Spectra (1A)

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Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = +1, +2, \dots$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k} \right)$$

$$k = +1, +2, \dots$$

$$\cos(\alpha + \beta) = \underline{\cos(\alpha)\cos(\beta)} - \underline{\sin(\alpha)\sin(\beta)}$$

$$g_k \cos(k\omega_0 t + \phi_k) = \underline{g_k \cos(\phi_k)}\cos(k\omega_0 t) - \underline{g_k \sin(\phi_k)}\sin(k\omega_0 t)$$

$$\underline{\qquad} a_k \cos(k\omega_0 t) + \underline{\qquad} b_k \sin(k\omega_0 t)$$

$$a_k = g_k \cos(\phi_k)$$

$$-b_k = g_k \sin(\phi_k)$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

Two-Sided Spectrum

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2} g_k\, e^{+j\phi_k} & (k > 0) \\ \frac{1}{2} g_k\, e^{-j\phi_k} & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} \tan^{-1}(-b_k/a_k) & (k > 0) \\ \tan^{-1}(+b_k/a_k) & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} +\phi_k & (k > 0) \\ -\phi_k & (k < 0) \end{cases}$$

Power Spectrum   *Two-Sided*

$$|C_k|^2 + |C_{-k}|^2 = \tfrac{1}{2}|g_k|^2 = \tfrac{1}{2}(a_k^2 + b_k^2)$$

Periodogram   *One-Sided*

$$2 \cdot |C_k| = |g_k| = \sqrt{a_k^2 + b_k^2}$$

Power Spectrum (1)

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k \, e^{+jk\omega_0 t}$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ ...$$

$$c_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = -2,\ -1,\ 0,\ +1,\ +2,\ ...$$

$$c_k = \begin{cases} a_0 & (k=0) \\ \frac{1}{2}(a_k - jb_k) & (k>0) \\ \frac{1}{2}(a_k + jb_k) & (k<0) \end{cases}$$

Average Power $\qquad P = \frac{1}{T} \int_0^T |x(t)|^2\, dt$

$$P = |a_0|^2 + \frac{1}{2} \sum_{k=1}^{+\infty} \left(|a_k|^2 + |b_k|^2 \right)$$

Power Spectrum

$$P_k = \begin{cases} |a_0|^2 & (k=0) \\ \frac{1}{2}\left(|a_k|^2 + |b_k|^2 \right) & (k>0) \end{cases}$$

Average Power $\qquad P = \frac{1}{T} \int_0^T |x(t)|^2\, dt$

$$P = |c_0|^2 + \sum_{k=1}^{+\infty} \left(|c_k|^2 + |c_{-k}|^2 \right)$$

Power Spectrum

$$P_k = \begin{cases} |c_0|^2 & (k=0) \\ \left(|c_k|^2 + |c_{-k}|^2 \right) & (k>0) \end{cases}$$

Power Spectrum (2)

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k\, e^{+jk\omega_0 t}$$

Average Power
$$P = \frac{1}{T} \int_0^T |x(t)|^2\, dt$$

$$P = |a_0|^2 + \frac{1}{2} \sum_{k=1}^{+\infty} \left(|a_k|^2 + |b_k|^2 \right)$$

Average Power
$$P = \frac{1}{T} \int_0^T |x(t)|^2\, dt$$

$$P = |c_0|^2 + \sum_{k=1}^{+\infty} \left(|c_k|^2 + |c_{-k}|^2 \right)$$

$$|c_k|^2 + |c_{-k}|^2$$
$$= \frac{1}{2}\left(|a_k|^2 + |b_k|^2 \right)$$

$$|c_k|^2 = \tfrac{1}{4}\left(|a_k|^2 + |b_k|^2 \right)$$
$$|c_{-k}|^2 = \tfrac{1}{4}\left(|a_k|^2 + |b_k|^2 \right)$$

$$c_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

Power Spectrum

$$P_k = \begin{cases} |a_0|^2 & (k = 0) \\ \dfrac{1}{2}\left(|a_k|^2 + |b_k|^2 \right) & (k > 0) \end{cases}$$

Power Spectrum

$$P_k = \begin{cases} |c_0|^2 & (k = 0) \\ \left(|c_k|^2 + |c_{-k}|^2 \right) & (k > 0) \end{cases}$$

Spectra

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Periodogram

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right)$$

An estimate of the spectral density of a signal

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t)\, dt$$

$$n = 1,\ 2,\ \dots$$

$$\frac{T}{2} a_n = \int_{t_1}^{t_1+T} x(t) \cos(k t)\, dt$$

$$\frac{T}{2} b_n = \int_{t_1}^{t_1+T} x(t) \sin(k t)\, dt$$

T: integer multiple $\dfrac{2\pi}{k}$

$$T = \frac{2\pi}{k} \cdot n$$
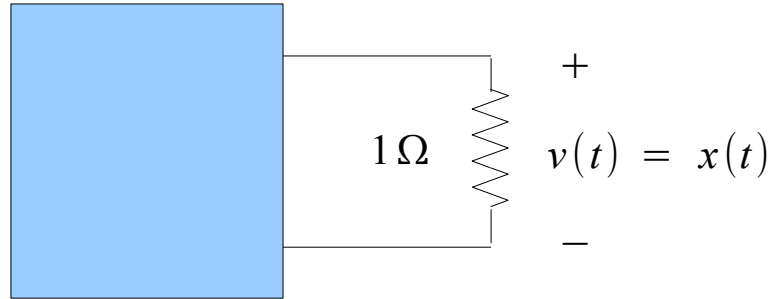
$$k = \frac{2\pi}{T} \cdot n = n\omega_0$$

Positive Frequency Only

$$f = \frac{n}{T} \qquad n = 1,\ 2,\ \dots$$

Single-sided Spectrum

$$a_n^2 + b_n^2$$

abscissa $\dfrac{2\pi}{k}$

ordinates $r = \sqrt{a_n^2 + b_n^2}$

Continuous Periodic Signal

instantaneous power $\quad x^2(t)$

$1\,\Omega \quad v(t) = x(t)$

average power $\qquad \dfrac{1}{T}\displaystyle\int_0^T x^2(t)\,dt$

$+$

$-$

$T \ : \ period$

$v(t) = x(t)$ Continuous Periodic

$\Rightarrow$ CTFS (Fourier Series)

Parseval's Theorem

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n\, e^{+jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$n = -2,\ -1,\ 0,\ +1,\ +2,\ \dots$$

$$\underline{\frac{1}{T}\int_0^T x^2(t)\,dt} = \underline{\sum_{n=-\infty}^{+\infty} |C_n|^2}$$

average power $\qquad$ sum of

power spectrum C_n

CTFS and CTFT

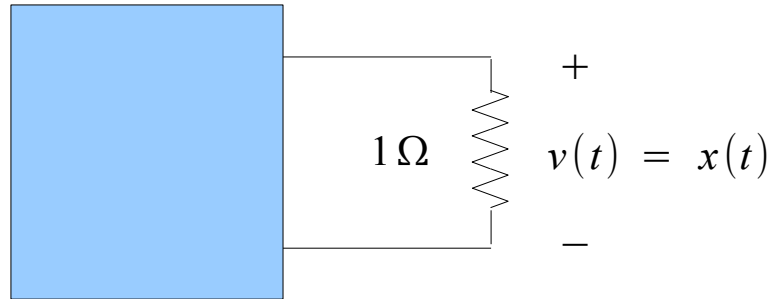**Continuous Time Fourier <u>Series</u>**

$$C_n = \frac{1}{T} \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_n\, e^{+jn\omega_0 t}$$

Continuous Time Fourier <u>Transform</u>

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt \quad \Longleftrightarrow \quad x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{+j2\pi f t}\, df$$

Continuous Aperiodic Signal

instantaneous power $\qquad x^2(t)$

$1\,\Omega \qquad v(t) = x(t)$

total energy $\qquad \displaystyle\int_{-\infty}^{+\infty} x^2(t)\,dt$

$v(t) = x(t)$ Continuous Aperiodic

$\Rightarrow$ CTFT (Fourier Integral)   Parseval's Theorem

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{+j2\pi f t}\, d f$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, d t$$

$$\int_{-\infty}^{+\infty} x^2(t)\,dt \;=\; \int_{-\infty}^{\infty} |X(f)|^2\, df$$

total energy   integral of
energy spectral
density $\qquad |X(f)|^2$

Average Power of Random Signals

A truncated sample function

$$x_T(t) = x(t) \quad -\frac{T}{2} < t < +\frac{T}{2}$$

$$= 0 \qquad otherwise$$

Fourier Transform

$$X_T(\omega) = \int_{-\infty}^{+\infty} x_T(t)\, e^{-j\omega t}\, dt \qquad \Longleftrightarrow \qquad x_T(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_T(\omega)\, e^{+j\omega t}\, d\omega$$

$$X_T(f) = \int_{-\infty}^{+\infty} x_T(t)\, e^{-j2\pi f t}\, dt \qquad \Longleftrightarrow \qquad x_T(t) = \int_{-\infty}^{+\infty} X_T(f)\, e^{+j2\pi f t}\, df$$

Parseval's Therorem

$$\int_{-\infty}^{+\infty} |x_T(t)|^2\, dt = \int_{-\infty}^{+\infty} |X_T(f)|^2\, df \qquad \text{total energy}$$

Average Power

$$\frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t)\, dt = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\, dt \qquad \text{total energy / T}$$

Power Spectral Density of Random Signals

Average Power

$$\frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t)\, dt \quad = \quad \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\, df \quad \Rightarrow \quad \int_{-\infty}^{+\infty} \hat{S}_{xx}(f)\, df$$

Raw Power Spectral Density

$$\frac{|X_T(f)|^2}{T} = \hat{S}_{xx}(f)$$

Power Spectral Density

$$\lim_{T \to \infty} \frac{E\left[|X_T(f)|^2\right]}{T} = S_{xx}(f)$$

$$\lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t)\, dt \quad = \quad \lim_{T \to \infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\, df \qquad \blacktriangleright \quad \textit{not converge}$$

$$E\left[\lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t)\, dt\right] \quad = \quad E\left[\lim_{T \to \infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\, df\right] \qquad \textit{Random Signal}$$

$$Var(x(t)) = \sigma_x^2 \quad = \quad \int_{-\infty}^{+\infty} \lim_{T \to \infty} \frac{E\left[|X_T(f)|^2\right]}{T}\, df \Rightarrow \int_{-\infty}^{+\infty} S_{xx}(f)\, df$$

Parseval's Theorem – CTFS

Average Power

$$E[x^2(t)] = \frac{1}{T}\int_{-\infty}^{+\infty} x^2(t)\, dt$$

CTFS

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k\, e^{+jk\omega t}$$

$$x^2(t) = x(t)\cdot x^*(t) = \left(\sum_{k=-\infty}^{+\infty} c_k\, e^{+jk\omega t}\right)\left(\sum_{l=-\infty}^{+\infty} c_l^*\, e^{-jl\omega t}\right)$$

$$x^2(t) = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} c_k c_l^*\, e^{+j(k-l)\omega t}$$

$$\frac{1}{T}\int_{-\infty}^{+\infty} x^2(t)\, dt = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} c_k c_l^* \left(\frac{1}{T}\int_{-\infty}^{+\infty} e^{+j(k-l)\omega t}\, dt\right)$$

Average Power

$$\frac{1}{T}\int_{-\infty}^{+\infty} x^2(t)\, dt = \sum_{k=-\infty}^{+\infty} |c_k|^2$$

Power Spectrum

$$|c_k|^2$$

power associated with individual frequency components

Parseval's Theorem – CTFT

Total Energy

$$\int\limits_{-\infty}^{+\infty} x^2(t)\, dt$$

CTFT

$$x(t) = \int\limits_{-\infty}^{+\infty} X(f)\, e^{+j2\pi ft}\, df$$

$$x^2(t) = x(t)\cdot x^*(t) = \left(\int\limits_{-\infty}^{+\infty} X(f)\, e^{+j2\pi ft}\, df \right) \left(\int\limits_{-\infty}^{+\infty} X^*(\nu)\, e^{-j2\pi\nu t}\, d\nu \right)$$

$$x^2(t) = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} |X(f)|^2\, e^{+j2\pi(f-\nu)t}\, df\, d\nu$$

$$\int\limits_{-\infty}^{+\infty} x^2(t)\, dt = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} |X(f)|^2 \int\limits_{-\infty}^{+\infty} e^{+j2\pi(f-\nu)t}\, dt\, df\, d\nu$$

Total Energy

$$\int\limits_{-\infty}^{+\infty} x^2(t)\, dt = \int\limits_{-\infty}^{+\infty} |X(f)|^2\, df \qquad = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} |X(\omega)|^2\, d\omega$$

Energy Spectral Density

$$|X(f)|^2$$

energy associated with individual frequency components

Parseval's Theorem – DTFT

Total Energy

$$\int_{-\infty}^{+\infty} x^2(t)\, dt$$

CTFT

Total energy over all discrete time n
Total energy in fundamental period of frequency

$$x[n] = \int_{-0.5}^{+0.5} X(e^{j\hat{f}})\, e^{+j2\pi \hat{f} n}\, d\hat{f}$$

$$x^2[n] = x[n] \cdot x^*[n] = \left(\int_{-0.5}^{+0.5} X(e^{j\hat{f}})\, e^{+j2\pi \hat{f} n}\, d\hat{f} \right) \left(\int_{-0.5}^{+0.5} X^*(e^{j\hat{v}})\, e^{+j2\pi \hat{v} n}\, d\hat{v} \right)$$

$$x^2[n] = \int_{-0.5}^{+0.5} \int_{-0.5}^{+0.5} |X(e^{j\hat{f}})|^2\, e^{+j2\pi(\hat{f}-\hat{v})n}\, df\, dv$$

$$\sum_{n=-\infty}^{+\infty} x^2[n]\, dt = \int_{-0.5}^{+0.5} \int_{-0.5}^{+0.5} |X(e^{j\hat{f}})|^2 \sum_{n=-\infty}^{+\infty} e^{+j2\pi(\hat{f}-\hat{v})n}\, df\, dv$$

Total Energy

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \int_{-0.5}^{+0.5} |X(\hat{f})|^2\, d\hat{f} \qquad = \frac{1}{2\pi} \int_{-\pi}^{+\pi} |X(\hat{\omega})|^2\, d\hat{\omega}$$

Energy Spectral Density

$$|X(\hat{f})|^2$$

energy associated with individual frequency components

Parseval's Theorem – DFT

Average Power

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] \quad \approx \quad \frac{1}{NT}\sum_{0}^{NT} x^2(t)\, dt$$

CTFT

$$x[n] \;=\; \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j\left(\frac{2\pi}{N}\right)kn} \qquad x^2[n] = x[n]\cdot x^*[n] = \frac{1}{N^2}\left(\sum_{k=0}^{N-1} X[k]\, e^{+j\left(\frac{2\pi}{N}\right)kn}\right)\left(\sum_{l=0}^{N-1} X^*[l]\, e^{-j\left(\frac{2\pi}{N}\right)ln}\right)$$

$$x^2[n] \;=\; \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} X[k]X^*[l]\, e^{+j\left(\frac{2\pi}{N}\right)(k-l)n}$$

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n]\, dt \;=\; \frac{1}{N^3}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} X[k]X^*[l]\sum_{n=0}^{N-1} e^{+j\left(\frac{2\pi}{N}\right)(k-l)n}$$

Average Power

$$\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2 \qquad\qquad \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

Periodogram

$$\frac{1}{N}|X[k]|^2$$

Power and Power Density Spectra

Average Power

$$E[x^2(t)] = \frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t)\, dt$$

Average Power of N sample of x(t)

$$\frac{1}{N} \sum_{n=0}^{N-1} x_n^2 \approx \frac{1}{N\Delta t} \int_0^{N\Delta t} x^2(t)\, dt = \frac{1}{N^2} \sum_{m=0}^{N-1} |X_m|^2$$

$$\sum_{n=0}^{N-1} x_n^2 = \frac{1}{N} \sum_{m=0}^{N-1} |X_m|^2 \quad m = 0, 1, \cdots, N-1$$

Periodogram:

$$P_{xx}(m) = \frac{1}{N} |X_m|^2$$

$$m = 0, 1, \cdots, N-1$$

Average Power

$$\sum_{n=0}^{N-1} x_n^2 = \frac{1}{N} \sum_{m=0}^{N-1} P_{xx}(m)$$

Total Energy

$$T \sum_{n=0}^{N-1} x_n^2 = T \sum_{m=0}^{N-1} P_{xx}(m)$$

$$\approx \int_0^{N\Delta t} x^2(t)\, dt$$

Spectra

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	Periodic Signals	Aperiodic Signals	Random Signals	
Frequency Spacing	$\Delta f = \frac{1}{N\Delta t}$	$\Delta f = \frac{1}{N\Delta t}$	$\sum S\,\Delta f = \frac{1}{N\Delta t}\sum S$	$\frac{1}{N\Delta t}\sum x^2 \Delta t$
Two Sided	$\frac{1}{N}X(k)$	$\frac{\Delta t}{N}X(k)$	$S(k)=\frac{\Delta t}{N}\lvert X(k)\rvert^2$	$P=\sum_{k=0}^{N-1}S(k)\Delta f$
One Sided				
$k=0,\ \frac{N}{2}$	$\frac{1}{N}X(k)$	$\frac{\Delta t}{N}X(k)$	$S_1(k)=2S(k)$	$P=\sum_{k=0}^{N/2}S_1(k)\Delta f$
$k=1,\cdots,\frac{N}{2}-1$	$\frac{2}{N}X(k)$	$\frac{2\Delta t}{N}X(k)$	$S_1(k)=S(k)$	
Frequency Scale	$k\,\Delta f$	$k\,\Delta f$	$k\,\Delta f$	

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] K.H. Shin, J.K. Hammond, Fundamentals of Signal Processing for Sound and Vibration Engineers, Wiley, 2008