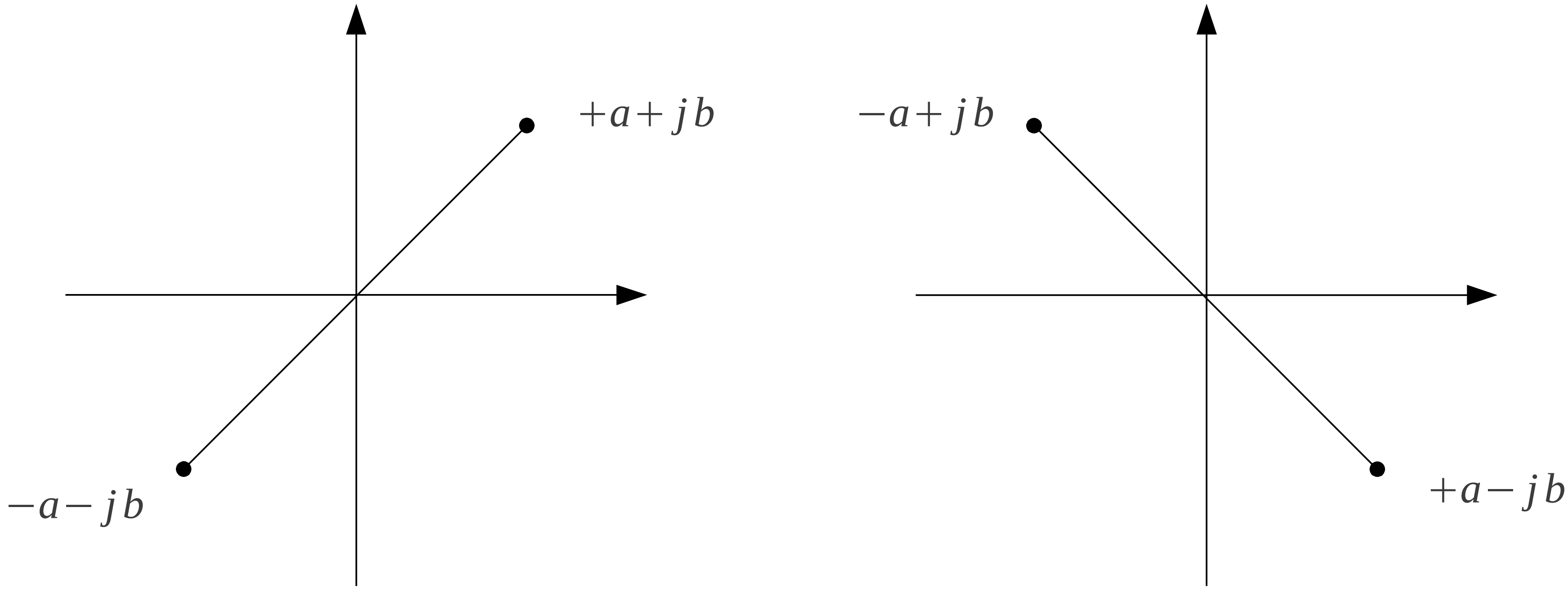
All Pass Filter (2A)

Young Won Lim
8/4/12

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Phase and $\tan^{-1}(b/a)$



$$\tan^{-1}\left(\frac{-b}{-a}\right) = \tan^{-1}\left(\frac{+b}{+a}\right) \qquad arg\{-a-jb\} = arg\{+a+jb\} + \pi$$

$$\tan^{-1}\left(\frac{+b}{-a}\right) = \tan^{-1}\left(\frac{-b}{+a}\right) \qquad arg\{-a+jb\} = arg\{+a-jb\} - \pi$$

Example - All Pass Filter (1)

$$H_{all}(s) = \frac{1-2s}{1+2s}$$

Flat Magnitude

$$\left|\frac{1-j2\omega}{1+j2\omega}\right| = \frac{|1-j2\omega|}{|1+j2\omega|}$$

$$= \frac{\sqrt{1+4\omega^2}}{\sqrt{1+4\omega^2}} = 1$$

$$|H_{all}(j\omega)| = \frac{\sqrt{1+4\omega^2}}{\sqrt{1+4\omega^2}} = 1$$

A Pure Phase Shifter

$$arg\left\{\frac{1-j2\omega}{1+j2\omega}\right\} = arg\{1-j2\omega\} - arg\{1+j2\omega\}$$

$$= \tan^{-1}\left(\frac{-2\omega}{1}\right) - \tan^{-1}\left(\frac{2\omega}{1}\right) = -2\tan^{-1}(2\omega)$$

$$arg\{H_{all}(j\omega)\} = -2\tan^{-1}(2\omega)$$

$$\frac{1-j2\omega}{1+j2\omega} = \frac{1-j2\omega}{1+j2\omega} \cdot \frac{1-j2\omega}{1-j2\omega}$$

$$= \frac{(1-4\omega^2) - j4\omega}{1+4\omega^2}$$

$$arg\{H_{all}(j\omega)\} = -\tan^{-1}\left(\frac{4\omega}{1-4\omega^2}\right)$$

Example - All Pass Filter (2)

$$H_{all}(s) = -\frac{s-0.5}{s+0.5} \quad \Leftarrow \quad H_{all}(s) = \frac{1-2s}{1+2s}$$

Flat Magnitude

$$\left|\frac{j\omega - 0.5}{j\omega + 0.5}\right| = \frac{|\, j\omega - 0.5 \,|}{|\, j\omega + 0.5 \,|}$$

$$= \frac{\sqrt{\omega^2 + 0.25}}{\sqrt{\omega^2 + 0.25}} = 1$$

$$|H_{all}(j\omega)| = \frac{\sqrt{1+4\omega^2}}{\sqrt{1+4\omega^2}} = 1$$

A Pure Phase Shifter

$$-arg\left\{\frac{j\omega - 0.5}{j\omega + 0.5}\right\} = -arg\{j\omega - 0.5\} + arg\{j\omega + 0.5\}$$

$$= -\tan^{-1}\left(\frac{\omega}{-0.5}\right) + \tan^{-1}\left(\frac{\omega}{0.5}\right) = -(-\tan^{-1}(2\omega) - \pi) + \tan^{-1}(2\omega)$$

$$arg\{H_{all}(j\omega)\} = \pi + 2\tan^{-1}(2\omega)$$

$$\frac{1-j2\omega}{1+j2\omega} = \frac{1-j2\omega}{1+j2\omega} \cdot \frac{1-j2\omega}{1-j2\omega}$$

$$= \frac{(1-4\omega^2) - j4\omega}{1+4\omega^2}$$

$$arg\{H_{all}(j\omega)\} = -\tan^{-1}\left(\frac{4\omega}{1-4\omega^2}\right)$$

Example - All Pass Filter (2)

$$H_{all}(s) = \frac{s - 0.5}{s + 0.5}$$

$$= \frac{s + 0.5 - 1}{s + 0.5}$$

$$H(s) = 1 - \frac{2}{(s + 0.5)}$$

Inverse Laplace Transform

$$h(t) = \delta(t) - e^{-0.5t}$$

$$H_{all}(j\omega) = \frac{j\omega - 0.5}{j\omega + 0.5}$$

Flat Magnitude

$$|H_{all}(j\omega)| = \frac{\sqrt{\omega^2 + 0.25}}{\sqrt{\omega^2 + 0.25}} = 1$$

Phase Shifter

$$arg\{H_{all}(j\omega)\} = -2\tan^{-1}\left(\frac{\omega}{0.5}\right)$$

Group Delay

$$-\frac{d}{d\omega}\left(arg\{H_{all}(j\omega)\}\right)$$

$$= -\frac{d}{d\omega}\left(-2\tan^{-1}\left(\frac{\omega}{0.5}\right)\right)$$

$$= \frac{4}{(1 + \omega^2/0.25)} > 0$$

Example - All Pass Filter (2)

$$H_{all}(s) = \frac{s - 0.5}{s + 0.5}$$

$$= \frac{s + 0.5 - 1}{s + 0.5}$$

$$H(s) = 1 - \frac{2}{(s + 0.5)}$$

Inverse Laplace Transform

$$h(t) = \delta(t) - e^{-0.5t}$$

$$H_{all}(j\omega) = \frac{j\omega - 0.5}{j\omega + 0.5}$$

Flat Magnitude

$$|H_{all}(j\omega)| = \frac{\sqrt{\omega^2 + 0.25}}{\sqrt{\omega^2 + 0.25}} = 1$$

Phase Shifter

$$arg\{H_{all}(j\omega)\} = -2\tan^{-1}\left(\frac{\omega}{0.5}\right)$$

Group Delay

$$-\frac{d}{d\omega}\left(arg\{H_{all}(j\omega)\}\right)$$

$$= -\frac{d}{d\omega}\left(-2\tan^{-1}\left(\frac{\omega}{0.5}\right)\right)$$

$$= \frac{4}{(1 + \omega^2/0.25)} > 0$$

All Pass Filter

$$G_{all}(s) = \pm \frac{(s-\bar{s}_1)(s-\bar{s}_2)\ \cdots\ (s-\bar{s}_n)}{(s-s_1)(s-s_2)\ \cdots\ (s-s_n)}$$

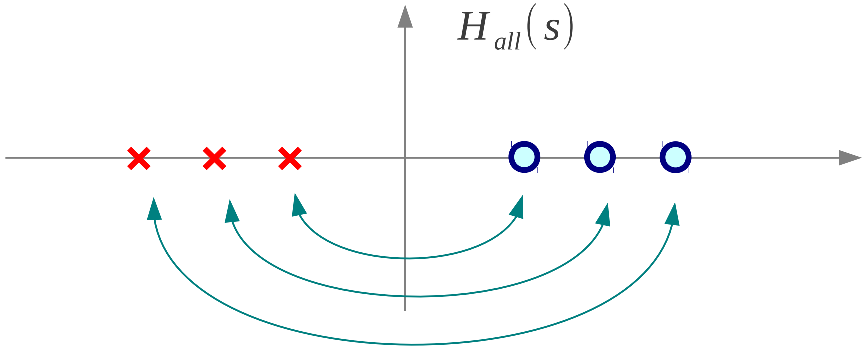
Flat Magnitude

A Pure Phase Shifter

zero $\bar{s}_i$

pole s_i

not complex conjugate



$s_i = +a + jb$

$\bar{s}_i = -a + jb$

only differ in the signs
of their real parts

All Pass Filter

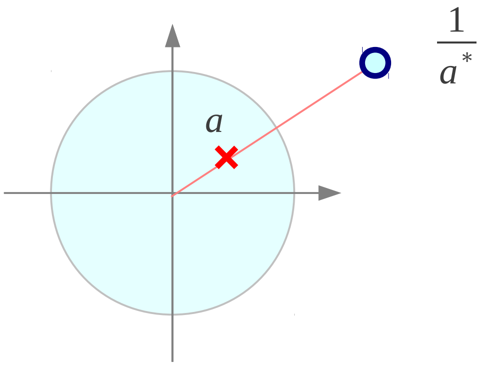
$$H(z) = \frac{z^{-1} - a^*}{1 - a z^{-1}}$$

$z^{-1} = a^* \rightarrow z = \frac{1}{a^*}$

$a z^{-1} = 1 \rightarrow z = a$

Flat Magnitude

A Pure Phase Shifter



$$H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - a e^{-j\omega}} = e^{-j\omega} \frac{1 - a^* e^{+j\omega}}{1 - a e^{-j\omega}}$$

$$(1 - a^* e^{+j\omega})^* = (1 - a e^{-j\omega})$$

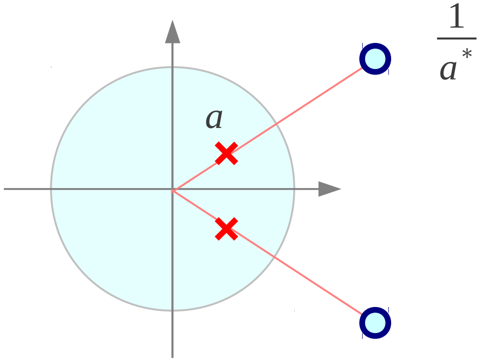
$$|H(e^{j\omega})| = |e^{-j\omega}| \left| \frac{1 - a^* e^{+j\omega}}{1 - a e^{-j\omega}} \right| = 1$$

All Pass Filter

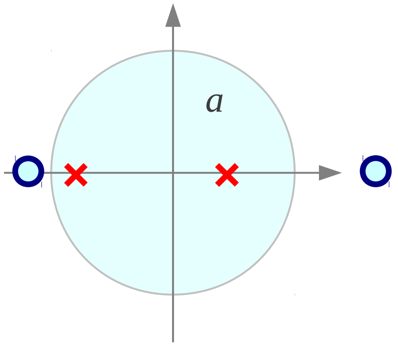
Flat Magnitude

A Pure Phase Shifter

$$H_{all}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

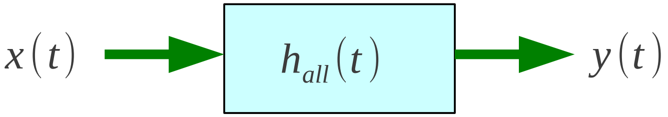


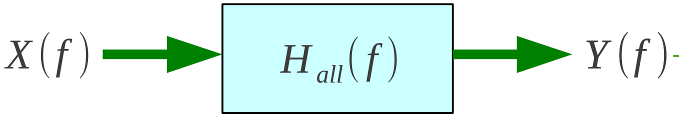
Cascade form of all pass system for real-valued impulse response system



Conjugate symmetric $H(e^{j\omega})$

All Pass Filter (4)





Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |y(t)|^2 \, dt$$

Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |X(f)|^2 \, dt$$

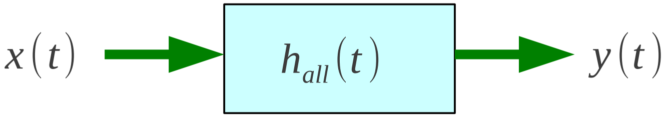
$$\int_{-\infty}^{+\infty} |y(t)|^2 \, dt = \int_{-\infty}^{+\infty} |Y(f)|^2 \, dt$$

$$Y(f) = H_{all}(f) X(f) \quad \Rightarrow \quad = \int_{-\infty}^{+\infty} |H_{all}(f)|^2 |X(f)|^2 \, dt$$

$$|H_{all}(f)| = 1 \quad \Rightarrow \quad = \int_{-\infty}^{+\infty} |X(f)|^2 \, dt$$

Energy Compaction

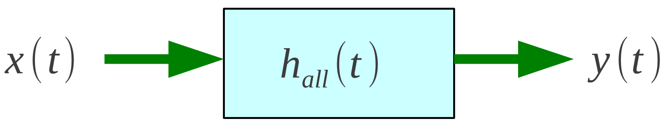
$$\int_{-\infty}^{t_0} |x(t)|^2 \, dt \geq \int_{-\infty}^{t_0} |y(t)|^2 \, dt$$



The energy build-up in the input is more **rapid** than in the output

Allpass Filter $\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |y(t)|^2 \, dt$

All Pass Filter (5)

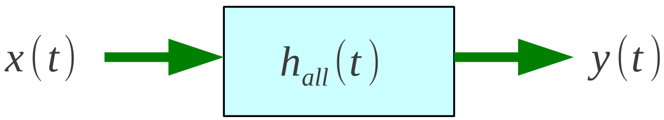


Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |y(t)|^2\, dt$$

Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2\, dt \geq \int_{-\infty}^{t_0} |y(t)|^2\, dt$$



The energy build-up in the input is more **rapid** than in the output

Truncated input

$$x_1(t) = \begin{cases} x(t) & (t \leq t_0) \\ 0 & (t > t_0) \end{cases}$$

For $t \leq t_0$ ➡ $x_1(t) = x(t)$ ➡

$$y_1(t) = \int_{-\infty}^{t_0} h(t-\tau)x_1(\tau)\,d\tau = \int_{-\infty}^{t} h(t-\tau)x(\tau)\,d\tau = y(t)$$

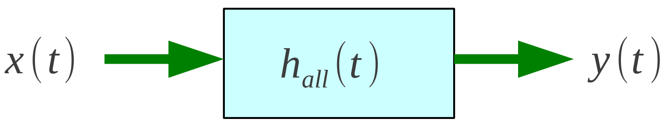
$$\int_{-\infty}^{+\infty} |x_1(t)|^2\, dt = \int_{-\infty}^{+\infty} |y_1(t)|^2\, dt$$

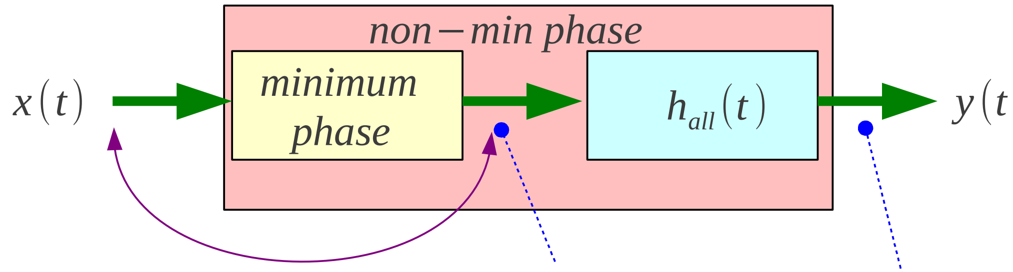
For $t > t_0$ ➡ $x_1(t) = 0$ ➡

$$\int_{-\infty}^{t_0} |x_1(t)|^2\, dt = \int_{-\infty}^{+\infty} |y_1(t)|^2\, dt = \int_{-\infty}^{t_0} |y_1(t)|^2\, dt + \int_{t_0}^{+\infty} |y_1(t)|^2\, dt$$

$$\int_{-\infty}^{t_0} |x(t)|^2\, dt \geq \int_{-\infty}^{t_0} |y(t)|^2\, dt \qquad \textit{For} \quad t \leq t_0$$

All Pass Filter (6)





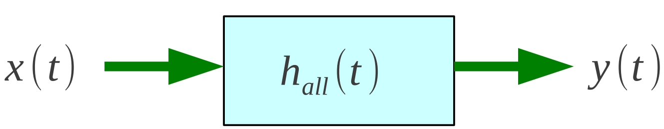
rapid energy
build-up than the output

Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |y(t)|^2 \, dt$$

Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2 \, dt \geq \int_{-\infty}^{t_0} |y(t)|^2 \, dt$$



The energy build-up in the input
is more **rapid** than in the output

The signal energy until t_0 of the minimum phase
$\geq$ any other causal signal
with the same magnitude response

Thus minimum phase signals
➡ maximally concentrated toward time 0
when compared against all causal signals
having the same magnitude response

minimum phase signals
➡ minimum delay signals

Properties of a Minimum Phase System

Properties of a Minimum Phase System

Properties of a Minimum Phase System

Properties of a Minimum Phase System

Properties of a Minimum Phase System

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] http://www.libinst.com/tpfd.htm
[4] K.Shin, J.K. Hammond, “Fundamentals of Signal Processing for Sound and Vibration Engineers”
[5] www.radiolab.com.au/DesignFile/DN004.pdf