Resistive damping

$$u_{xx} = \frac{1}{c^2} u_{tt} + k u_t$$

$u(0,t) = 0 \qquad u(a,t) = 0 \qquad$ BCs

$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad$ ICs

Homogeneous BCs so no need for $v(x)$.

$u(x,t) = \varphi(x) T(t)$

from PDE

$$\varphi'' T = \frac{1}{c^2} \varphi T'' + k \varphi T'$$

$$\frac{\varphi''}{\varphi} = \frac{\left(\frac{1}{c^2} T'' + kT'\right)}{T} = -\lambda^2$$

gives $\varphi(x) = A \cos \lambda x + B \sin \lambda x$

$\varphi(0) = 0 \qquad \varphi(a) = 0$

$\varphi_n(x) = \sin(\lambda_n x)$ with $\lambda_n = \frac{n\pi}{a}$

and $\frac{T''}{c^2} + kT' + \lambda^2 T = 0$

$$\underline{T'' + c^2 k T' + \lambda^2 c^2 T = 0}$$

$r^2 + c^2 k r + \lambda_n^2 c^2 = 0$ characteristic polynomial

quadratic formula.

$$r = -\frac{c^2 k}{2} \pm \frac{\sqrt{(c^2 k)^2 - 4(\lambda_n^2 c^2)}}{2}$$

$\frac{(c^2 k)^2}{4} - \lambda_n^2 c^2$ is discriminant

if $\left(\frac{c^2 k}{2}\right)^2 - (\lambda_n c)^2 < 0$ r has an imaginary component.

$$\left(\frac{c^2 k}{2}\right)^2 < (\lambda_n c)^2$$

$$\frac{c^2 k}{2} < \lambda_n c$$

$$\boxed{\frac{ck}{2} < \lambda_n}$$

when $\lambda_n > \frac{ck}{2}$ get $r = \alpha \pm \gamma_n i$

where $\alpha = -\frac{c^2 k}{2}$ $\gamma_n = \frac{\sqrt{|(c^2 k)^2 - 4\lambda_n^2 c^2|}}{2}$

then $T_n(t) = Ae^{(\alpha + \gamma_n i)t} + Be^{(\alpha - \gamma_n i)t}$

$= Ae^{\alpha t} e^{\gamma_n i t}$

$$\boxed{T_n(t) = e^{\alpha t}(a_n \cos \gamma_n t + b_n \sin \gamma_n t)}$$

$$u_n(x,t) = \varphi_n(x)\, T_n(t);$$

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n(x) e^{\alpha t} (a_n \cos \gamma_n t + b_n \sin \gamma_n t)$$

use ICs

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \varphi_n(x)\, a_n$$

$$a_n = \frac{\int_l^r f(x)\, \varphi_n(x)\, p(x)\, dx}{\int_l^r \varphi_n^2(x)\, p(x)\, dx}$$

$p(x) = 1$ for this problem. $l = 0,\ r = L$

$$\frac{\partial u(x,0)}{\partial t} = g(x) = \sum_{n=1}^{\infty} \varphi_n(x) \left.\frac{\partial T_n(t)}{\partial t}\right|_{t=0}$$

$$g(x) = \sum_{n=1}^{\infty} \varphi_n(x) \left[\left.\frac{d e^{\alpha t}}{dt} (a_n \cos \gamma_n t + b_n \sin \gamma_n t)\right|_{t=0} + \left. e^{\alpha t} \frac{d}{dt} (a_n \cos \gamma_n t + b_n \sin \gamma_n t)\right|_{t=0} \right]$$

$$g(x) = \sum_{n=1}^{\infty} \varphi_n(x) \left[\alpha e^{\alpha \cdot 0} (a_n \cdot 1 + b_n \cdot 0) + e^{\alpha \cdot 0} (a_n \gamma_n (-\sin(0)) + b_n \gamma_n \cos(0)) \right]$$

$$g(x) = \sum_{n=1}^{\infty} \varphi_n(x) \left[\alpha a_n + b_n \gamma_n \right]$$

$$g(x) = \sum_{n=1}^{\infty} \varphi_n(x)\left[\alpha a_n + b_n \gamma_n\right]$$

$$g(x)\,\varphi_m(x) = \sum_{n=1}^{\infty} \varphi_n(x)\cdot\varphi_m(x)\left[\alpha a_n + b_n \gamma_n\right]$$

$$\int_e^r g(x)\,\varphi_m(x)\,dx = \sum_{n=1}^{\infty} \int_e^r \varphi_n(x)\,\varphi_m(x)\left[\alpha a_n + b_n \gamma_n\right] dx$$

$$\int_e^r g(x)\,\varphi_m(x)\,dx = \sum_{n=1}^{\infty} \left[\alpha a_n + b_n \gamma_n\right] \int_e^r \varphi_n(x)\,\varphi_m(x)\,dx$$

when $n \neq m$ $\quad \int_e^r \varphi_n(x)\,\varphi_m(x)\,dx = 0$

$$\int_e^r g(x)\,\varphi_m(x)\,dx = \left[\alpha a_m + b_m \gamma_m\right] \int_e^r \varphi_m(x)\,\varphi_m(x)\,dx$$

$$\frac{\int_e^r g(x)\,\varphi_m(x)\,dx}{\int_e^r \varphi_m^2(x)\,dx} = \alpha a_m + b_m \gamma_m$$

$$\frac{\dfrac{\int_{\ell}^{r} g(x)\,\varphi_m(x)\,dx}{\int_{\ell}^{r} \varphi_m^2(x)\,dx} - \alpha a_m}{\gamma_m} = b_m$$

relabel m as n.

O.K. $u_n(x,t) = \varphi_n(x)\,T_n(t)$

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n(x)\, e^{\alpha t}\left(a_n \cos \gamma_n t + b_n \sin \gamma_n t\right)$$

<u>maple</u>