(f - n f_0)$$

Energy Spectral Density

$$E_x = \int_{-\infty}^{+\infty} x^2(t)\, dt$$

$$= \int_{-\infty}^{+\infty} \Psi(f)\, df$$

$$\Psi(f) = |X(f)|^2$$

Power Spectral Density

$$P_x = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x^2(t)\, dt$$

$$= \int_{-\infty}^{+\infty} G_x(f)\, df$$

$$G_x(f) = \sum_{n=-\infty}^{+\infty} |c_n|^2\, \delta(f - n f_0)$$

$$G_x(f) = \lim_{T \to +\infty} \frac{1}{T} |X_T(f)|^2$$

Autocorrelation of Energy and Power Signals

Autocorrelation of an Energy Signal

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau)\,dt$$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \le R_x(0)$$

$$R_x(\tau) \Leftrightarrow \Psi(f)$$

$$R_x(0) = \int_{-\infty}^{+\infty} x^2(t)\,dt$$

Autocorrelation of a Power Signal

$$R_x(\tau) = \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{+T/2} x(t)x(t+\tau)\,dt$$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \le R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x^2(t)\,dt$$

Ensemble Average

Random Variable

$$m_x = \boldsymbol{E}\{X\}$$

$$= \int_{-\infty}^{+\infty} x \, p_X(x) \, dx$$

$$\boldsymbol{E}\{X^2\}$$

$$= \int_{-\infty}^{+\infty} x^2 p_X(x) \, dx$$

Random Process

$$m_x(t_k) = \boldsymbol{E}\{X(t_k)\}$$

$$= \int_{-\infty}^{+\infty} x \, p_{X_k}(x) \, dx$$

$$R_x(t_1, \; t_2) = \boldsymbol{E}\{X(t_1) \; X(t_2)\}$$

$$p_{X_1}(x) p_{X_2}(x)$$

WSS (Wide Sense Stationary)

Random Process

$$m_x(t_k) = \boldsymbol{E}\{X(t_k)\}$$

$$= \int_{-\infty}^{+\infty} x\, p_{X_k}(x)\, dx$$

$$R_x(t_1, \; t_2) = \boldsymbol{E}\{X(t_1)\, X(t_2)\}$$

WSS Process

➡ m_x

➡ $R_x(t_1 - \; t_2)$

Autocorrelation of Random and Power Signals

Autocorrelation of an Random Signal

$$R_x(\tau) = \boldsymbol{E}\{X(t)\,X(t+\tau)\}$$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \le R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \boldsymbol{E}\{X^2(t)\}$$

Autocorrelation of a Power Signal

$$R_x(\tau) = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t) x(t+\tau)\, dt$$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \le R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x^2(t)\, dt$$

Time Averaging and Ergodicity

Random Process

$$m_x(t_k) = \boldsymbol{E}\{X(t_k)\} \qquad R_x(t_1,\ t_2) = \boldsymbol{E}\{X(t_1)\,X(t_2)\}$$

$$= \int_{-\infty}^{+\infty} x\, p_{X_k}(x)\, dx$$

WSS Process

$\Rightarrow$ m_x $\qquad\Rightarrow$ $R_x(t_1 - t_2)$

Ergodic Process

$$\Rightarrow \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{+T/2} X(t)\, dt \qquad \Rightarrow \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{+T/2} X(t)\, X(t+\tau)\, dt$$

Autocorrelation of Random and Power Signals

Autocorrelation of a Random Signal

$$R_x(\tau) = \boldsymbol{E}\{X(t)\,X(t+\tau)\}$$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \le R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \boldsymbol{E}\{X^2(t)\}$$

Power Spectral Density of a Random Signal

$$G_x(f) = \lim_{T\to+\infty} \frac{1}{T}\left|X_T(f)\right|^2$$

$$G_x(f) = G_x(-f)$$

$$G_x(f) \ge 0$$

$$G_x(f) \Leftrightarrow R_x(\tau)$$

$$P_x(0) = \int_{-\infty}^{+\infty} G_X(f)\,df$$

Time Averaging and Ergodicity

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] B. Sklar, "Digital Communications: Fundamentals and Applications"