

DFT

- Discrete Fourier Transform

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CTFT and DFT

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} \quad \longleftrightarrow \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{+j(2\pi/N)kn}$$

From CTFT to DFT (1)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

$$t \rightarrow nT_s \quad dt \rightarrow T_s \quad \int \rightarrow \sum \quad T_s \rightarrow 0$$

$$\hat{X}(j\omega) = \sum_{n=-\infty}^{+\infty} x(nT_s) e^{-j\omega nT_s} \cdot T_s \quad \longleftrightarrow \quad x(nT_s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{X}(j\omega) e^{+j\omega nT_s} d\omega$$

$$\omega \rightarrow \omega_k \quad 0 \leq \omega_k < \frac{2\pi}{T_s} \quad 0 \leq k < N \quad 0 \leq n < L$$

$$\omega_k = \frac{2\pi}{T_s} \frac{k}{N}$$

$$\hat{X}(j\omega_k) = T_s \sum_{n=0}^{L-1} x[n] e^{-j\omega_k nT_s} \quad \longleftrightarrow \quad x[n] = \frac{1}{2\pi} \sum_{k=0}^{N-1} \hat{X}(j\omega_k) e^{+j\omega_k nT_s} \frac{2\pi}{T_s} \frac{1}{N}$$

From CTFT to DFT (2)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

$$t \rightarrow nT_s \quad dt \rightarrow T_s \quad \int \rightarrow \sum \quad T_s \rightarrow 0$$

$$\omega \rightarrow \omega_k \quad 0 \leq \omega_k < \frac{2\pi}{T_s} \quad 0 \leq k < N \quad 0 \leq n < L$$

$$\omega_k = \frac{2\pi}{T_s} \frac{k}{N}$$

$$\hat{X}(j\omega_k) = T_s \sum_{n=0}^{L-1} x[n] e^{-j\omega_k n T_s} \quad \longleftrightarrow \quad x[n] = \frac{1}{2\pi} \sum_{k=0}^{N-1} \hat{X}(j\omega_k) e^{+j\omega_k n T_s} \frac{2\pi}{T_s} \frac{1}{N}$$

$$\omega_k T_s \rightarrow \frac{2\pi}{N} k$$

$$\omega_k = \frac{2\pi}{T_s} \frac{k}{N}$$

$$\frac{1}{T_s} \hat{X}(j\omega_k) = \sum_{n=0}^{L-1} x[n] e^{-j\left(\frac{2\pi}{N}\right)kn} \quad \longleftrightarrow \quad x[n] = \sum_{k=0}^{N-1} \frac{1}{T_s} \hat{X}(j\omega_k) e^{+j\left(\frac{2\pi}{N}\right)kn}$$

From CTFT to DFT (3)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

$$t \rightarrow nT_s \quad dt \rightarrow T_s \quad \int \rightarrow \sum \quad T_s \rightarrow 0$$

$$\omega \rightarrow \omega_k \quad 0 \leq \omega_k < \frac{2\pi}{T_s} \quad 0 \leq k < N \quad 0 \leq n < L$$

$$\omega_k T_s \rightarrow \frac{2\pi}{N} k$$

$$\omega_k = \frac{2\pi}{T_s} \frac{k}{N}$$

$$\omega_k = \frac{2\pi}{T_s} \frac{k}{N}$$

$$\frac{1}{T_s} \hat{X}(j\omega_k) = \sum_{n=0}^{L-1} x[n] e^{-j(\frac{2\pi}{N})kn} \quad \longleftrightarrow \quad x[n] = \sum_{k=0}^{N-1} \frac{1}{T_s} \hat{X}(j\omega_k) e^{+j(\frac{2\pi}{N})kn}$$

From CTFT to DFT (4)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

$$\omega_k T_s \rightarrow \frac{2\pi}{N} k$$

$$\omega_k = \frac{2\pi}{T_s} \frac{k}{N}$$

$$\frac{1}{T_s} \hat{X}(j\omega_k) = \sum_{n=0}^{L-1} x[n] e^{-j(\frac{2\pi}{N})kn} \quad \longleftrightarrow \quad x[n] = \sum_{k=0}^{N-1} \frac{1}{T_s} \hat{X}(j\omega_k) e^{+j(\frac{2\pi}{N})kn}$$

Discrete Fourier Transform

$$L = N$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} \quad \longleftrightarrow \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{+j(2\pi/N)kn}$$

DTFT and CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s) e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n}$$

DTFT of a sampled signal

$$X(e^{j\hat{\omega}})$$

$$\hat{\omega} = \omega T_s$$

$$= X(e^{j\omega T_s}) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

CTFT of a sampled signal

DFT and DTFT

DTFT of a sampled signal

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n}$$

$$\hat{\omega} = \omega T_s$$

$$\hat{\omega} \rightarrow \hat{\omega}_k \quad 0 \leq \hat{\omega}_k < 2\pi \quad 0 \leq k < N \quad 0 \leq n < L$$

$$X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n] e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = \left(\frac{2\pi}{N}\right) k$$

DFT of a sampled signal

$$X[k] =$$

$$X(e^{j(2\pi/N)k}) = \sum_{n=0}^{L-1} x[n] e^{-j(2\pi/N)kn}$$

DTFT sampled in frequency

DFT and CTFT

DFT of a sampled signal

$$X[k]$$

$$= X(e^{j(2\pi/N)k}) = \sum_{n=0}^{L-1} x[n] e^{-j(2\pi/N)kn}$$

DTFT sampled in frequency

$$X(e^{j\omega T_s}) \Big|_{\omega = \frac{2\pi k}{NT_s}}$$

CTFT evaluated at $\omega = \frac{2\pi k}{NT_s}$

$$= \frac{1}{T_s} \sum_{l=-\infty}^{+\infty} X_c(j(\omega - l\omega_s)) \Big|_{\omega = \frac{2\pi k}{NT_s}}$$

$$= \frac{1}{T_s} \sum_{l=-\infty}^{+\infty} X_c(j(\omega - l\frac{2\pi}{T_s})) \Big|_{\omega = \frac{2\pi k}{NT_s}}$$

References

[1] <http://en.wikipedia.org/>

[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003