

# Complex Integration (2A)

---

Copyright (c) 2012 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

Please send corrections (or suggestions) to [youngwlim@hotmail.com](mailto:youngwlim@hotmail.com).

This document was produced by using OpenOffice and Octave.

# Contour Integrals

$f(z)$  defined at points  
of a **smooth curve**  $C$

a **smooth curve**  $C$  is defined by

$$\begin{aligned}x &= x(t) \\ y &= y(t) \quad a \leq t \leq b\end{aligned}$$

The contour integral of  $f$  along  $C$

$$\begin{aligned}\int_C f(z) dz &= \int_C (u + iv)(dx + idy) = \int_C \underline{u dx} - \underline{v dy} + i \int_C \underline{v dx} + \underline{u dy} \\ &= \int_a^b [\underline{u x'(t)} - \underline{v y'(t)}] dt + i \int_a^b [\underline{v x'(t)} + \underline{u y'(t)}] dt \\ &= \int_a^b (u + iv)(x'(t) + iy'(t)) dt\end{aligned}$$

$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt$$

$$z(t) = x(t) + iy(t)$$

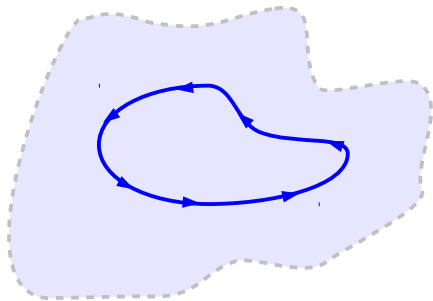
$$z'(t) = x'(t) + iy'(t)$$

$$a \leq t \leq b$$

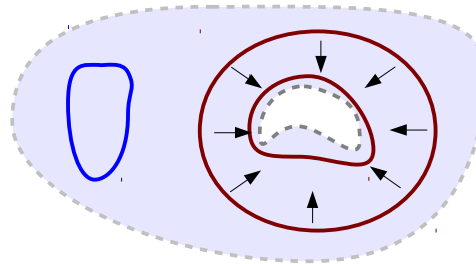
# Connected Region

## Connected Domains

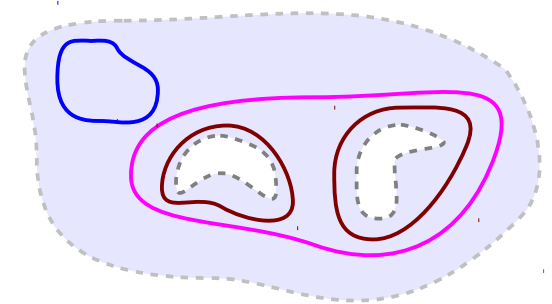
### Simply Connected



### Doubly Connected

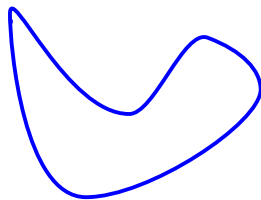


### Triply Connected

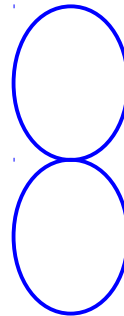


## Closed Paths

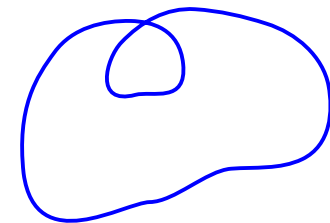
### simple closed path



### not simple closed path



### not simple closed path



# Cauchy's Theorem

$f(z)$  : **analytic** in a **simply connected domain**  $D$

$f'(z)$  : **continuous** in a **simply connected domain**  $D$



for every **simple closed contour**  $C$  in  $D$

$$\oint_C f(z) dz = 0$$

$$\int_C f(z) dz = \int_C (u+iv)(dx+idy) = \int_C u dx - v dy + i \int_C v dx + u dy$$

$$= \iint_D \left( -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dA + i \iint_D \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dA = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

# Cauchy-Goursat Theorem

$f(z)$  : **analytic** in a **simply connected domain**  $D$



for every **simple closed contour**  $C$  in  $D$

$$\oint_C f(z) dz = 0$$

$f'(z)$  : ~~**continuous**~~ in a simply connected domain  $D$

simple closed curve

a continuously turning tangent

except possibly at a finite number of points

allow a finite number of corners (**not smooth**)

# Cauchy's Integral Formula

$f(z)$  : **analytic** on and inside simple close curve  $C$

➔ 
$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$

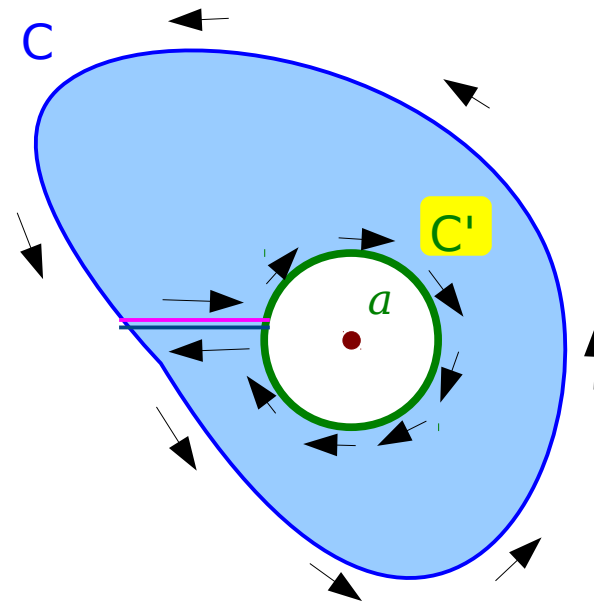
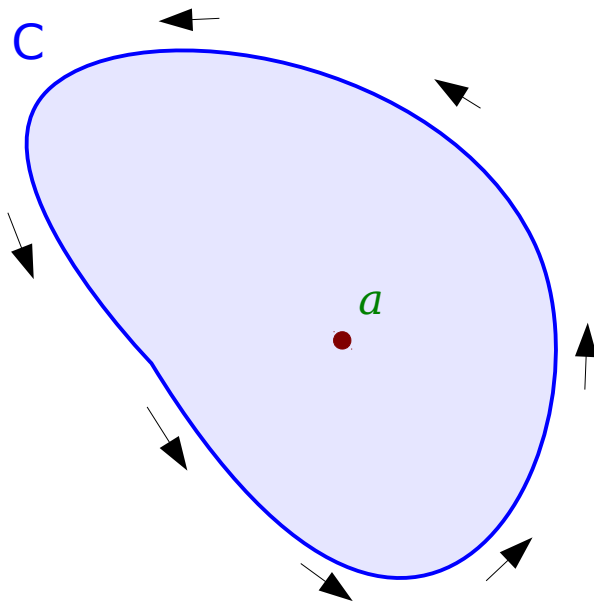
the value of  $f(z)$   
at a point  $z = a$  inside  $C$

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw$$

# Cauchy's Integral Formula

$f(z)$  : **analytic** on and inside simple close curve  $C$

➔ 
$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$



$$\begin{aligned} & \oint_{\text{ccw } C} \frac{f(z) dz}{z-a} \\ &= \oint_{\text{ccw } C'} \frac{f(z) dz}{z-a} \end{aligned}$$

$$\oint_C f(z) dz = 0$$

$$\oint_{\text{ccw } C} \frac{f(z) dz}{z-a} + \oint_{\text{cw } C'} \frac{f(z) dz}{z-a} = 0$$



# Cauchy's Integral Formula

$f(z)$  : **analytic** on and inside simple close curve  $C$

➔  $f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$

along  $C'$   $z - a = \rho e^{i\theta}$

$z = a + \rho e^{i\theta}$

$dz = i\rho e^{i\theta} d\theta$

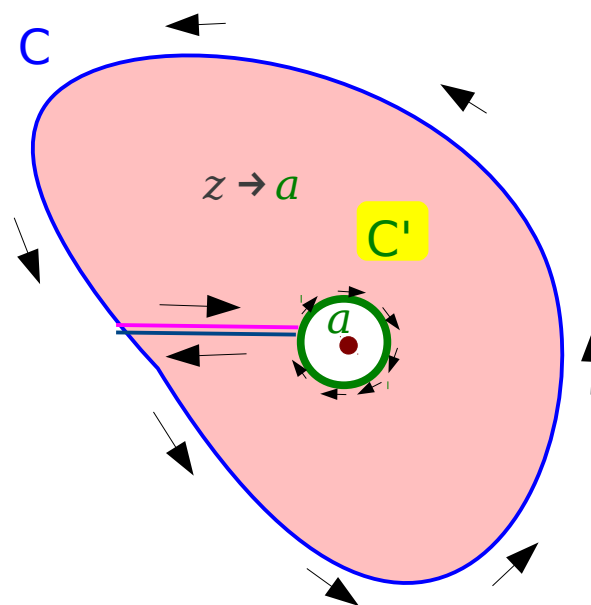
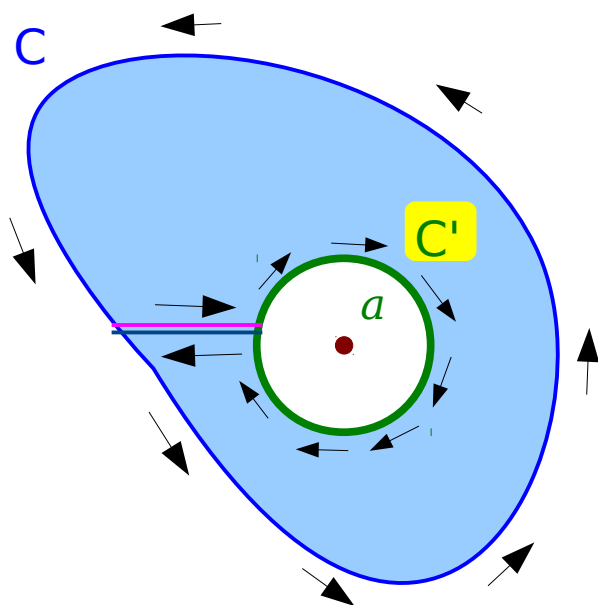
$\frac{dz}{z-a} = \frac{i\rho e^{i\theta} d\theta}{\rho e^{i\theta}}$

$\oint_{\text{ccw } C} \frac{f(z) dz}{z-a}$

$= \int_0^{2\pi} f(z) i d\theta$

$= 2\pi i f(a)$

as  $z \rightarrow a$  ➔  $\rho \rightarrow 0$



$\oint_{\text{ccw } C} \frac{f(z) dz}{z-a} = \oint_{\text{ccw } C'} \frac{f(z) dz}{z-a}$

$= 2\pi i f(a)$

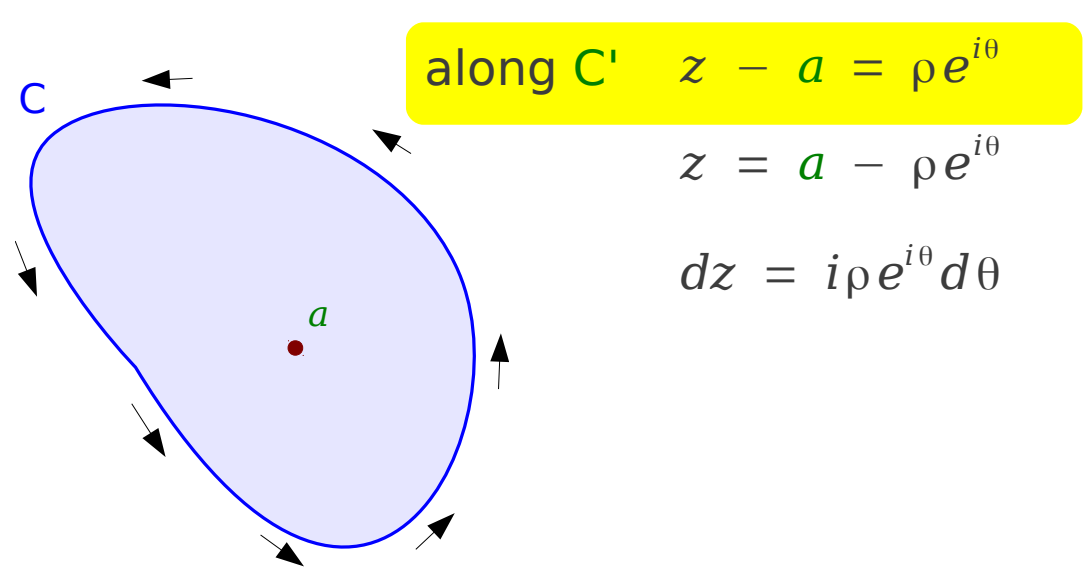
# Cauchy's Integral Formula

$$\frac{dz}{(z-a)^2} = \frac{i\rho e^{i\theta} d\theta}{(\rho e^{i\theta})^2}$$

$$\oint_{\text{ccw } C} \frac{f(z) dz}{(z-a)^2} = \int_0^{2\pi} \frac{f(z)i}{\rho e^{i\theta}} d\theta$$

$$= \int_0^{2\pi} \frac{f(z)}{\rho} i e^{-i\theta} d\theta = \left[ -\frac{f(z)}{\rho} e^{-i\theta} \right]_0^{2\pi}$$

$$= -\frac{f(z)}{\rho} (e^{-i2\pi} - e^{-i0}) = 0$$



$$dz = i\rho e^{i\theta} d\theta$$

$$\oint_{\text{ccw } C} f(z) dz = \int_0^{2\pi} f(z) i\rho e^{i\theta} d\theta$$

$$= \left[ f(z) \rho e^{i\theta} \right]_0^{2\pi}$$

$$= f(z) \rho (e^{-i2\pi} - e^{-i0}) = 0$$

$$(z-a) dz = \rho e^{i\theta} i\rho e^{i\theta} d\theta$$

$$\oint_{\text{ccw } C} (z-a) f(z) dz = \int_0^{2\pi} f(z) i(\rho e^{i\theta})^2 d\theta$$

$$= \int_0^{2\pi} f(z) \rho^2 i e^{i2\theta} d\theta = \left[ f(z) \frac{\rho}{2} e^{i2\theta} \right]_0^{2\pi}$$

$$= f(z) \frac{\rho}{2} (e^{-i4\pi} - e^{-i0}) = 0$$

## References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”