CLTI Differential Equation

-

Copyright (c) 2011 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Causal LTI Systems (1)

$$a_N \frac{d^N y(t)}{d t^N} + a_{N-1} \frac{d^{N-1} y(t)}{d t^N} + \cdots + a_1 \frac{d\, y(t)}{d t} + a_0 y(t) = b_M \frac{d^M x(t)}{d t^M} + b_{M-1} \frac{d^{M-1} x(t)}{d t^M} + \cdots + b_1 \frac{d\, x(t)}{d t} + b_0 x(t)$$

$$\frac{d^N y(t)}{d t^N} + a_1 \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{d t} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{d t^M} + b_{N-M+1} \frac{d^M x(t)}{d t^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{d t} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$$Q(D) y(t) = P(D) x(t)$$

$$M = N$$

$$\frac{d^N y(t)}{d t^N} + a_1 \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{d t} + a_N y(t) = b_0 \frac{d^M x(t)}{d t^M} + b_1 \frac{d^M x(t)}{d t^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{d t} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$$Q(D) y(t) = P(D) x(t)$$

Causal LTI Systems (2)

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^N} + \cdots + a_1 \frac{d\, y(t)}{dt} + a_0\, y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^M} + \cdots + b_1 \frac{d\, x(t)}{dt} + b_0\, x(t)$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N\, x(t)$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) y(t) = \left(D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N\right) x(t)$$

$$Q(D)\, y(t) = P(D)\, x(t)$$

- **Zero Input Response**
- **Zero State Response (Convolution with h(t))**

- **Natural Response (Homogeneous Solution)**
- **Forced Response (Particular Solution)**

Zero Input Response $y_0(t)$ – (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1}\frac{d\,y(t)}{dt} + a_N y(t) = b_{N-M}\frac{d^M x(t)}{dt^M} + b_{N-M+1}\frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1}\frac{d\,x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$Q(D)\, y_0(t) = 0 \quad \Rightarrow \quad (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_0(t) = 0$$

Linear combination of $y_0(t)$ and its derivatives = 0

if and only if

$$y_0(t) = ce^{\lambda t}$$
$$\dot{y}_0(t) = c\lambda e^{\lambda t}$$
$$\ddot{y}_0(t) = c\lambda^2 e^{\lambda t}$$
$$\cdots$$

$$Q(\lambda) = 0 \quad \Leftrightarrow \quad \underbrace{(\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N)}_{=0}\ \underbrace{ce^{\lambda t}}_{\neq 0} = 0$$

Young Won Lim
11/12/11

Zero Input Response $y_0(t)$ – (2)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$Q(D)\, y_0(t) = 0 \quad \Rightarrow \quad (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_0(t) = 0$$

$$Q(\lambda) = 0 \quad \Leftrightarrow \quad \underbrace{(\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1} \lambda + a_N)}_{=\,0} \ \underbrace{c e^{\lambda t}}_{\neq\,0} = 0$$

$$Q(\lambda) = (\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1} \lambda + a_N) = 0$$

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N)$$

λ_i characteristic roots

$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \cdots + c_N e^{\lambda_N t} = y_0(t)$$

$e^{\lambda_i t}$ characteristic modes

ZIR: a linear combination of the characteristic modes of the system

Zero State Response y(t) – (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

All initial conditions are zero

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) y(t-\tau)\, d\tau$$

Impulse response $h(t)$

$$y(t) = \int_{0^-}^{+t} x(\tau) y(t-\tau)\, d\tau \quad , \quad t \geq 0$$

Causality

causal system: Response cannot begin before the input

causal input: The input starts at t=0 $\quad h(\tau) = 0 \quad \tau < 0$

causal h(t): The causal system's response to a unit impulse cannot begin before t=0

$$h(t-\tau) = 0 \quad t-\tau < 0$$

Total Response

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1}\frac{d\,y(t)}{dt} + a_N y(t) = b_{N-M}\frac{d^M x(t)}{dt^M} + b_{N-M+1}\frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1}\frac{d\,x(t)}{dt} + b_N x(t)$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) \cdot y(t) = \left(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N\right) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$y(t) = \underline{\sum_{k=1}^{N} c_k e^{\lambda_k t}} + \underline{x(t) * h(t)}$$

Zero Input Response — Zero State Response

$$y(t) = \underline{y_n(t)} + \underline{y_\Phi(t)}$$

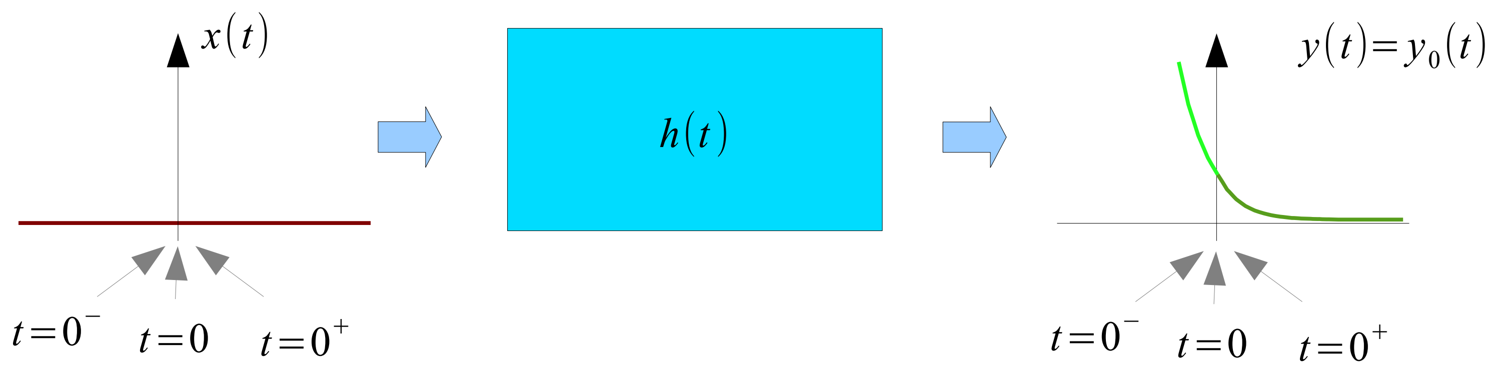
Natural Response — Forced Response

Zero Input Response

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) \;=\; b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) \cdot y(t) \;=\; \left(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N\right) \cdot x(t)$$

$$Q(D) \cdot y(t) \;=\; P(D) \cdot x(t)$$



Input is zero

Only initial conditions drives the system

$$y_0(0^-) \;=\; y_0(0) \;=\; y_0(0^+)$$

$$\dot{y}_0(0^-) \;=\; \dot{y}_0(0) \;=\; \dot{y}_0(0^+)$$

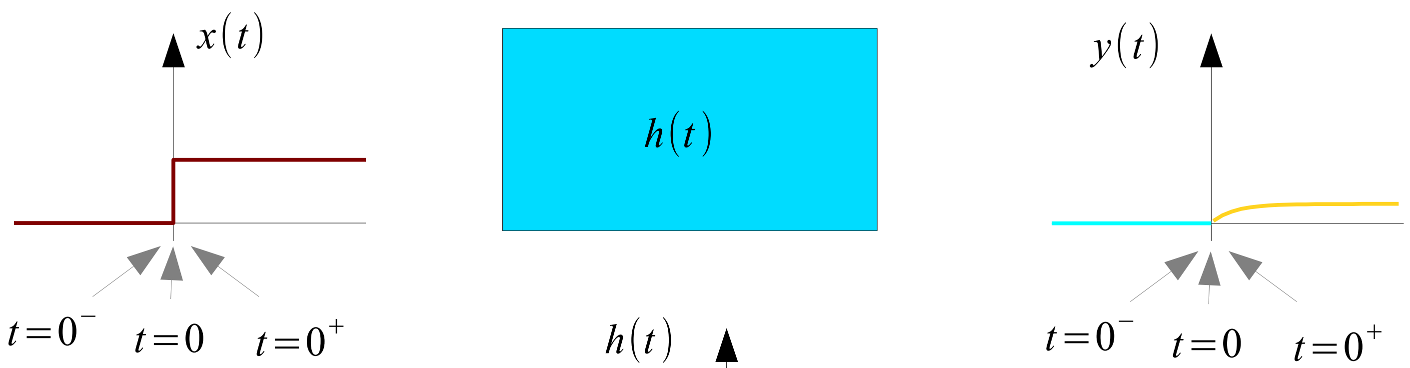
$$\ddot{y}_0(0^-) \;=\; \ddot{y}_0(0) \;=\; \ddot{y}_0(0^+)$$

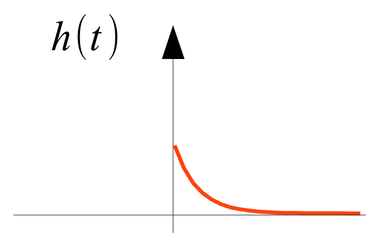
Zero State Response

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$





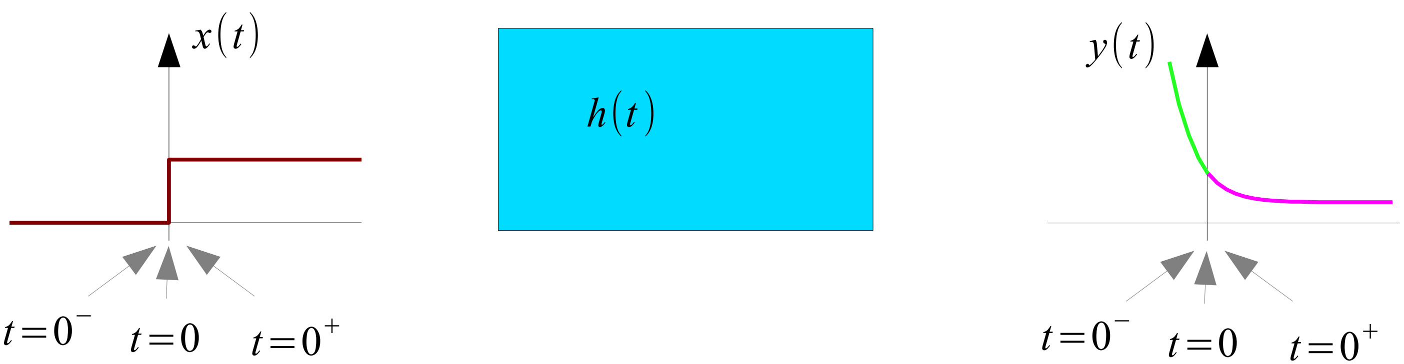
All initial conditions are zero

Total Response y(t)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) \;=\; b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) \cdot\; y(t) \;=\; \left(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N\right) \cdot\; x(t)$$

$$Q(D) \quad\cdot\quad y(t) \;=\; P(D) \quad\cdot\quad x(t)$$



$$y(t) \;=\; y_0(t) \qquad t \le 0^-$$

zero input response

+

zero state response

because the input has not started yet

$$y(0^-) \;=\; y_0(0^-)$$

$$\dot{y}(0^-) \;=\; \dot{y}_0(0^-)$$

In general, the total response

$$y(0^-) \;\neq\; y(0^+)$$

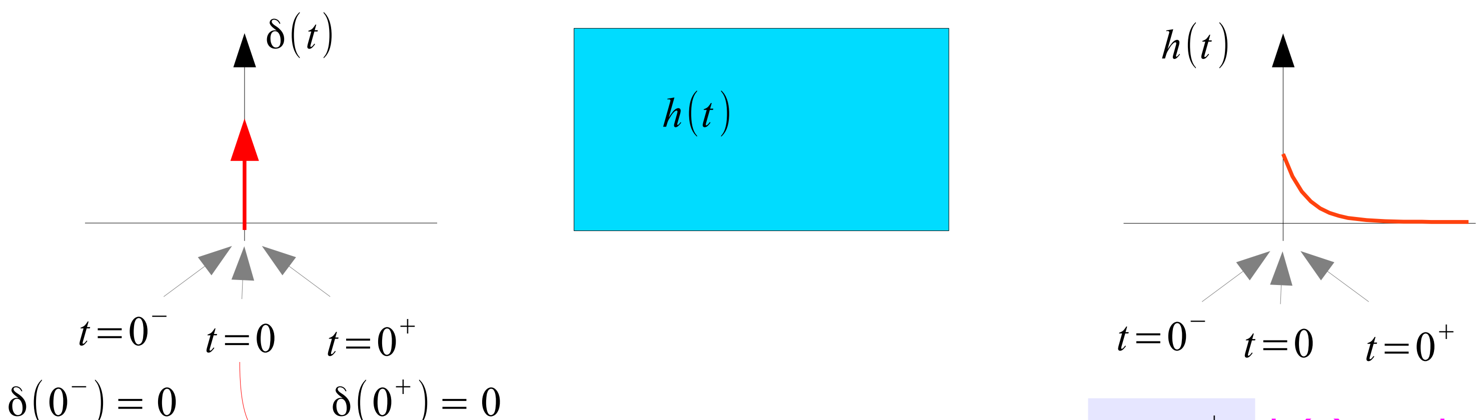
$$\dot{y}(0^-) \;\neq\; \dot{y}(0^+)$$

Impulse Response h(t)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$



All init conditions are zero at $t=0^-$

Generates energy storage
Creates nonzero initial condition at $t=0^+$

$t \geq 0^+$ $(t \neq 0)$	h(t) = characteristic mode terms
$t=0$	h(t) can have at most an impulse $A_0 \delta(t)$

$$h(t) = A_0 \delta(t) + \text{char mode terms} \quad t \geq 0$$

h(t) can have at most a δ(t)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$M = N$

$$Q(D) y(t) = P(D) x(t)$$

If $\delta(t)$ is included in $h(t)$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) h(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N) \delta(t) \qquad M = N$$

The highest order term $\delta^{(N+1)}(t)$ ⟷ $\delta^{(N)}(t)$ contradiction

h(t) cannot contain $\delta^{(i)}(t)$ at all

h(t) can contain at most δ(t)

Simplified Impulse Matching Method (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$M = N$

$$Q(D) y(t) = P(D) x(t)$$

$$h(t) = b_0 \delta(t) + [P(D) y_n(t)] u(t)$$

$y_n(t)$ *linear combination of characteristic modes with the following initial conditions*

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) \cdots = y_n^{(N-2)}(0) = 0 \qquad y_n^{(N-1)}(0) = 1$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y_n(t) = \delta(t)$$

$$y_n^{(N)}(t) + a_1 y_n^{(N-1)}(t) + \cdots + a_{N-1} y_n^{(1)}(t) + y_n(t) = \delta(t)$$

$$Q(D) y(t) = P(D) x(t)$$

$$Q(D) w(t) = x(t)$$

$$Q(D) y_n(t) = \delta(t)$$

$$Q(D) w(t) = x(t)$$

$$Q(D) P(D) w(t) = P(D) x(t)$$

$$Q(D) y(t) = P(D) x(t)$$

$$y(t) = P(D) w(t)$$

Simplified Impulse Matching Method (2)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$M = N$

$$Q(D) y(t) = P(D) x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y_n(t) = \delta(t)$$

$$y_n^{(N)}(t) + a_1 y_n^{(N-1)}(t) + \cdots + a_{N-1} y_n^{(1)}(t) + y_n(t) = \delta(t)$$

$$h(t) = P(D)[y_n(t)u(t)]$$

$$h(t) = b_o \delta(t) + P(D) y_n(t), \qquad t \geq 0$$

$$h(t) = b_o \delta(t) + [P(D) y_n(t)] u(t)$$

$$Q(D) w(t) = x(t)$$

$$Q(D) \boxed{P(D) w(t)} = \boxed{P(D) x(t)}$$

$$y(t) = P(D) w(t)$$

$$Q(D) y_n(t) = \delta(t)$$

$$Q(D) \boxed{P(D) y_n(t)} = \boxed{P(D) \delta(t)}$$

$$h(t) = P(D) y_n(t)$$

causal $y_n(t)u(t)$

$$h(t) = P(D)[y_n(t)u(t)]$$

Classical Solution (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

When all the ***characteristic mode terms*** *of the total system response together, they form the system's* ***natural response*** $y_n(t)$ *(homogeneous, complementary solution)*

$$y(t) = y_n(t) + y_\Phi(t)$$

$$Q(D)\,[y_n(t) + y_\Phi(t)] = P(D)\,x(t)$$

The remaining portion of noncharacteristic mode terms form the system's ***forced response*** *(particular solution)* $y_\Phi(t)$

$$Q(D)\,y_n(t) = 0$$

$$Q(D)\,y_\Phi(t) = P(D)\,x(t)$$

Classical Solution (2)

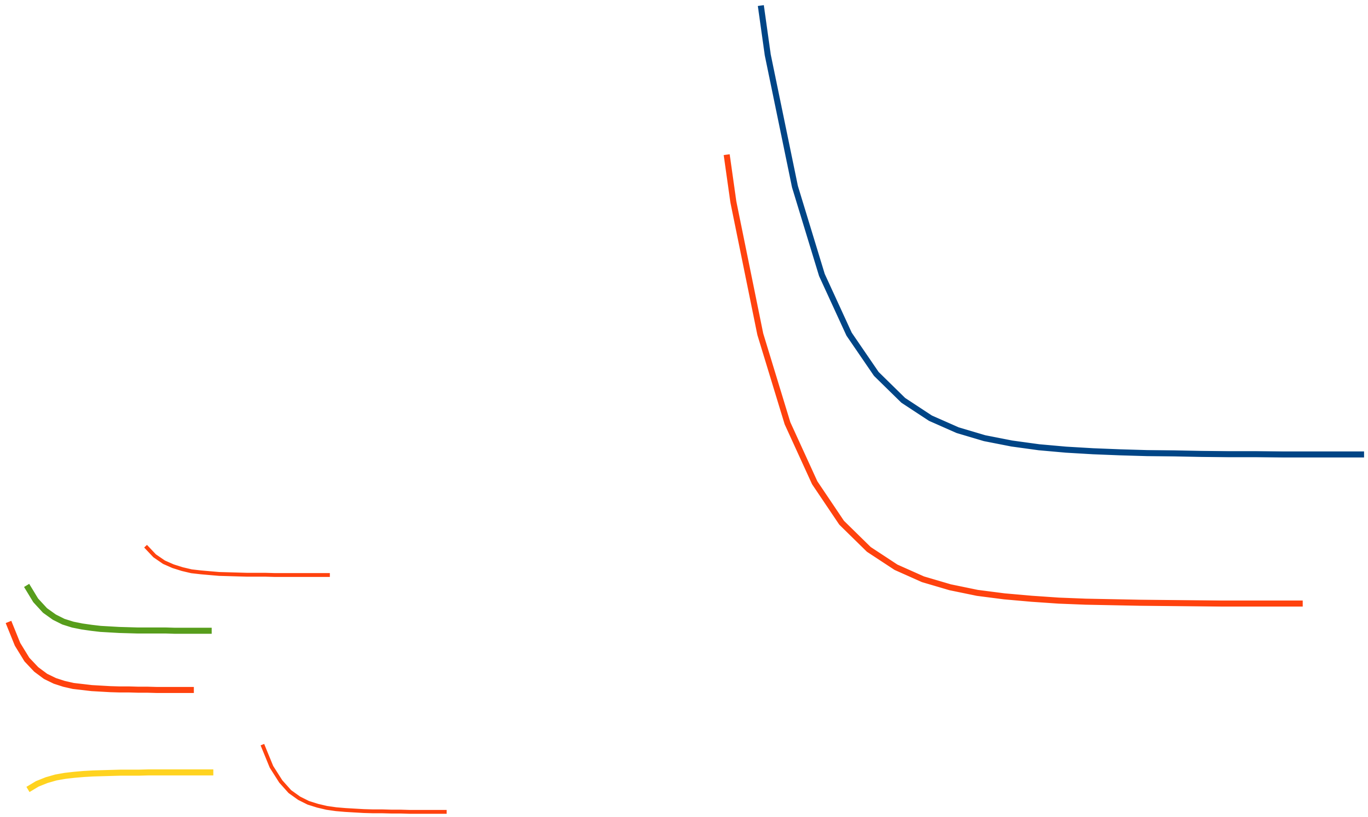
$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) \cdot y(t) = \left(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N\right) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

- *linear combination of the* ***characteristic modes***. $y_n(t)$
- *the same form as that of the* ***zero input response***
- *only its constants are different*
- *these constants are determined from the* ***auxiliary conditions***
- *initial conditions at* $t = 0^+$
- *at* $t = 0^-$ *only the* ***zero input response***
- *initial condition at* $t = 0^-$ $\rightarrow$ *applied to* ***zero input response***
- ***zir*** *and* ***zsr*** *cannot be separated*

Impulse Response h(t)



References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] B.P. Lathi, Linear Systems and Signals (2nd Ed)