CORDIC in VHDL (1A)

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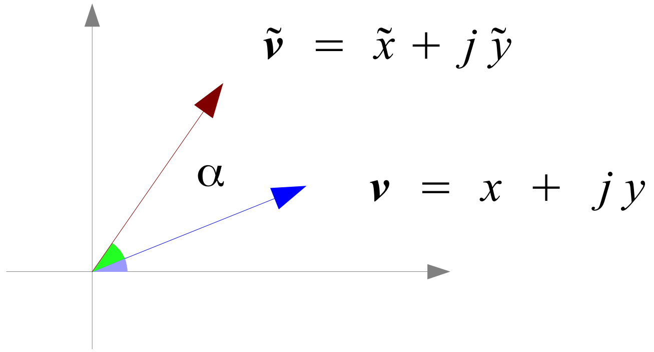
CORDIC Background

1. G Hampson,
A VHDL Implementation of a CORDIC Arithmetic Processor Chip
Monash University, Technical Report 94-9, 1994

Angle Expansion

$$\tilde{\boldsymbol{v}} = \boldsymbol{v}\, e^{j\alpha}$$



elementary angle

$$\alpha_0 = \tan^{-1}(2^0) =$$
$$\alpha_1 = \tan^{-1}(2^{-1}) =$$
$$\alpha_2 = \tan^{-1}(2^{-2}) =$$
$$\alpha_3 = \tan^{-1}(2^{-3}) =$$

α can be expanded by

a set of elementary angles α_i

pseudo-digits q_i

$$\alpha_i \begin{cases} \pi/2 & i = -1 \\ \tan^{-1}(2^{-i}) & i = 0, 1, 2, \cdots, n-1 \end{cases}$$

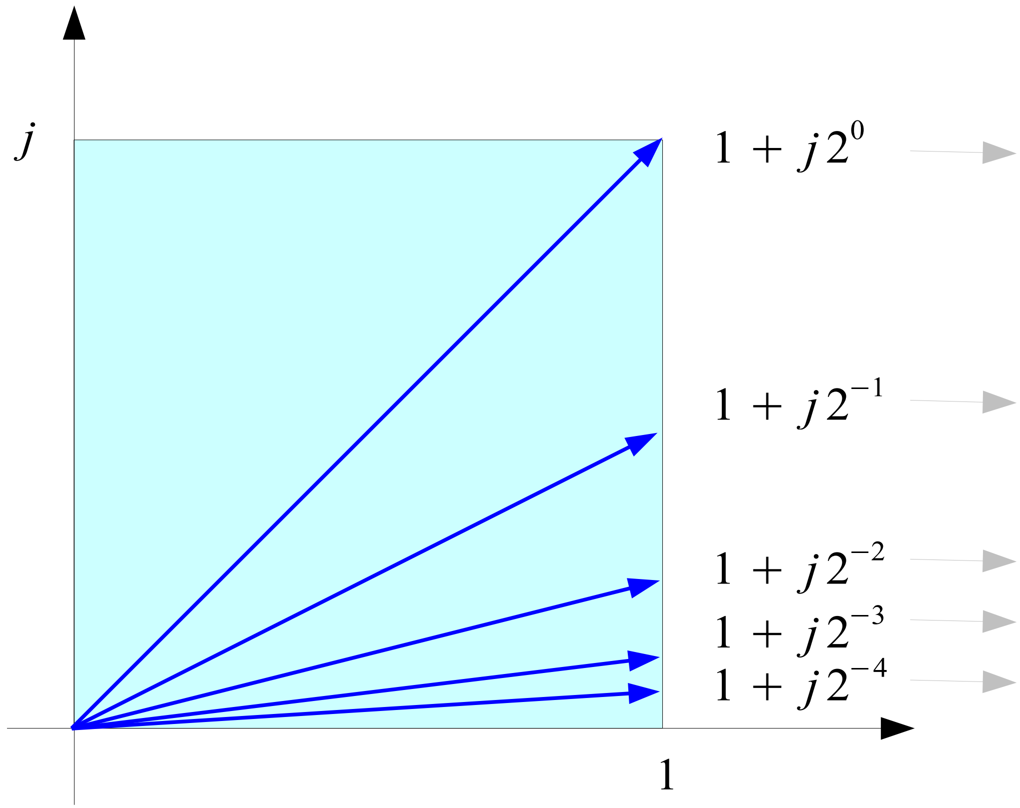
$$q_i \begin{cases} -1 \\ +1 \end{cases}$$

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

angle expansion error z_n

$$|z_n| \leq 2^{-(n-1)}$$

Elementary Angle: $\tan^{-1}(K)$



$$\alpha_L = \tan^{-1}(2^{-L}) = \tan^{-1}(K)$$

$$\alpha_0 = \tan^{-1}(2^{0}) = 45.00000$$

$$\alpha_1 = \tan^{-1}(2^{-1}) = 26.56505$$

$$\alpha_2 = \tan^{-1}(2^{-2}) = 14.03624$$

$$\alpha_3 = \tan^{-1}(2^{-3}) = 7.12502$$

$$\alpha_4 = \tan^{-1}(2^{-4}) = 3.57633$$

Represent arbitrary angle α

in terms of $\pm\alpha_0, \pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \cdots, \pm\alpha_L, \cdots$ $\left(K = \frac{1}{2^L}, \quad L = 0, 1, 2, \cdots\right)$

Phase and Magnitude of 1 + jK (1)

Cumulative Magnitude

L	$K = \frac{1}{2^L}$	$R = 1 + jK$	*Phase of R*	*Magnitude of R*	*CORDIC Gain*
0	1.0	1 + j1.0	45°	1.41421356	1.414213562
1	0.5	1 + j0.5	26.56505°	1.11803399	1.581138830
2	0.25	1 + j0.25	14.03624°	1.03077641	1.629800601
3	0.125	1 + j0.125	7.12502°	1.00778222	1.642484066
4	0.0625	1 + j0.0625	3.57633°	1.00195122	1.645688916
5	0.03125	1 + j0.03125	1.78991°	1.00048816	1.646492279
6	0.015625	1 + j0.015625	0.89517°	1.00012206	1.646693254
7	0.007813	1 + j0.007813	0.44761°	1.00003052	1.646743507
•••	•••	•••	•••	•••	•••
					1.647 ←

$$R = 1 + jK \xrightarrow[L \,=\, 0,1,2,\cdots]{K \,=\, 1/2^L} \sqrt{1^2 + K^2} > 1.0$$

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Rotating Vector

$$\tilde{\boldsymbol{v}} = \boldsymbol{v}\, e^{j\alpha}$$

$$= \boldsymbol{v}\, \exp\left(j \left(\sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n \right) \right)$$

$$= \boldsymbol{v} \cdot \left(\prod_{i=-1}^{n-1} e^{j q_i \alpha_i} \right) \cdot e^{j z_n}$$

$$= \boldsymbol{v} \cdot (j q_{-1}) \left(\prod_{i=0}^{n-1} e^{j q_i \alpha_i} \right) \cdot e^{j z_n}$$

$$= \boldsymbol{v} \cdot (j q_{-1}) \left(\prod_{i=0}^{n-1} \cos(\alpha_i) \cdot (1 + j\, q_i 2^{-i}) \right) \cdot e^{j z_n}$$

$$= \boldsymbol{v} \cdot (j q_{-1}) \left(\prod_{i=0}^{n-1} \cos(\alpha_i) \right) \cdot \left(\prod_{i=0}^{n-1} (1 + j\, q_i 2^{-i}) \right) \cdot e^{j z_n}$$

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

$$e^{j\theta} = \cos\theta + j \sin\theta$$

$$e^{j q_{-1} \alpha_0} = e^{j q_{-1} \frac{\pi}{2}} = j q_{-1} \quad \left(e^{\pm j \frac{\pi}{2}} = \pm j \right)$$

$$e^{j q_i \alpha_i} = \cos(q_i\, \alpha_i) + j \sin(q_i\, \alpha_i)$$

$$= \cos(q_i\, \alpha_i) \cdot (1 + j \tan(q_i\, \alpha_i))$$

$$= \cos(q_i\, \alpha_i) \cdot (1 + j\, q_i 2^{-i})$$

$$= \cos(\alpha_i) \cdot (1 + j\, q_i 2^{-i})$$

$$(\cos(\pm\alpha_i) = \cos(\alpha_i))$$

Rotating Via Elementary Angles

$$\tilde{\boldsymbol{v}} = \boldsymbol{v}\, e^{j\alpha}$$ rotated by α

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

$$= \boldsymbol{v} \cdot \left(\prod_{i=0}^{n-1} \cos(\alpha_i) \right) \cdot (jq_{-1}) \cdot \left(\prod_{i=0}^{n-1} (1 + j\, q_i 2^{-i}) \right) \cdot e^{jz_n}$$

$$K_n = \prod_{i=0}^{n-1} \frac{1}{\sqrt{1 + 2^{-2i}}}$$

series rotations of α_i

Angle Expansion Error

$$\alpha - \sum_{i=-1}^{n-1} q_i \cdot \alpha_i = z_n$$

n iterations

q_{-1}	q_0	q_1		$\cdots$	$\cdots$		q_{n-1}

Angle Expansion

$$((((\alpha - q_{-1}\alpha_{-1}) - q_0\alpha_0) - q_1\alpha_1) \quad \cdots \quad - q_{n-1}\alpha_{n-1})$$

$z_{-1}=\alpha$	z_0	z_1	z_2		z_n

$$z_{i+1} = z_i - q_i\,\alpha_i$$

$$\begin{aligned} &if\ (z_0 \ge 0)\ \ q_0 = +1 \quad && z_1 = z_0 - \alpha_0 \\ &if\ (z_0 < 0)\ \ q_0 = -1 \quad && z_1 = z_0 + \alpha_0 \end{aligned}$$

$$\begin{aligned} &if\ (z_i \ge 0)\ \ q_i = +1 \\ &if\ (z_i < 0)\ \ q_i = -1 \end{aligned}$$

$$\begin{aligned} &if\ (z_1 \ge 0)\ \ q_1 = +1 \quad && z_2 = z_1 - \alpha_1 \\ &if\ (z_1 < 0)\ \ q_1 = -1 \quad && z_2 = z_1 + \alpha_1 \end{aligned}$$

$$\begin{aligned} &if\ (z_2 \ge 0)\ \ q_2 = +1 \quad && z_3 = z_2 - \alpha_2 \\ &if\ (z_2 < 0)\ \ q_2 = -1 \quad && z_3 = z_2 + \alpha_2 \end{aligned}$$

$$\begin{aligned} &if\ (z_{-1} \ge 0)\ \ q_{-1} = +1 \\ &if\ (z_{-1} < 0)\ \ q_{-1} = -1 \end{aligned}$$

$$\begin{aligned} &if\ (z_{n-1} \ge 0)\ \ q_{n-1} = +1 \quad && z_n = z_{n-1} - \alpha_{n-1} \\ &if\ (z_{n-1} < 0)\ \ q_{n-1} = -1 \quad && z_n = z_{n-1} + \alpha_{n-1} \end{aligned}$$

q_{-1}	q_0	q_1	q_2		q_{n-1}

CORDIC Function (1)

$$\tilde{\boldsymbol{v}} = \boldsymbol{v}\, e^{j\alpha}$$ rotated by α

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

Rotated vector after n iteration

$$\boldsymbol{v}_n = \boldsymbol{v} \cdot e^{j\alpha} \cdot e^{-jz_n}$$

$$\alpha - z_n = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i$$

$$v_{i+1} = v_i \cdot (1 + j\, q_i \cdot 2^{-i})$$

$$v_0 = v_{-1} \cdot (j\, q_{-1})$$

$$x_{i+1} + jy_{i+1} = (x_i + jy_i) \cdot (1 + j\, q_i \cdot 2^{-i})$$
$$= (x_i - y_i \cdot q_i \cdot 2^{-i}) + j(y_i + x_i \cdot q_i \cdot 2^{-i})$$

$$x_0 + jy_0 = (x_{-1} + jy_{-1}) \cdot (j\, q_{-1})$$
$$= (-q_{-1} \cdot y_{-1}) + j(q_{-1} \cdot x_{-1})$$

$$\begin{cases} x_{i+1} = (x_i - y_i \cdot q_i \cdot 2^{-i}) \\ y_{i+1} = (y_i + x_i \cdot q_i \cdot 2^{-i}) \end{cases}$$

$$\begin{cases} x_0 = -q_{-1} \cdot y_{-1} \\ y_0 = q_{-1} \cdot x_{-1} \end{cases}$$

CORDIC Function (2)

$$\begin{aligned} &if\ (z_i \geq 0) \quad q_i = +1 \\ &if\ (z_i < 0) \quad q_i = -1 \end{aligned}$$

$$v_{i+1} = v_i \cdot (1 + j\, q_i \cdot 2^{-i})$$

$$\begin{cases} x_{i+1} = (x_i - y_i \cdot q_i \cdot 2^{-i}) \\ y_{i+1} = (y_i + x_i \cdot q_i \cdot 2^{-i}) \end{cases}$$

$$z_{i+1} = z_i - q_i\, \alpha_i$$

$$v_0 = v_{-1} \cdot (j\, q_{-1})$$

$$\begin{cases} x_0 = -q_{-1} \cdot y_{-1} \\ y_0 = q_{-1} \cdot x_{-1} \end{cases}$$

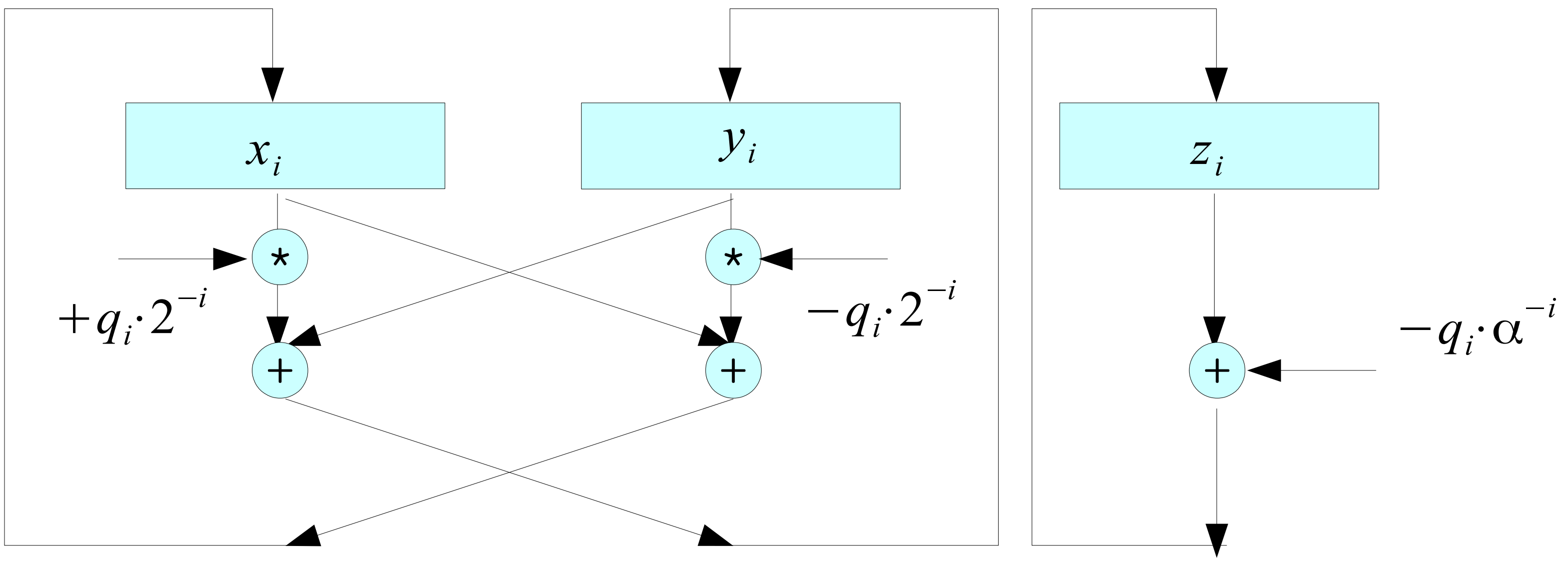
CORDIC Datapath (1)

$$\begin{array}{ll} if\ (z_i \ge 0) & q_i = +1 \\ if\ (z_i < 0) & q_i = -1 \end{array}$$

$$\begin{cases} x_{i+1} = (x_i - y_i \cdot q_i \cdot 2^{-i}) \\ y_{i+1} = (y_i + x_i \cdot q_i \cdot 2^{-i}) \end{cases}$$

$$\begin{cases} x_0 = -q_{-1} \cdot y_{-1} \\ y_0 = q_{-1} \cdot x_{-1} \end{cases}$$

$$z_{i+1} = z_i - q_i\,\alpha_i$$



Rotating Vector

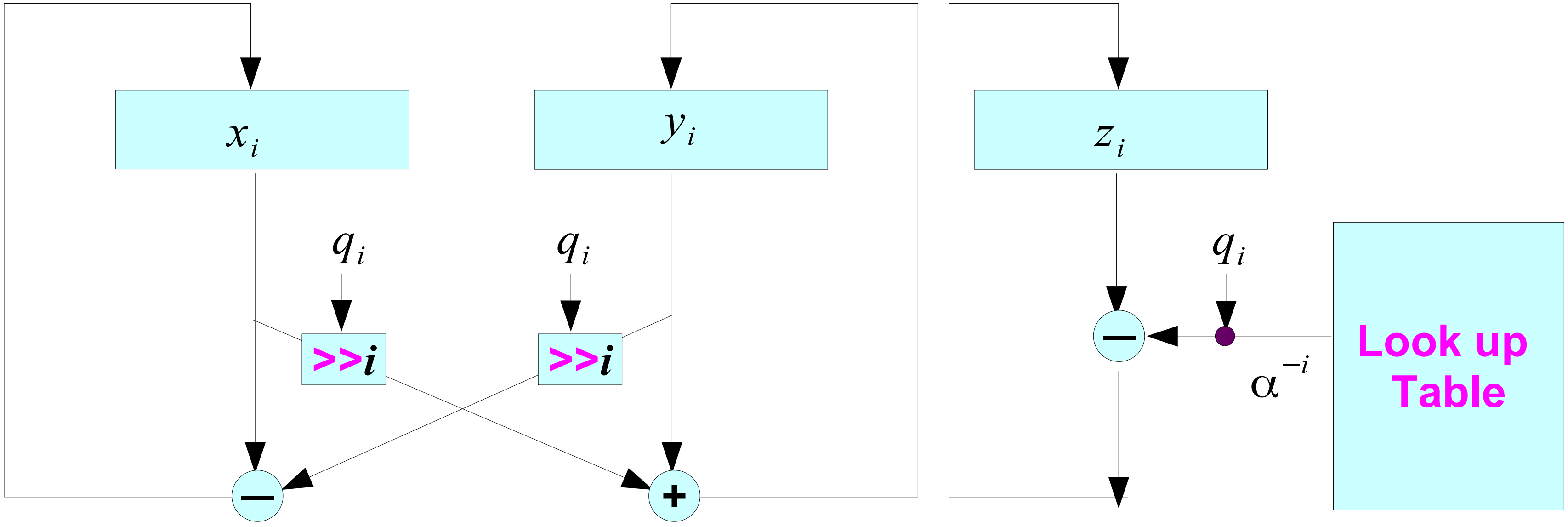
$$\begin{aligned} &if\ (z_i \geq 0) \quad q_i = +1 \\ &if\ (z_i < 0) \quad q_i = -1 \end{aligned}$$

! (msb of z_i) ➡ q_i

$$\begin{cases} x_{i+1} = (x_i - y_i \cdot q_i \cdot 2^{-i}) \\ y_{i+1} = (y_i + x_i \cdot q_i \cdot 2^{-i}) \end{cases}$$

$$\begin{cases} x_0 = -q_{-1} \cdot y_{-1} \\ y_0 = q_{-1} \cdot x_{-1} \end{cases}$$

$$z_{i+1} = z_i - q_i\, \alpha_i$$



Rotating Vector

$$\tilde{\boldsymbol{v}} = \boldsymbol{v}\, e^{j\alpha}$$ rotated by α

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

$$= \boldsymbol{v} \cdot \left(\prod_{i=0}^{n-1} \cos(\alpha_i) \right) \cdot (jq_{-1}) \cdot \left(\prod_{i=0}^{n-1} (1 + j\, q_i 2^{-i}) \right) \cdot e^{jz_n}$$

q_{-1}	q_0	q_1		$\cdots$	$\cdots$		q_{n-1}

References

[1] http://en.wikipedia.org/
[2] G Hampson, A VHDL Implementation of a CORDIC Arithmetic Processor Chip
Monash University, Technical Report 94-9, 1994