

Surface Integrals (6A)

- Surface Integral
- Stokes' Theorem

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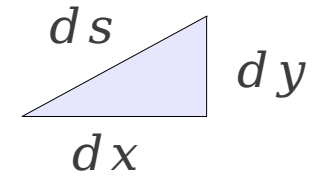
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Arc Length In the Plane

$$y = f(x)$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



Surface Area In the Space

$$z = f(x, y)$$

Area of the surface over R

$$\begin{aligned} A(S) &= \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA \\ &= \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dA \end{aligned}$$

Differential of surface area

$$dS = \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$

Differential of Surface Area (1)

Differential of surface area

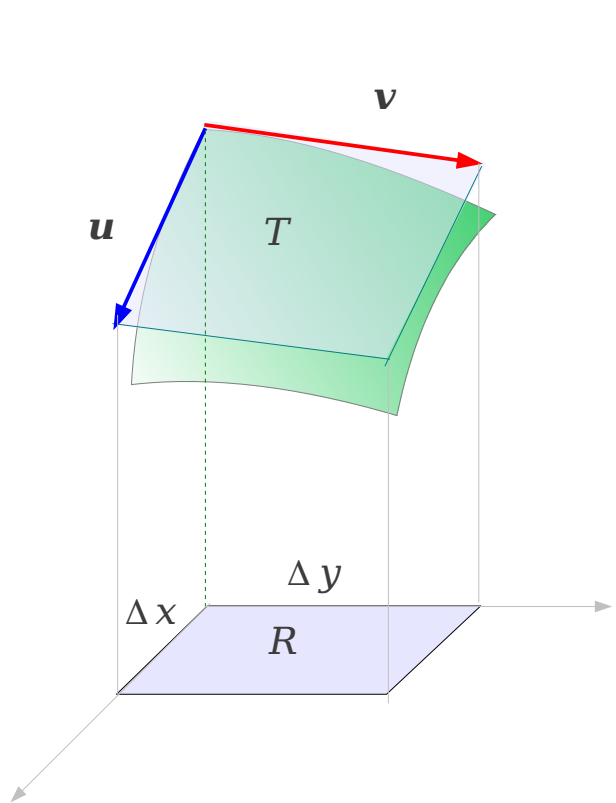
$$dS = \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$

Slope along
x direction

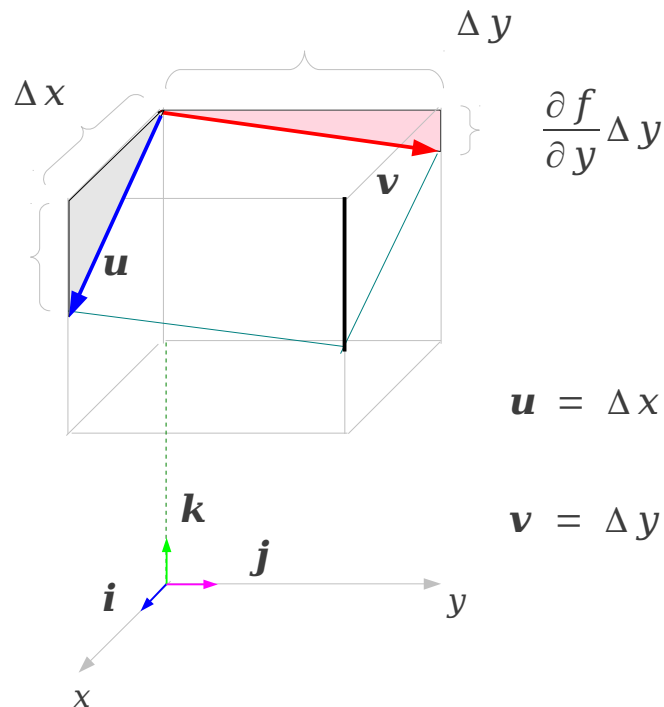
$$\frac{\partial f}{\partial x} = -2$$

Slope along
y direction

$$\frac{\partial f}{\partial y} = -1$$



$$\frac{\partial f}{\partial x} \Delta x$$



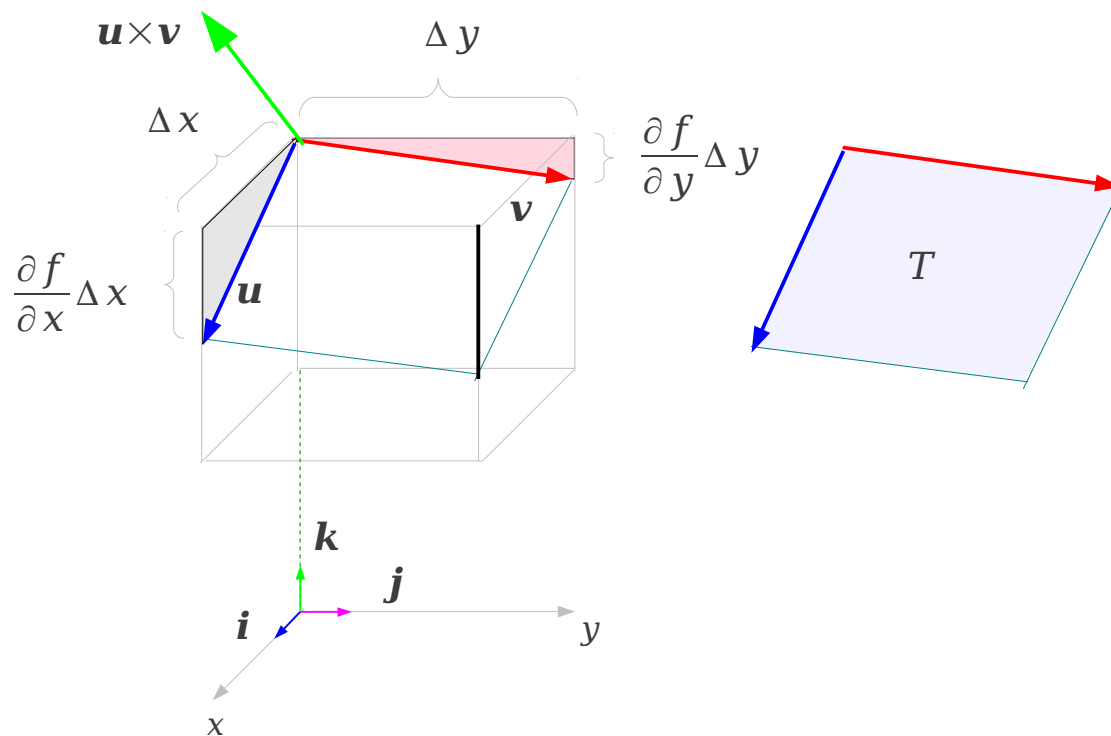
$$\mathbf{u} = \Delta x \mathbf{i} + \frac{\partial f}{\partial x} \Delta x \mathbf{k}$$

$$\mathbf{v} = \Delta y \mathbf{j} + \frac{\partial f}{\partial y} \Delta y \mathbf{k}$$

Differential of Surface Area (2)

Differential of surface area

$$dS = \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$



$$\mathbf{u} = \Delta x \mathbf{i} + \frac{\partial f}{\partial x} \Delta x \mathbf{k}$$

$$\mathbf{v} = \Delta y \mathbf{j} + \frac{\partial f}{\partial y} \Delta y \mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & \frac{\partial f}{\partial x} \Delta x \\ 0 & \Delta y & \frac{\partial f}{\partial y} \Delta y \end{vmatrix}$$

$$= \left[-\frac{\partial f}{\partial x} \mathbf{i} - \frac{\partial f}{\partial y} \mathbf{j} + \mathbf{k} \right] \Delta x \Delta y$$

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \Delta x \Delta y$$

$$T = \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \Delta x \Delta y$$

$$= \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \Delta A$$

Line Integral with an Explicit Curve Function

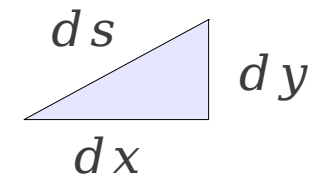
$$y = f(x) \quad \longrightarrow \quad \frac{dy}{dx} = f'(x) \quad \longrightarrow \quad dy = f'(x) dx$$

$$a \leq x \leq b$$

Curve C

$$ds = \sqrt{[dx]^2 + [dy]^2}$$

$$ds = \sqrt{1 + [f'(x)]^2} dx$$



$$\int_C G(x, y) dx = \int_a^b G(x, f(x)) dx$$

$$\int_C G(x, y) dy = \int_a^b G(x, f(x)) f'(x) dx$$

$$\int_C G(x, y) ds = \int_a^b G(x, f(x)) \sqrt{1 + [f'(x)]^2} dx$$

Surface Integral with an Explicit Surface Function

$$z = f(x, y) \quad \rightarrow \quad \frac{df}{dx} = f_x(x, y)$$

$$a \leq x \leq b$$

$$c \leq y \leq d$$

$$\frac{df}{dy} = f_y(x, y)$$

Region R

$$\rightarrow \quad dS = \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$

$$\iint_S G(x, y, z) dS = \iint_R G(x, y, f(x, y)) \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$

Surface Integral with an Explicit Surface Function

$$z = f(x, y) \quad \Rightarrow \quad \frac{df}{dx} = f_x(x, y) \quad \Rightarrow \quad dS = \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$

Region R

$$\frac{df}{dy} = f_y(x, y)$$

$$\iint_S G(x, y, z) dS = \iint_R G(x, y, f(x, y)) \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$

$$y = g(x, z) \quad \Rightarrow \quad \frac{dg}{dx} = g_x(x, z) \quad \Rightarrow \quad dS = \sqrt{1 + [g_x(x, z)]^2 + [g_z(x, z)]^2} dA$$

Region R

$$\frac{dg}{dz} = g_z(x, z)$$

$$\iint_S G(x, y, z) dS = \iint_R G(x, g(x, z), z) \sqrt{1 + [g_x(x, z)]^2 + [g_z(x, z)]^2} dA$$

$$x = h(y, z) \quad \Rightarrow \quad \frac{dh}{dy} = h_y(y, z) \quad \Rightarrow \quad dS = \sqrt{1 + [h_y(y, z)]^2 + [h_z(y, z)]^2} dA$$

Region R

$$\frac{dh}{dz} = h_z(y, z)$$

$$\iint_S G(x, y, z) dS = \iint_R G(h(y, z), y, z) \sqrt{1 + [h_y(y, z)]^2 + [h_z(y, z)]^2} dA$$

Mass of a Surface

density $\rho(x, y, z)$

mass $m = \iint_S \rho(x, y, z) dS$

Functions of Three Variables

Functions of three variables

$$w = F(x, y, z) \quad \mathbb{R}^4$$

Level Surface

$$c_0 = F(x, y, z) \quad \mathbb{R}^3$$

$$x = f(t)$$

$$y = g(t)$$

$$z = h(t)$$

$$\frac{dc_0}{dt} = \frac{dF}{dt}(x, y, z)$$

$$0 = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt}$$

$$0 = \left(\frac{\partial F}{\partial x} \mathbf{i} + \frac{\partial F}{\partial y} \mathbf{j} + \frac{\partial F}{\partial z} \mathbf{k} \right) \cdot \left(\frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \right)$$

$$0 = \nabla F(x, y, z) \cdot \mathbf{r}'(t)$$

∇F normal to the level surface at $P(x_0, y_0, z_0)$

Level Surface

Functions of three variables

$$w = G(x, y, z) \quad \mathbb{R}^4$$

Level Surface

$$c_0 = G(x, y, z) \quad \mathbb{R}^3$$

$$0 = G(x, y, z)$$

$$0 = \nabla G(x, y, z) \cdot \mathbf{r}'(t)$$

Functions of two variables

$$z = F(x, y) \quad \mathbb{R}^3$$

$$0 = z - F(x, y)$$

$$0 = F(x, y) - z$$

Level Surface

$$c_0 = F(x, y) \quad \mathbb{R}^2$$

$$0 = \nabla F(x, y) \cdot \mathbf{r}'(t)$$

Orientation of a Surface

Surface $g(x, y, z) = 0$

$$z = f(x, y) \quad g(x, y, z) = z - f(x, y) = 0$$

$$g(x, y, z) = f(x, y) - z = 0$$

Unit Normal Vector

$$\mathbf{n} = \frac{1}{\|\nabla g\|} \nabla g$$

Surface Integral over a 3-D Vector Field (1)

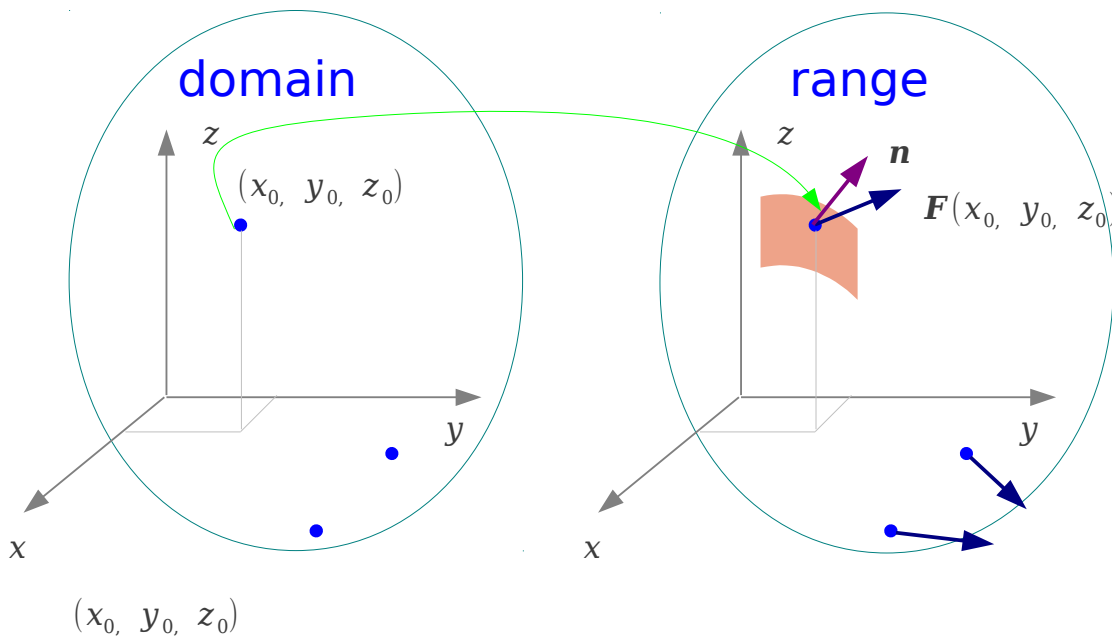
A given point in a 3-d space



A vector

$$(x_0, y_0, z_0)$$

$$\langle P(x_0, y_0, z_0), Q(x_0, y_0, z_0), R(x_0, y_0, z_0) \rangle$$



3 functions

$$(x_0, y_0, z_0) \longrightarrow P(x_0, y_0, z_0)$$

$$(x_0, y_0, z_0) \longrightarrow Q(x_0, y_0, z_0)$$

$$(x_0, y_0, z_0) \longrightarrow R(x_0, y_0, z_0)$$

only points that are
on the surface

$$\longrightarrow \mathbf{F}(x_0, y_0, z_0) = P(x_0, y_0, z_0)\mathbf{i} + Q(x_0, y_0, z_0)\mathbf{j} + R(x_0, y_0, z_0)\mathbf{k}$$

consider only the component of \mathbf{F} along $\mathbf{n} \longrightarrow \mathbf{F} \cdot \mathbf{n}$

Surface Integral over a 3-D Vector Field (2)

$$\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}$$

Line Integral over a 3-d Vector Field

$$\mathbf{r}(t) = f(t) \mathbf{i} + g(t) \mathbf{j} + h(t) \mathbf{k}$$

$$d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz$$

Line Integral over a 3-d Vector Field

$$\text{flux} = \iint_S (\mathbf{F} \cdot \mathbf{n}) dS$$

total volume of a fluid
passing through S
per unit time

\mathbf{F}

velocity field
of a fluid

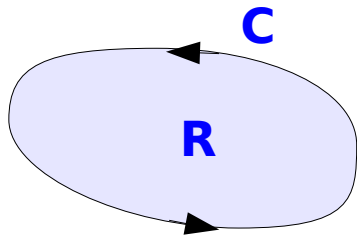
Vector Form of Green's Theorem

A force field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$

A smooth curve $C: x = f(t), y = g(t), a \leq t \leq b$

Work done by \mathbf{F} along C $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$

$$= \int_C P(x, y) dx + Q(x, y) dy$$



$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \quad (\nabla \times \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

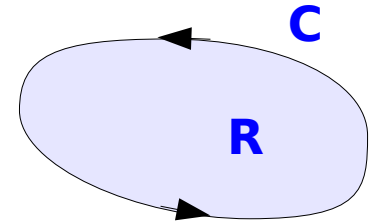
$$\oint_C P dx + Q dy = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA$$

Stokes' Theorem (1)

A force field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$

Work done by \mathbf{F} along C

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

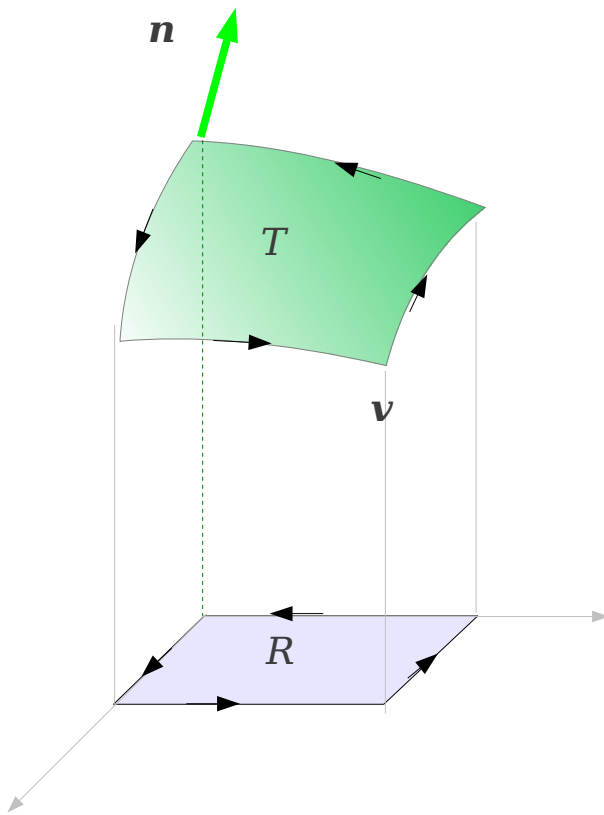


2-space

$$= \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA$$

3-space

$$= \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA$$



Stokes' Theorem (2)

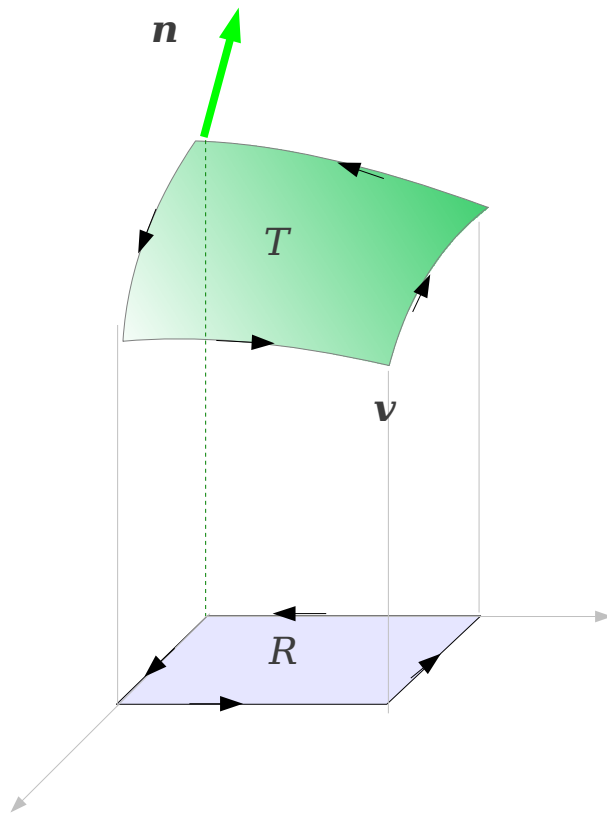
A force field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$

Work done by \mathbf{F} along C

$$W = \int_c \mathbf{F} \cdot d\mathbf{r} = \int_c \mathbf{F} \cdot \mathbf{T} ds$$

3-space

$$= \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA$$



$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

$$\mathbf{n} = \frac{\frac{\partial f}{\partial x} \mathbf{i} - \frac{\partial f}{\partial y} \mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2}} \quad \left. \begin{array}{l} \text{surface} \\ g(x, y, z) \\ = z - f(x, y) \end{array} \right\}$$

Gradient of a 2 Variable Function

Function of two variables $f(x, y)$

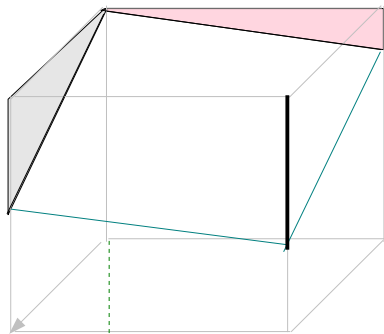
$$\nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

⇒ vector

Rate of change of f in the x direction
 Rate of change of f in the y direction

Slope in the x direction

$$\frac{\partial f}{\partial x} = -2$$

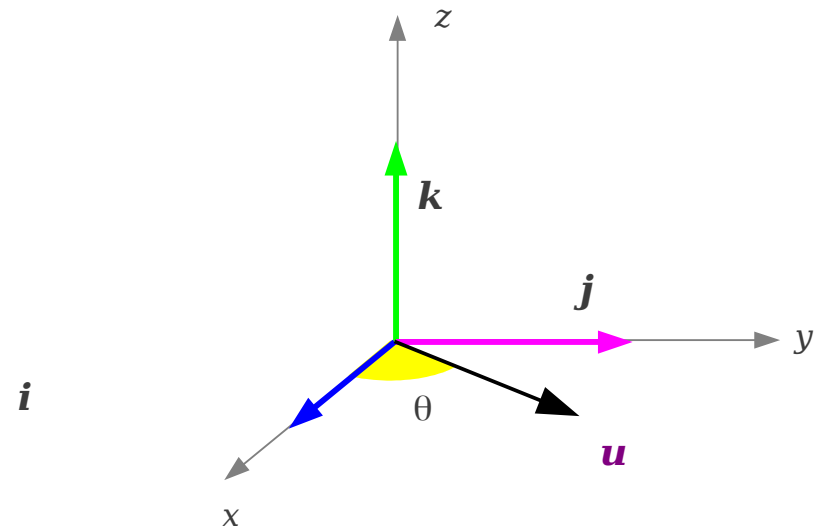
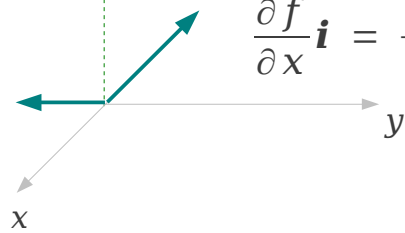


Slope in the y direction

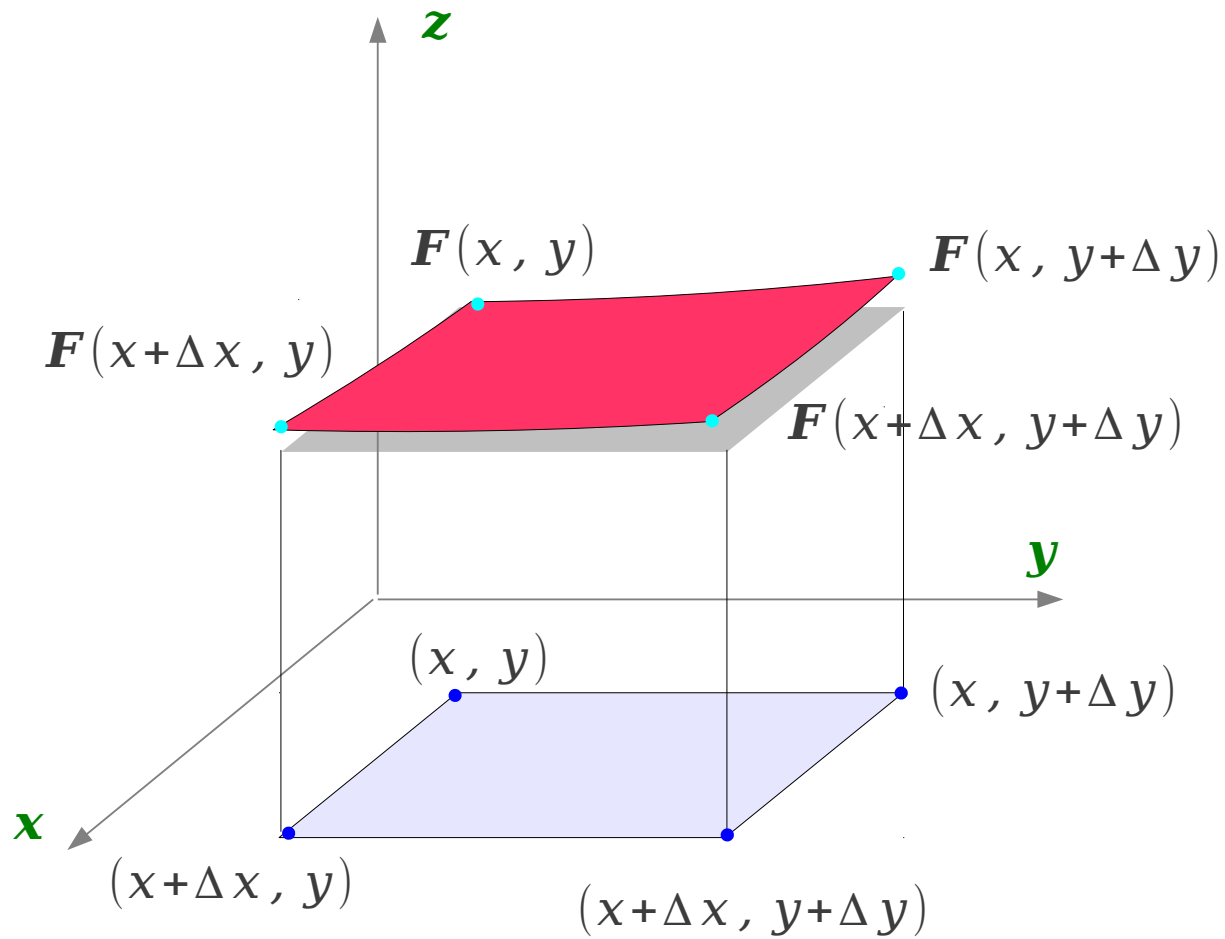
$$\frac{\partial f}{\partial y} = -1$$

$$\frac{\partial f}{\partial y} \mathbf{j} = -1 \mathbf{j}$$

$$\frac{\partial f}{\partial x} \mathbf{i} = -2 \mathbf{i}$$



2-D Divergence



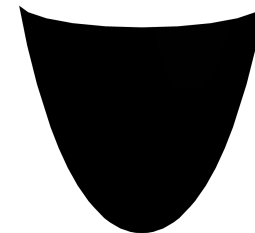
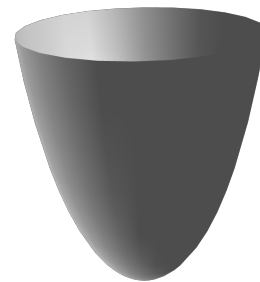
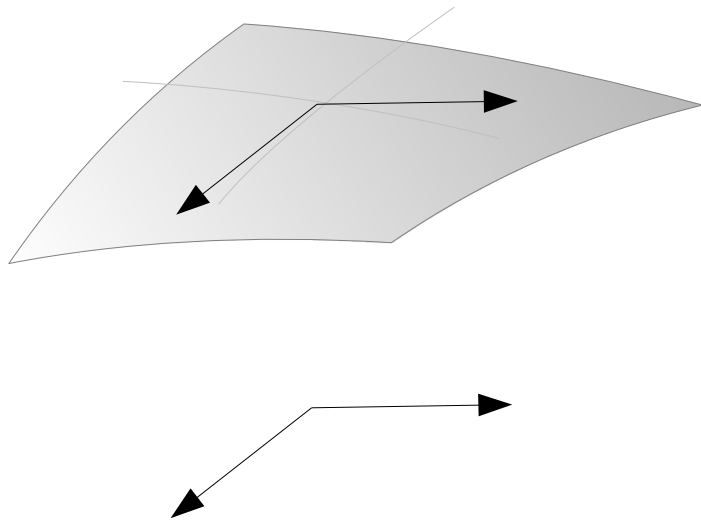
Chain Rule

Function of two variable

$$y = f(u, v)$$

$$u = g(x, y)$$

$$v = h(x, y)$$



References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [4] D.G. Zill, “Advanced Engineering Mathematics”