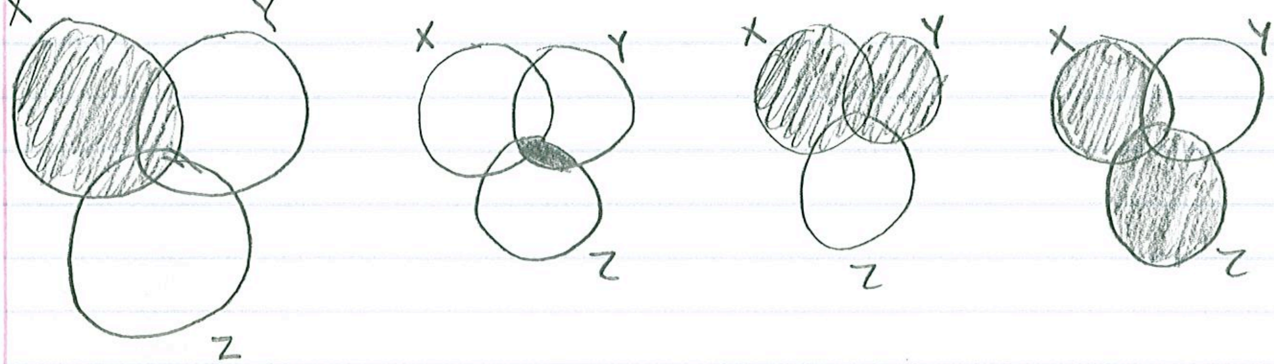
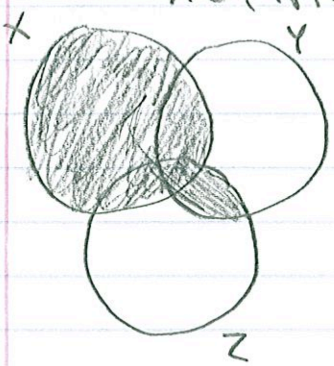
Proofs involving Sequences

When given a specific function, such as $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$, we can prove that this is true through the use of pictures. For example:

X $\quad$ $Y \cap Z$ $\quad$ $X \cup Y$ $\quad$ $X \cup Z$



$X \cup (Y \cap Z)$ $\quad$ $(X \cup Y) \cap (X \cup Z)$



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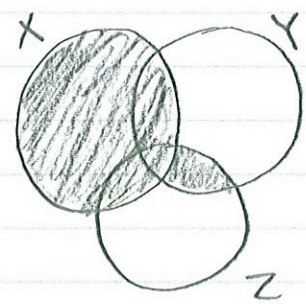


We can use this same technique in order to prove sequences. First, lets look at the sequence:

$$\frac{1}{3} = \frac{1}{3}$$

$$\frac{1+3}{5+7} = \frac{1}{3}$$

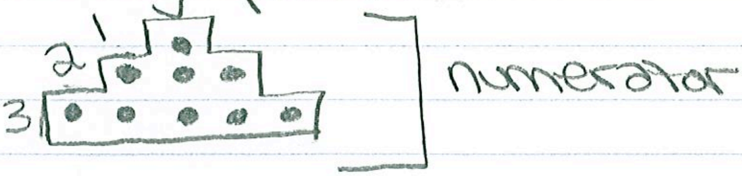
$$\frac{1+3+5}{7+9+11} = \frac{1}{3}$$

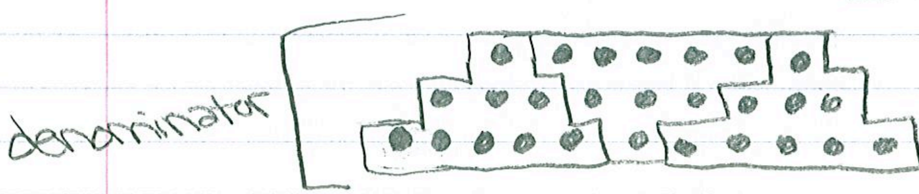
$$\frac{1+3+5+7}{9+11+13+15} = \frac{1}{3}$$

This sequence can be represented by

$$\frac{1+3+\ldots+(2n-1)}{(2n+1)+(2n+3)+\ldots+(4n-1)} = \frac{1}{3}$$

One of the first people to study this was Galileo and he proved this phenomenon by using the following picture:





This picture proves the sequence because each row within the pyramid (1, 2, 3) is added in the top picture, the numerator, and the bottom picture, the denominator. By adding up each set of numbers and then dividing the numerator by the denominator, we get 1/3 as our answer. By just looking at the picture, the top pyramid is clearly 1/3 of the bottom picture because we can see the bottom picture is 3 pyramids added together. Although this picture only works for the first couple of terms, you can add black dots as necessary. So, from the above picture we can see that:

$$\frac{1+3+\ldots+(2n-1)}{(2n+1)+(2n+3)+\ldots+(4n-1)} = \frac{1}{3}$$