

# Anti-aliasing Prefilter (6B)

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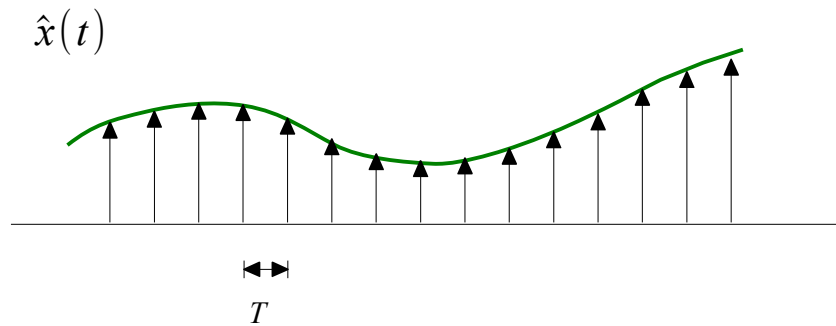
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# Sampler

## Ideal Sampling

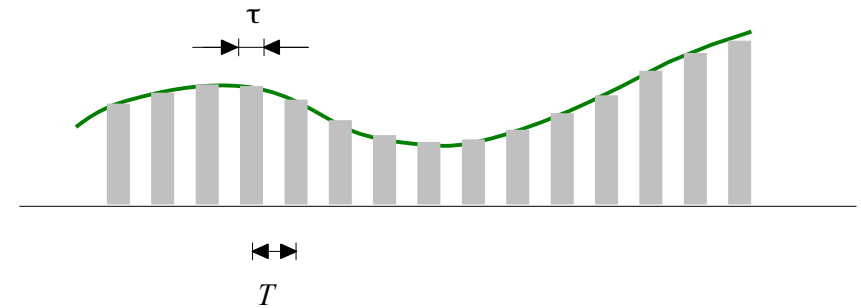


$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

↓ CTFT

$$\hat{X}(f) = \int_{-\infty}^{+\infty} \hat{x}(t) e^{-j2\pi ft} dt$$

## Practical Sampling

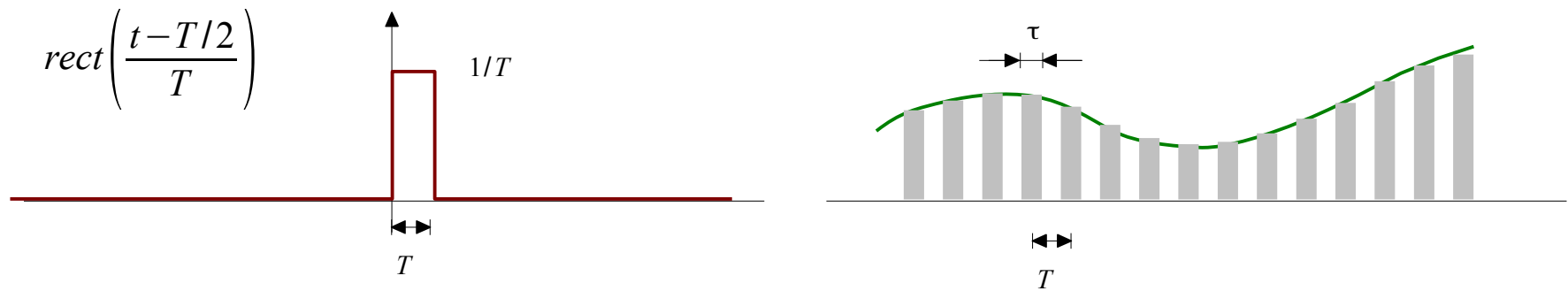


$$\hat{x}(t) \approx \sum_{n=-\infty}^{+\infty} x(nT) p(t-nT)$$

↓ CTFT

?

# Zero Order Hold (ZOH)



$$x_{ZOH}(t) = \sum_{n=-\infty}^{+\infty} x[n] \cdot \text{rect}\left(\frac{t - T/2 - nT}{T}\right)$$

# Square Wave CTFT

## Continuous Time Fourier Series

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt \quad \longleftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_k e^{+jk\omega_0 t}$$

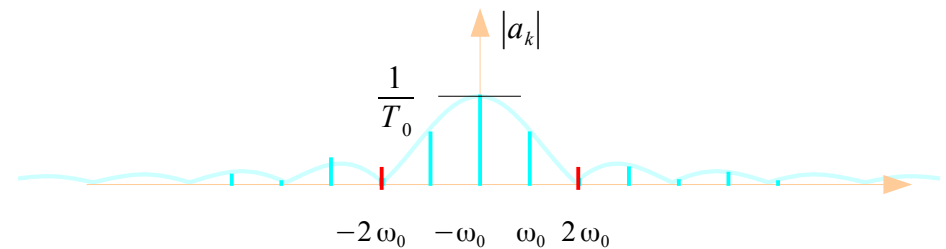
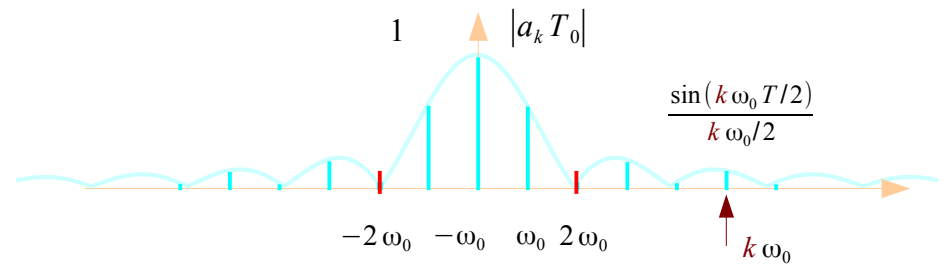
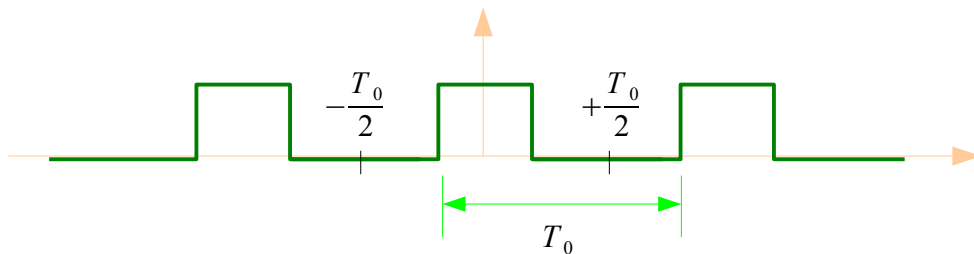
$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(t) e^{-jk\omega_0 t} dt$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t) e^{-jk\omega_0 t} dt$$

$$= \int_{-T_0/2}^{+T_0/2} e^{-jk\omega_0 t} dt = \left[ \frac{-1}{jk\omega_0} e^{-jk\omega_0 t} \right]_{-T_0/2}^{+T_0/2}$$

$$= \frac{e^{-jk\omega_0 T_0/2} - e^{+jk\omega_0 T_0/2}}{jk\omega_0} = \frac{\sin(k\omega_0 T_0/2)}{k\omega_0/2}$$

$$\omega_0 = \frac{2\pi}{T_0} \quad \text{Fundamental Frequency}$$

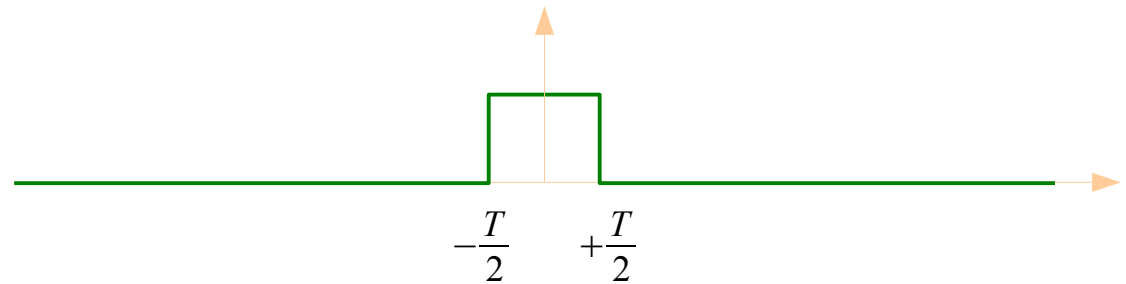


# CTFT and CTFS

## Continuous Time Fourier Transform

## Aperiodic Continuous Time Signal

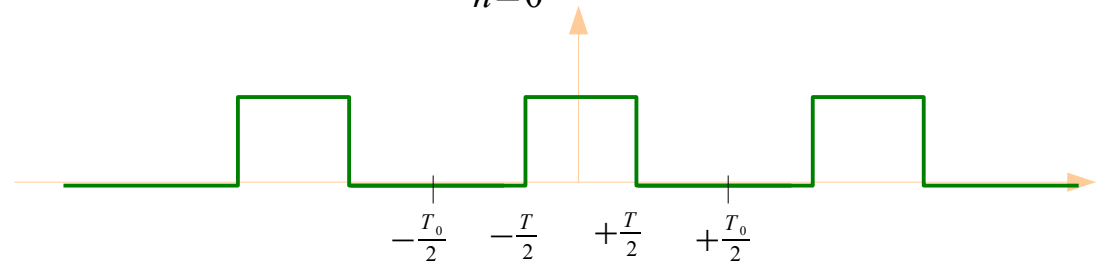
$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$



## Continuous Time Fourier Series

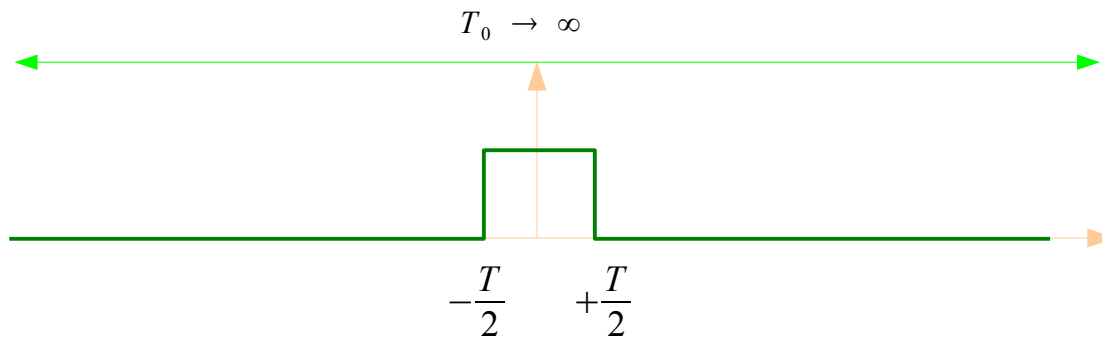
## Periodic Continuous Time Signal

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt \quad \longleftrightarrow \quad x(t) = \sum_{n=-\infty}^{\infty} C_k e^{+jk\omega_0 t}$$

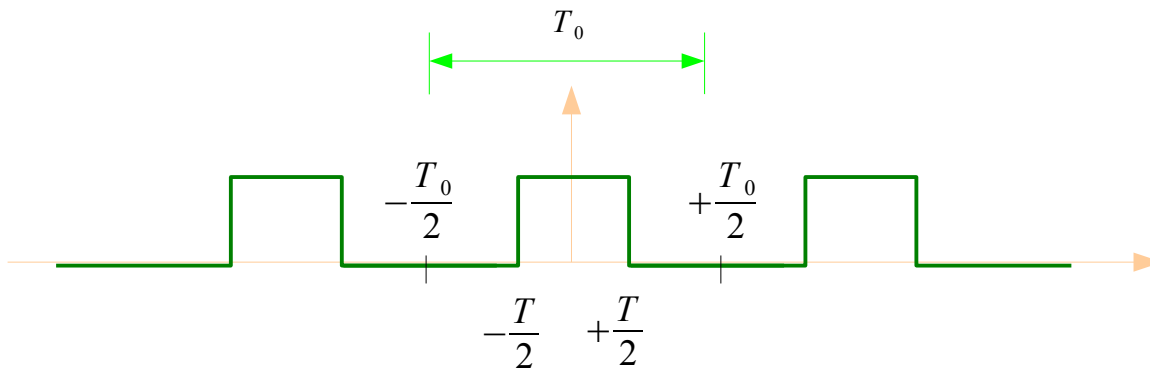


# CTFT ← CTFS

## Aperiodic Continuous Time Signal Continuous Time Fourier Transform



## Periodic Continuous Time Signal Continuous Time Fourier Series



$$x(t)$$

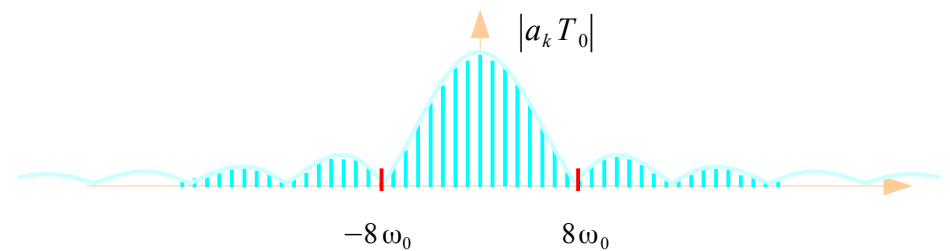
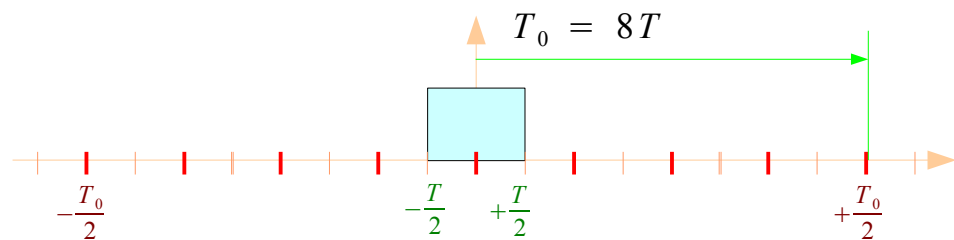
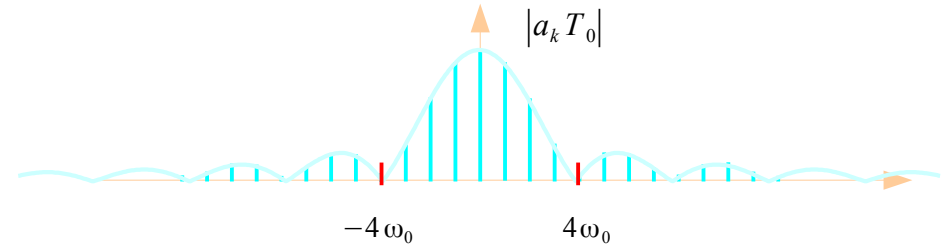
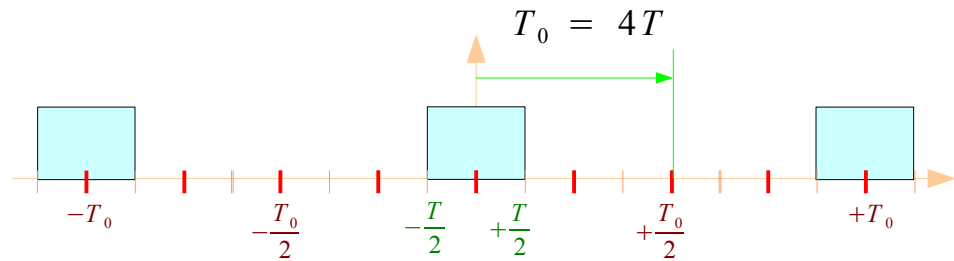
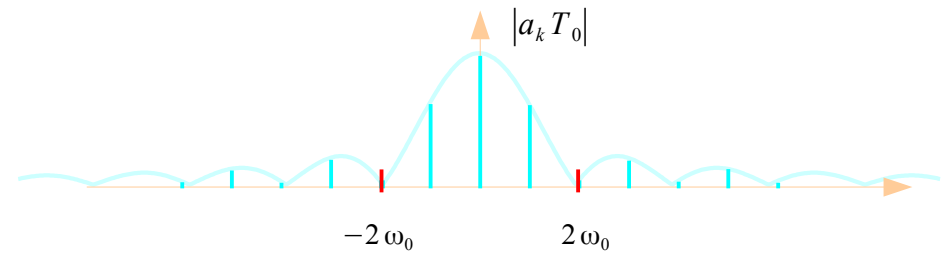
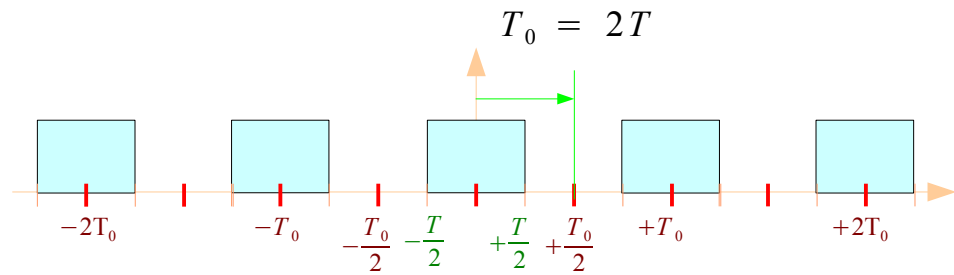
As  $T_0 \rightarrow \infty$ ,

$$x_{T_0}(t) \rightarrow x(t)$$

$$\omega_0 = \frac{2\pi}{T_0} \rightarrow 0$$

$$x_{T_0}(t) = \sum_{n=-\infty}^{+\infty} x(t - nT_0)$$

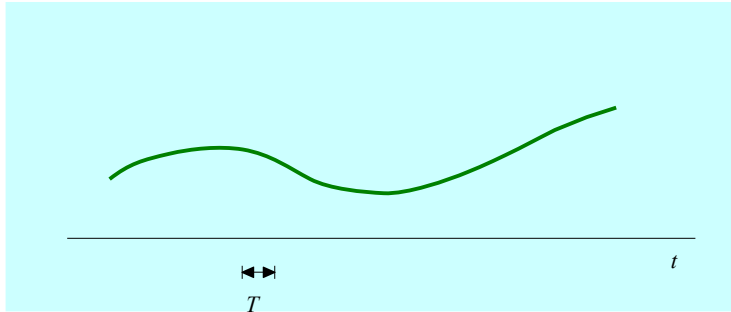
# CTFT and CTFS as $T_0 \rightarrow \infty$ ,



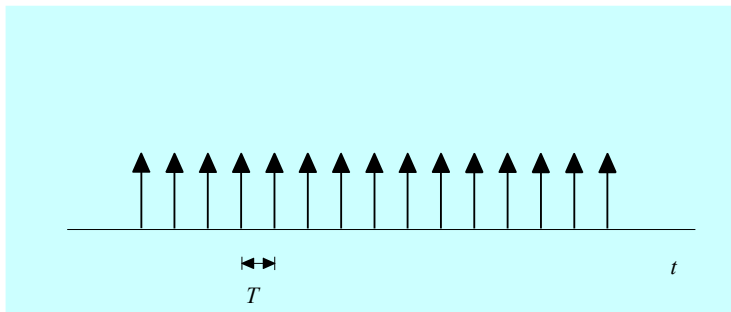


# Sampling (1)

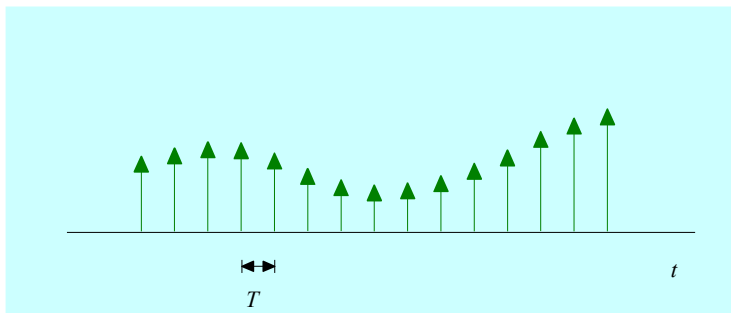
## Ideal Sampling



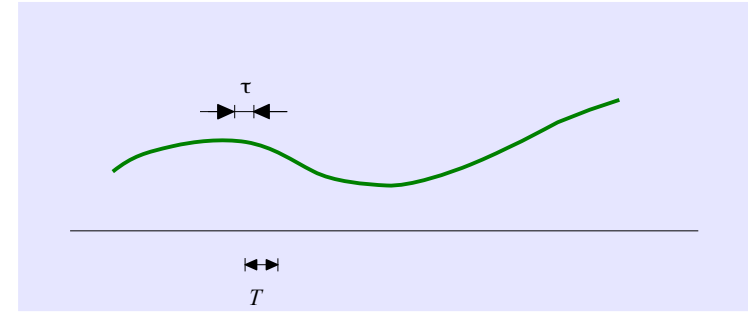
X



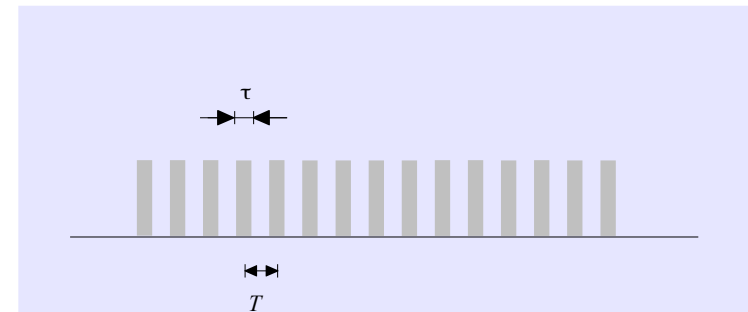
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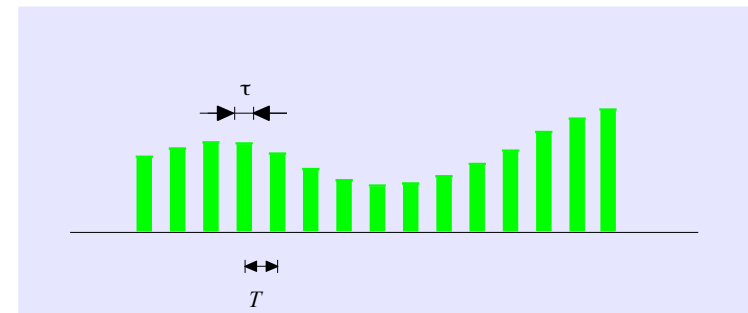
## Practical Sampling



X

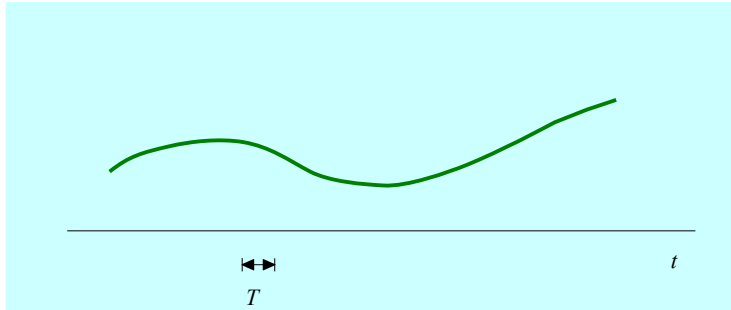


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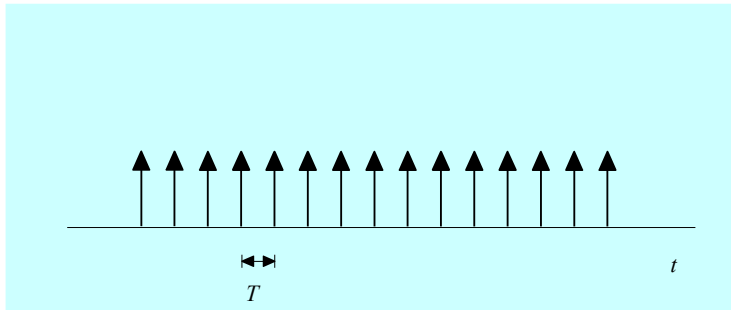


# Sampling (2)

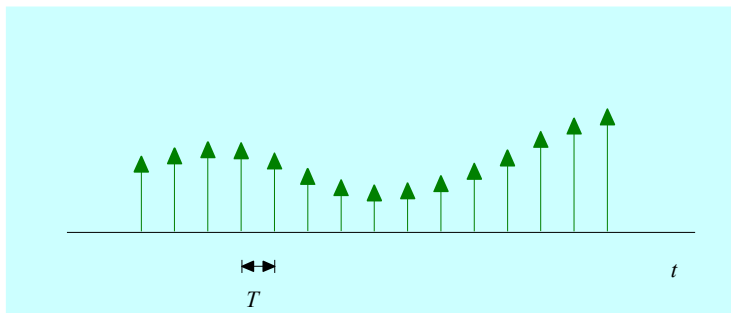
## Ideal Sampling



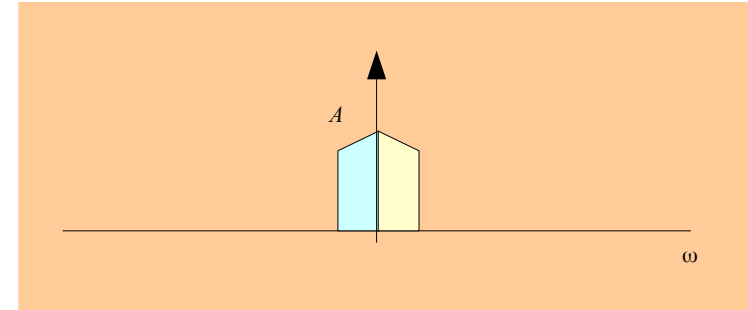
X



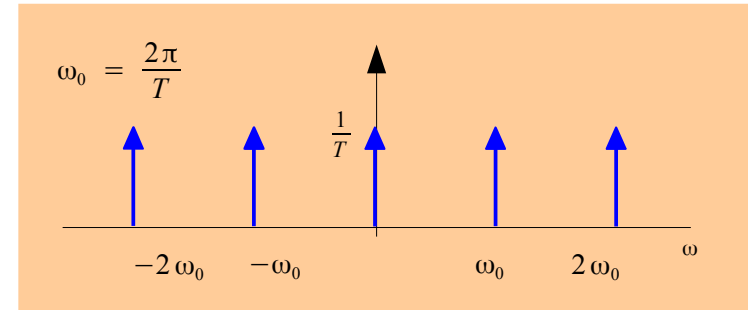
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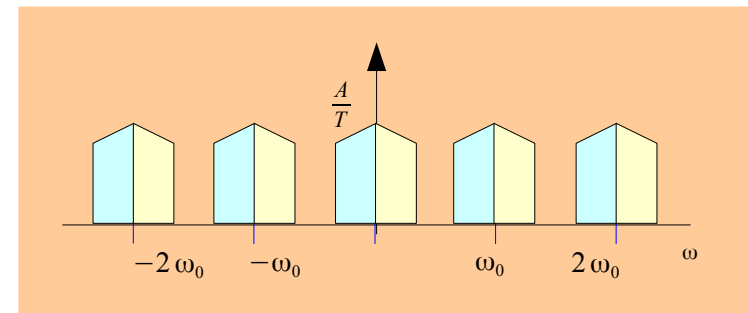
## Frequency Domain



\*



||



CTFT



CTFT

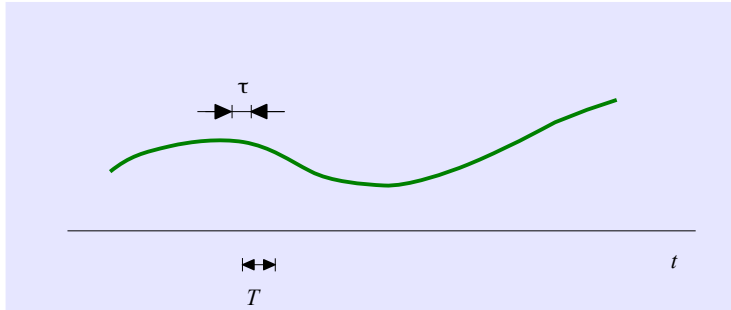


CTFT

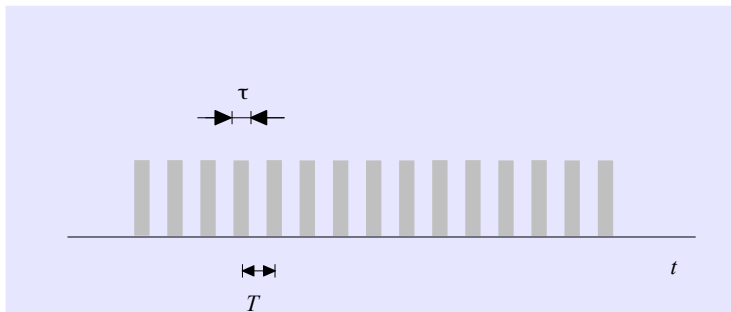


# Sampling (3)

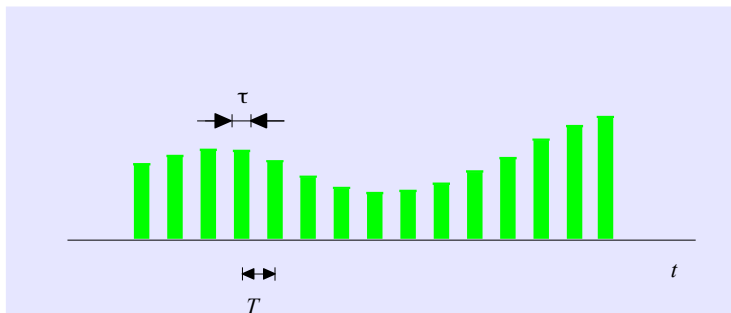
## Practical Sampling



X

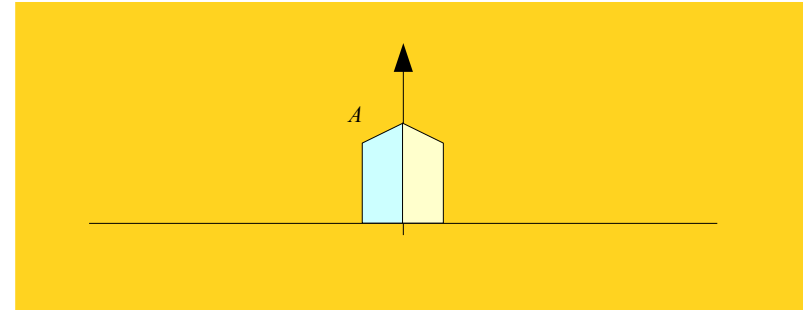


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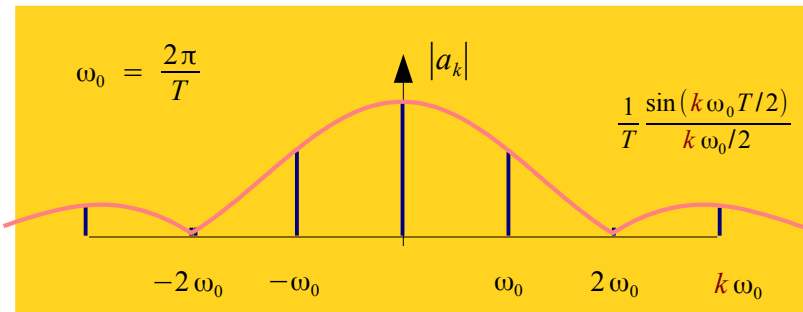
## Frequency Domain

CTFT



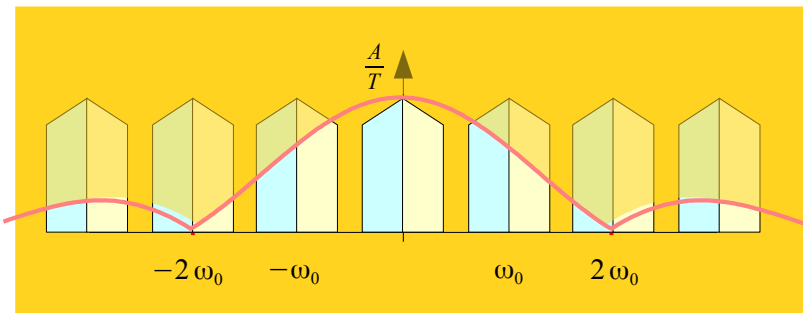
\*

CTFT

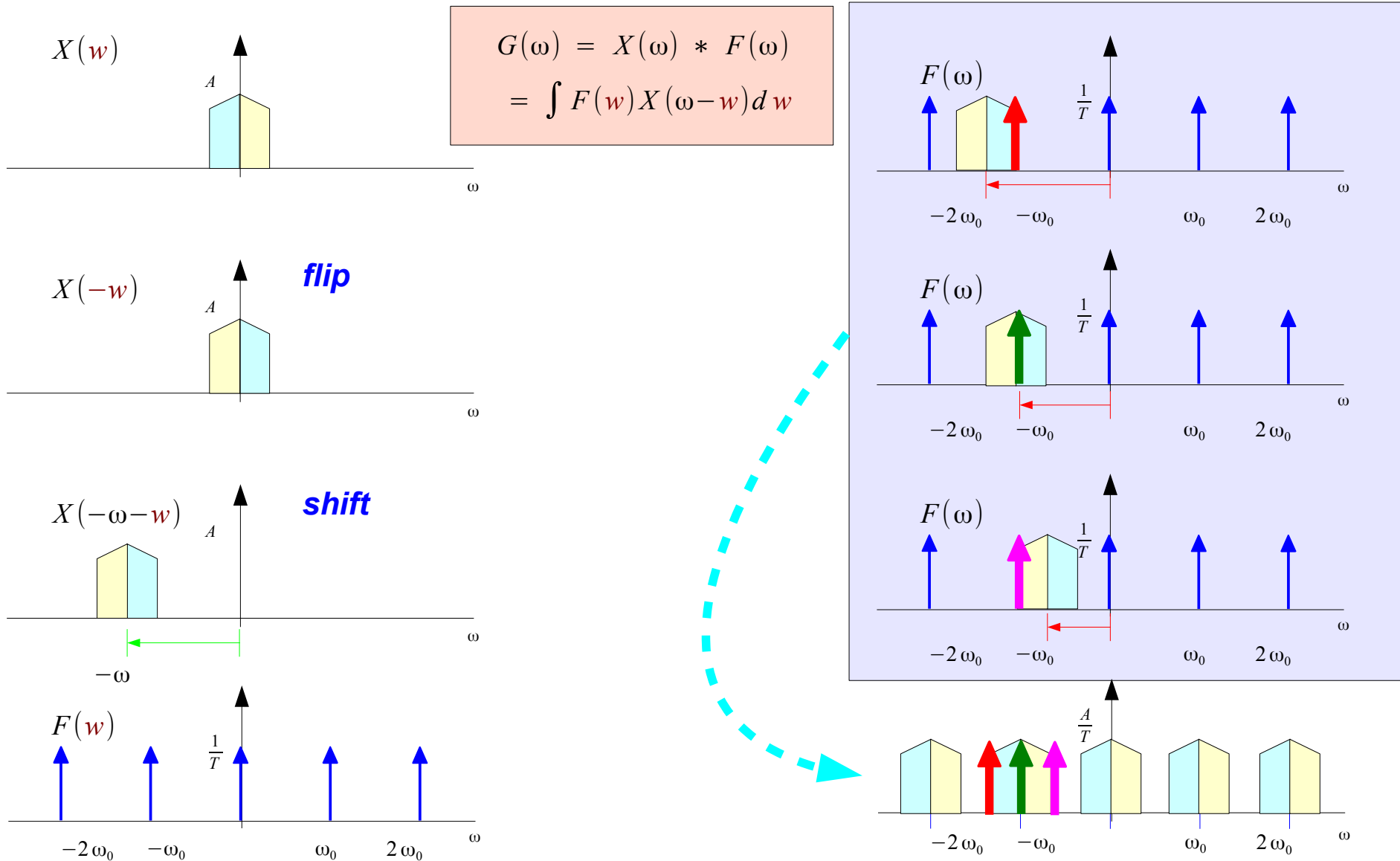


||

CTFT

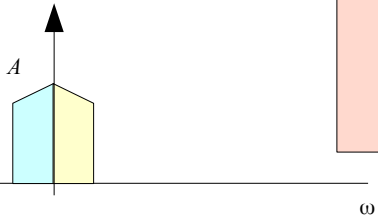


# Convolution with Impulse Train



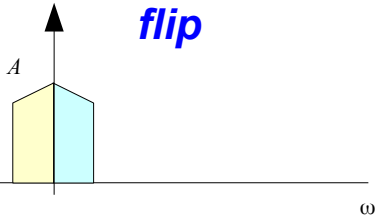
# Convolution with Sinc Impulse Train

$X(\omega)$

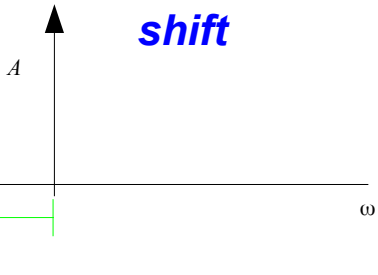


$$G(\omega) = X(\omega) * F(\omega) = \int F(\omega') X(\omega - \omega') d\omega'$$

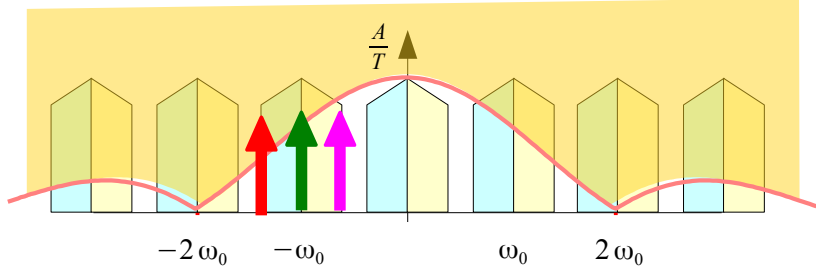
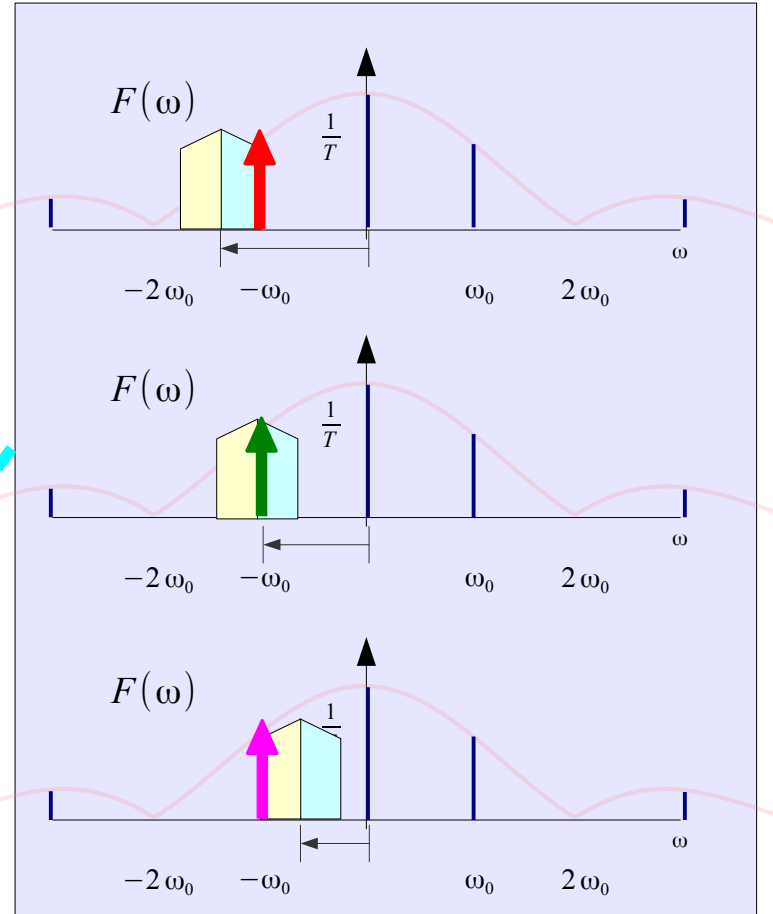
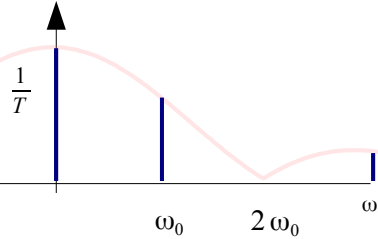
$X(-\omega)$



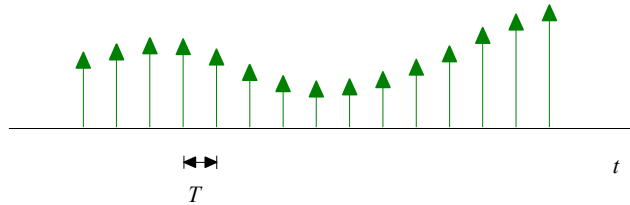
$X(-\omega - \omega')$



$F(\omega)$

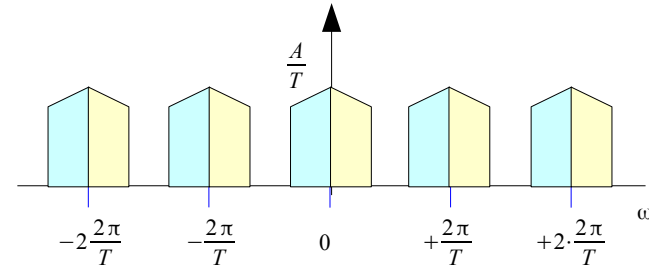


# CTFT of Sampled Signal



$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

CTFT



$$\begin{aligned} \hat{X}(f) &= \int_{-\infty}^{+\infty} \hat{x}(t) e^{-j2\pi f t} dt \\ &= \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT) e^{-j2\pi f t} dt \\ &= \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi f T n} \end{aligned}$$

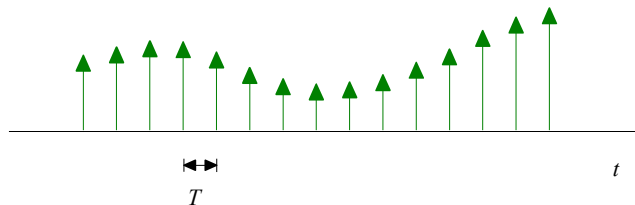
$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

CTFT



$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi f T n}$$

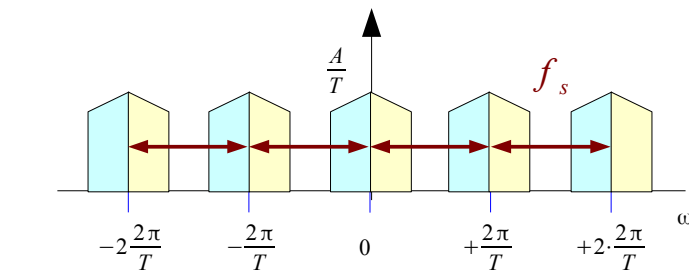
# Periodicity in Frequency



$$f_s = \frac{1}{T} \quad 2\pi f_s = \frac{2\pi}{T} = \omega_0$$

$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

CTFT



$$2\pi f = \omega$$

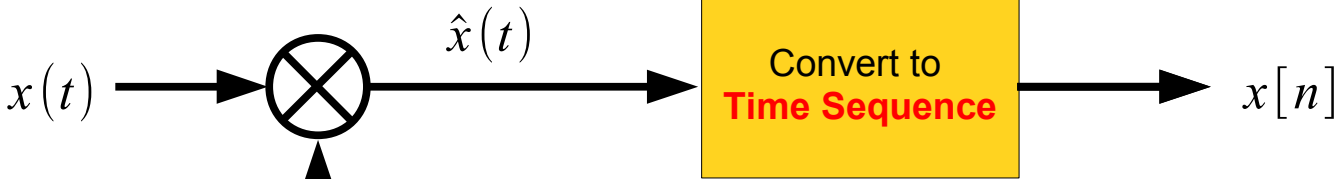
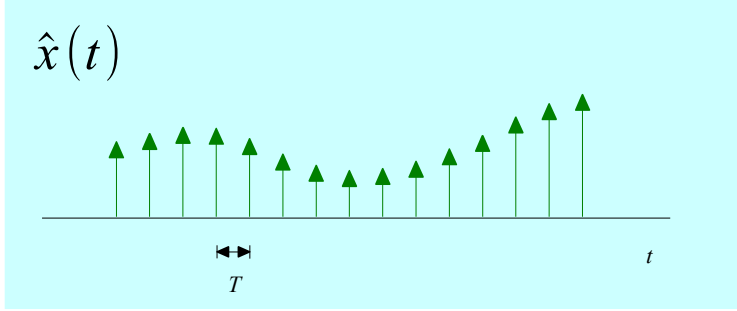
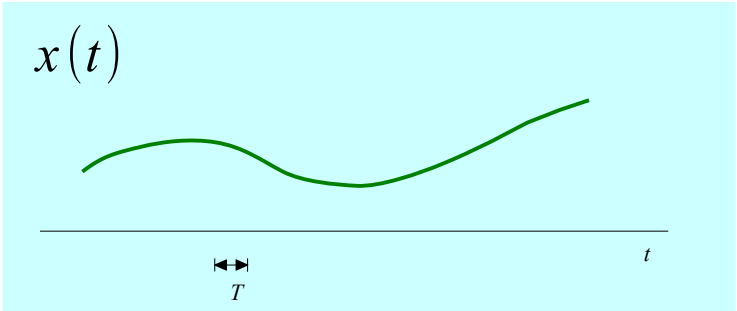
$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi f T n}$$

$$e^{-j2\pi(f+f_s)Tn} = e^{-j2\pi(f)Tn} \quad \leftarrow f_s T = 1$$

Period = Sampling Frequency  $f_s$

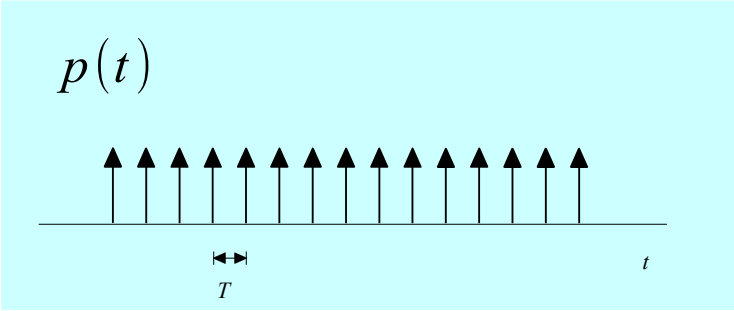
$$\hat{X}(f) = \hat{X}(f + f_s)$$

# Time Sequence

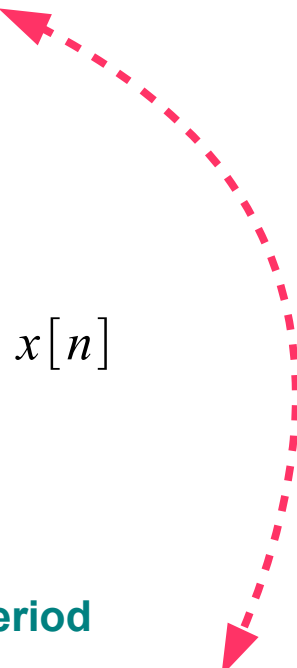
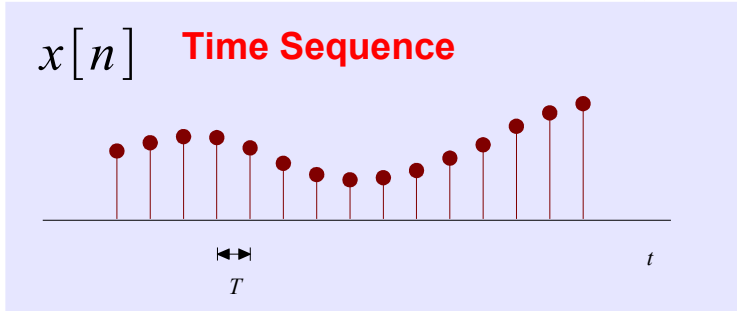


**Ideal Sampling**

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

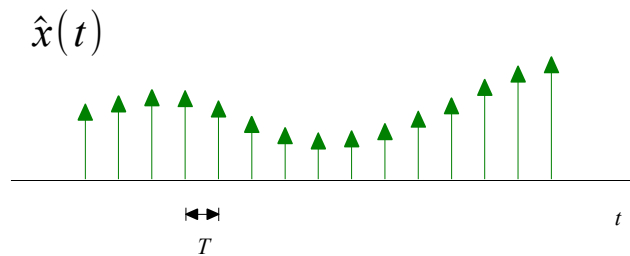


$T$  Sampling Period



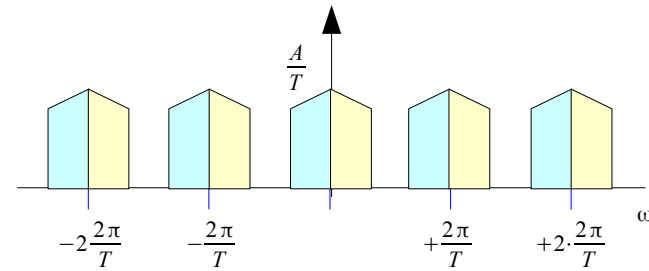


# DTFT of a Time Sequence

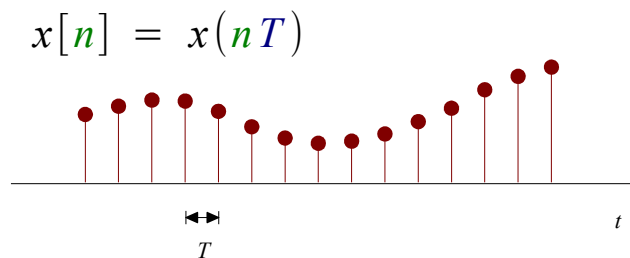


$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

CTFT

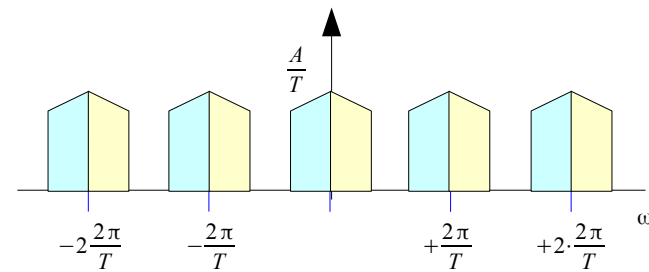


$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi fTn}$$



$$x[n], \text{ Sampling Period } T$$

DTFT

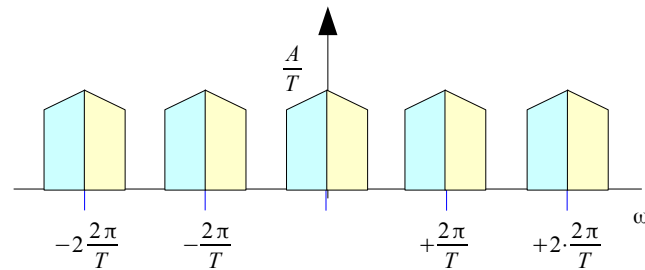
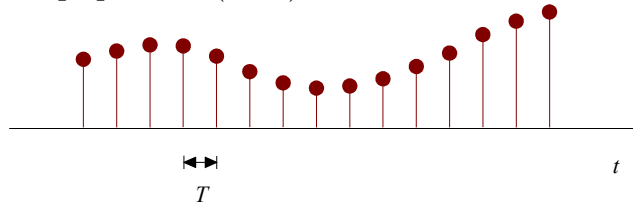


$$X(f) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi fTn}$$

Here,  $X(f)$  does not denote the CTFT of  $x(t)$

# Discrete Time Fourier Transform (1)

$$x[n] = x(nT)$$



$x[n]$  , Sampling Period  $T$

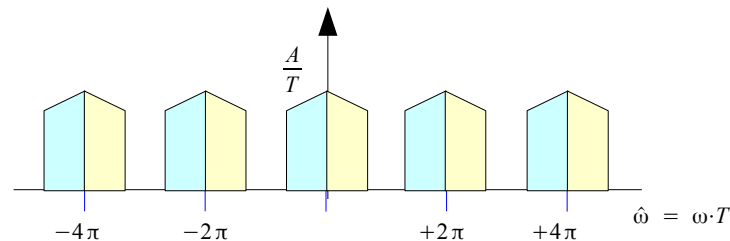
**DTFT**



$$X(f) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi f T n}$$

**Normalized Angular Frequency**

$$2\pi f T = \frac{2\pi f}{1/T} = 2\pi \frac{f}{f_s} = \hat{\omega}$$



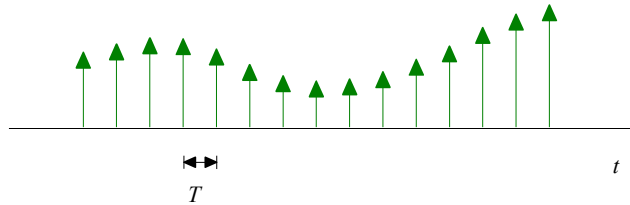
$x[n]$  , Sampling Period  $T$

**DTFT**



$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega} n}$$

# Discrete Time Fourier Transform (2)



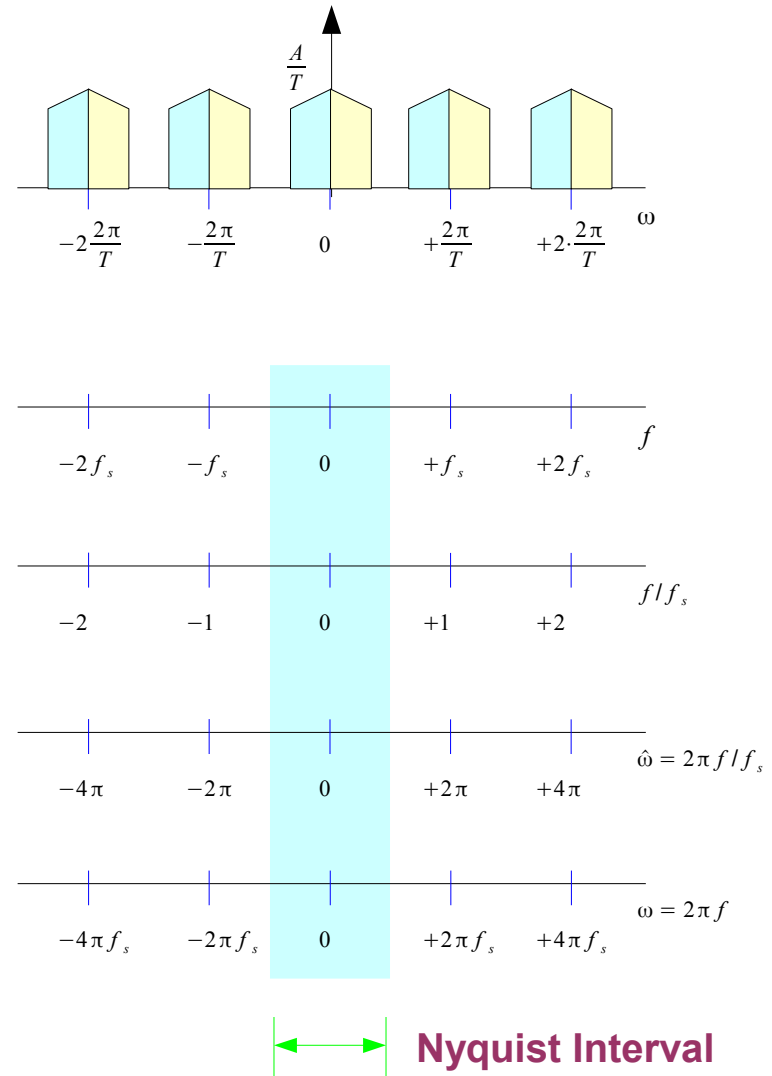
$$f_s = \frac{1}{T} \quad 2\pi f_s = \frac{2\pi}{T} = \omega_0$$

$$X(f) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi f T n}$$

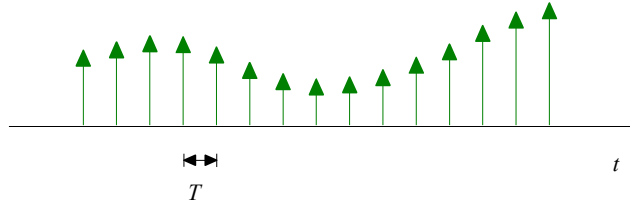
Normalized Angular Frequency

$$2\pi f T = \frac{2\pi f}{1/T} = 2\pi \frac{f}{f_s} = \hat{\omega}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega} n}$$



# Discrete Time Fourier Transform (3)



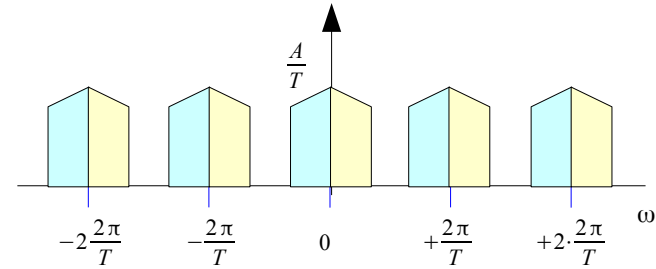
$$f_s = \frac{1}{T} \quad 2\pi f_s = \frac{2\pi}{T} = \omega_0$$

$$X(f) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi f T n}$$

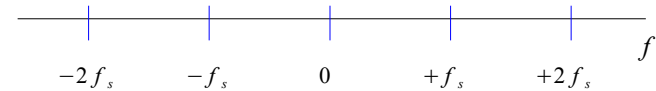
Normalized Angular Frequency

$$2\pi f T = \frac{2\pi f}{1/T} = 2\pi \frac{f}{f_s} = \hat{\omega}$$

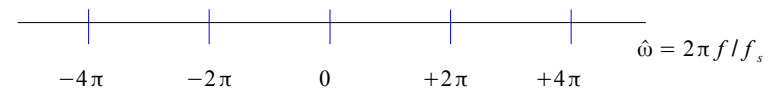
$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega} n}$$



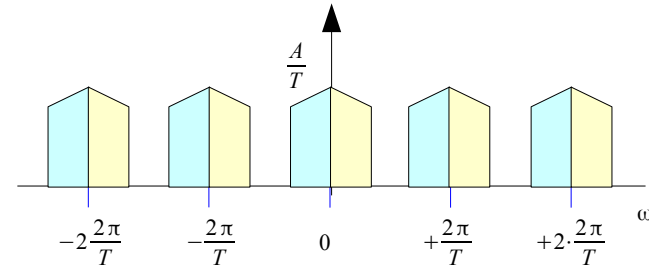
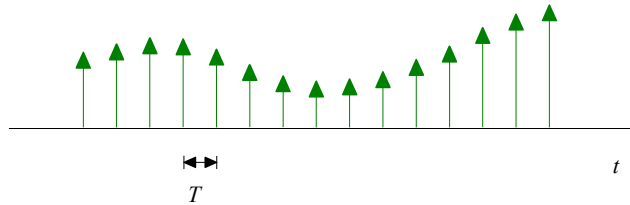
$\hat{X}(f)$  Absolute Frequency



$\hat{X}(e^{j\hat{\omega}})$  Normalized Angular Frequency  
unit circle →  
emphasize the periodic nature



# Fourier Series Interpretation



$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

**CTFT**



$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi fTn}$$

$$x(nT) = \frac{1}{f_s} \int_{-f_s/2}^{+f_s/2} \hat{X}(f) e^{+j2\pi fTn} df$$

$$= \int_{-\pi}^{+\pi} \hat{X}(\omega) e^{+j\omega n} \frac{d\omega}{2\pi}$$

**CTFS**



$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi fTn}$$

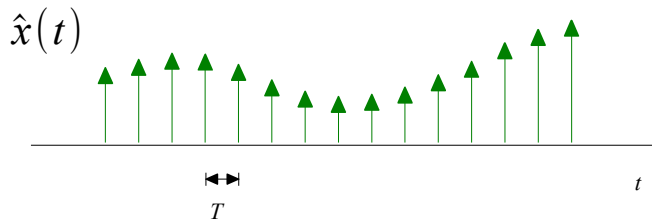
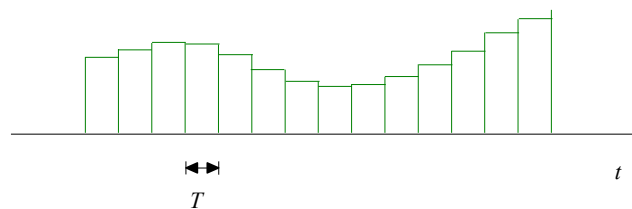
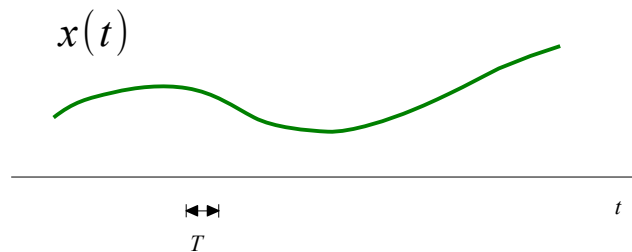
Fourier Series Coefficients  $x(nT)$   
 $\hat{X}(f)$  Continuous Periodic Function

$$\omega = 2\pi f / f_s \quad \frac{df}{f_s} = \frac{d\omega}{2\pi}$$

**View as a Fourier Series Expansion**

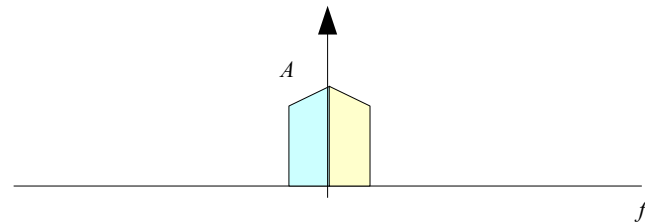
# Numerical Approximation

$$X(f) = \lim_{T \rightarrow 0} T \hat{X}(f)$$



$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

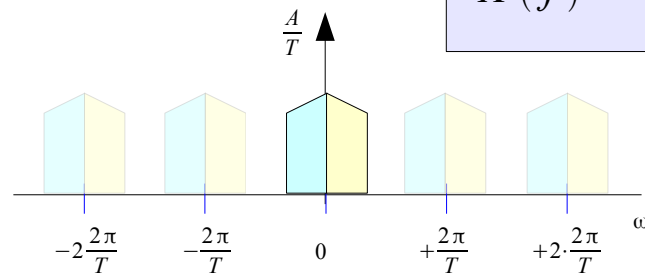
CTFT



$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{+j2\pi f t} dt$$

$$\approx \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi f T n} \cdot T$$

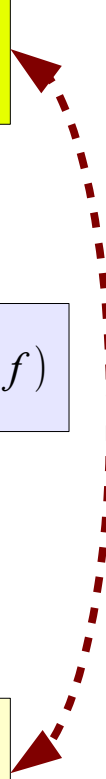
$$X(f) \approx T \hat{X}(f)$$



CTFT

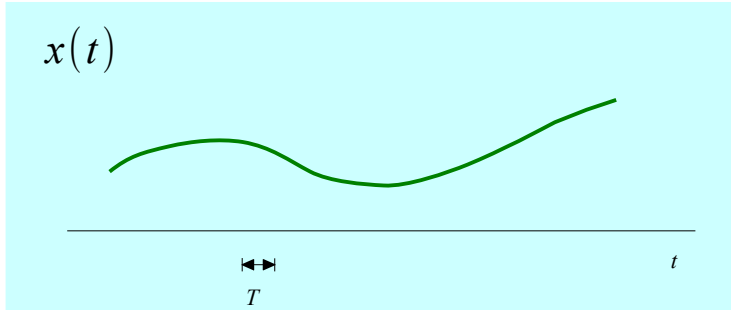


$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi f T n}$$

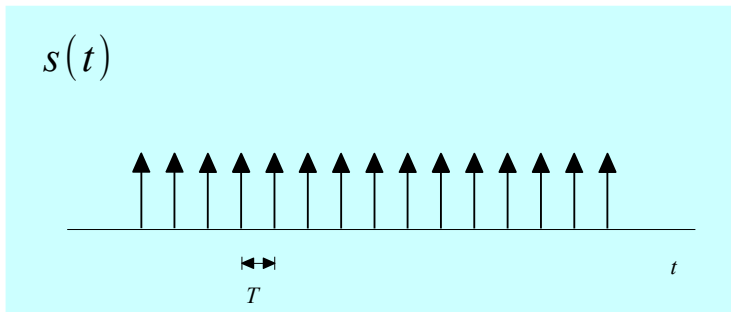


# Spectrum Replication (1)

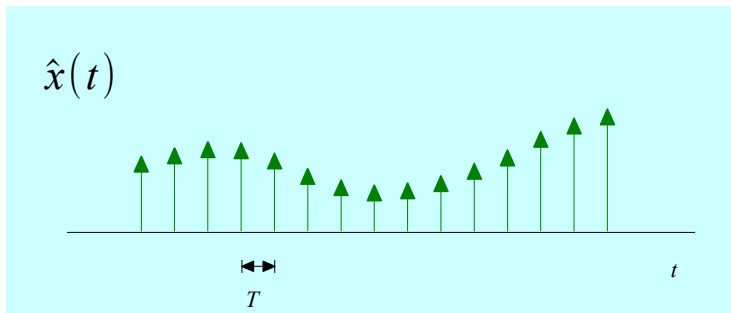
## Ideal Sampling



X



||



$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

$$s(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT)$$

$$= \frac{1}{T} \sum_{m=-\infty}^{+\infty} e^{+j2\pi m f_s t}$$

$$\hat{x}(t) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} x(t) e^{+j2\pi m f_s t}$$

Shift Property



$$\hat{X}(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X(f - m f_s)$$

# Spectrum Replication (2)

$$S(f) = \frac{1}{T} \sum_{m=-\infty}^{+\infty} \delta(f - m f_s)$$

## Convolution in Frequency

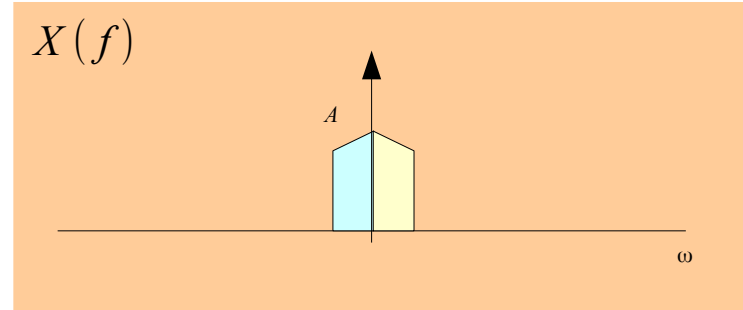
$$\hat{X}(f) = X(f) * S(f)$$

$$= \int_{-\infty}^{+\infty} X(f - f') S(f') d f'$$

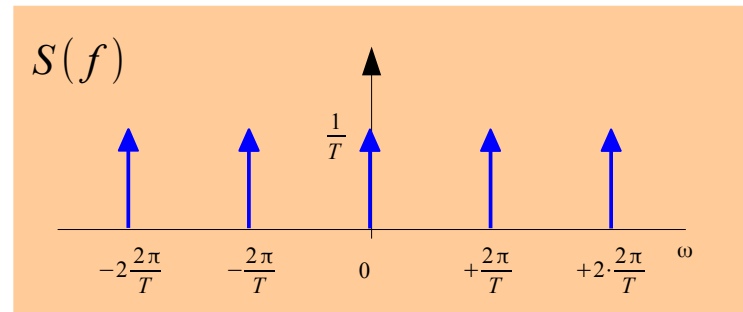
$$= \frac{1}{T} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(f - f') \delta(f' - m f_s) d f'$$

$$\hat{X}(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X(f - m f_s)$$

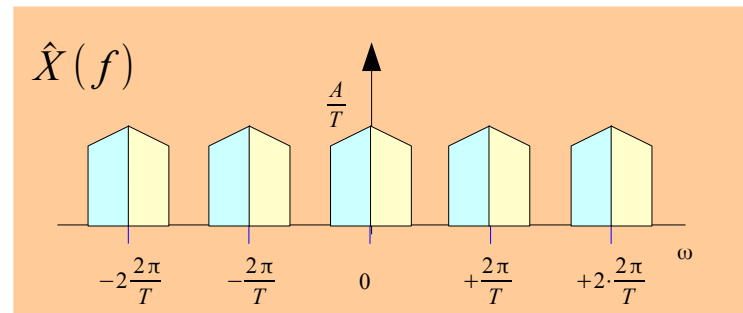
## Frequency Domain



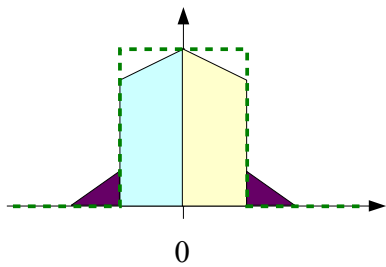
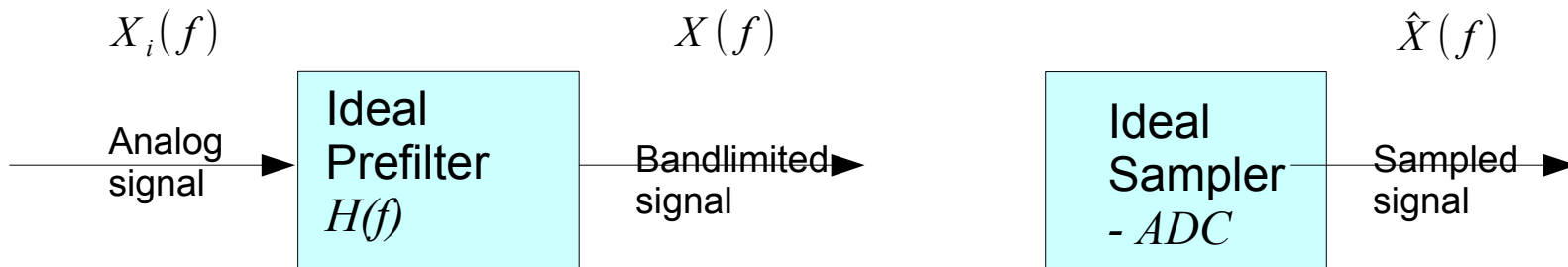
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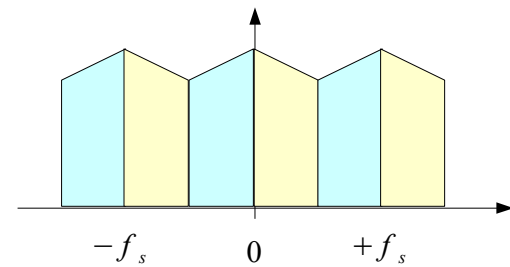
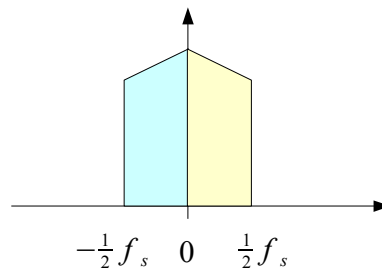
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$$\frac{2}{4}f_s \quad \frac{3}{4}f_s \quad f_s$$







## References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] A "graphical interpretation" of the DFT and FFT, by Steve Mann
- [4] R. G. Lyons, Understanding Digital Signal Processing, 1997
- [5] AVR121: Enhancing ADC resolution by oversampling
- [6] S.J. Orfanidis, Introduction to Signal Processing  
[www.ece.rutgers.edu/~orfanidi/intro2sp](http://www.ece.rutgers.edu/~orfanidi/intro2sp)