Downsampling (4B)

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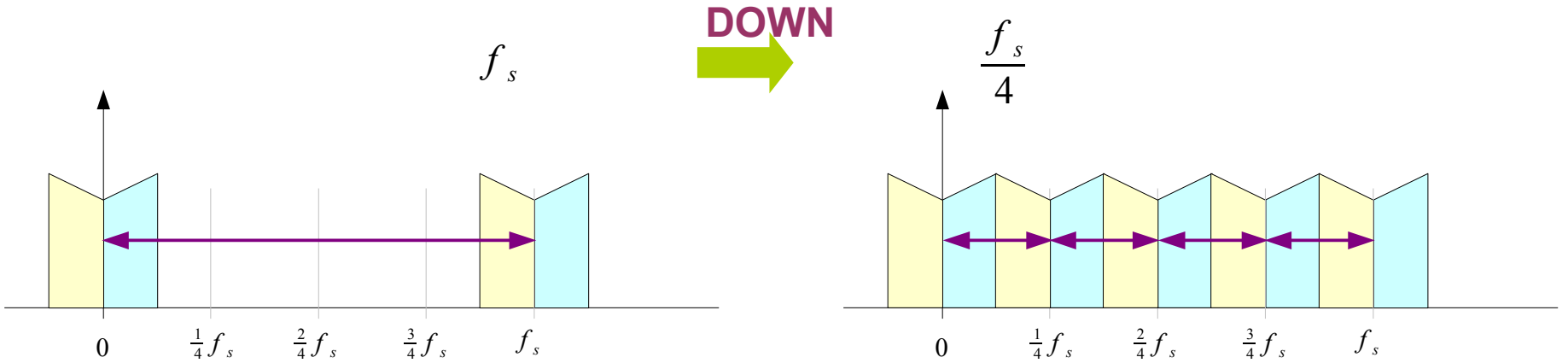
Band-limited Signal

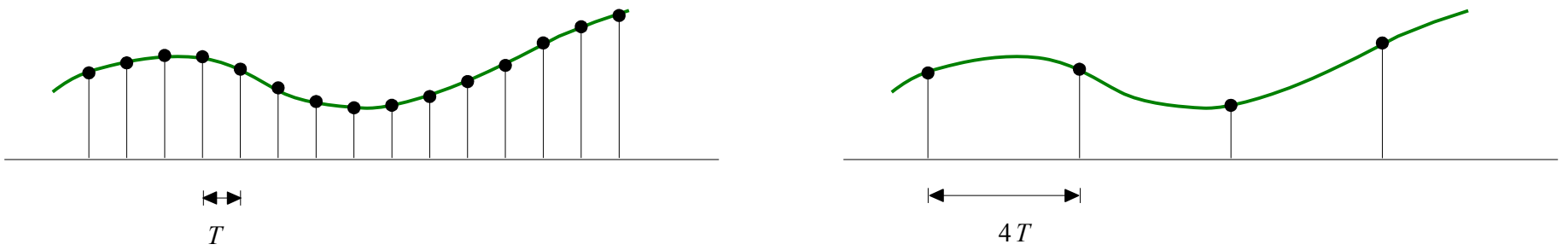


Sampling Frequency f_s

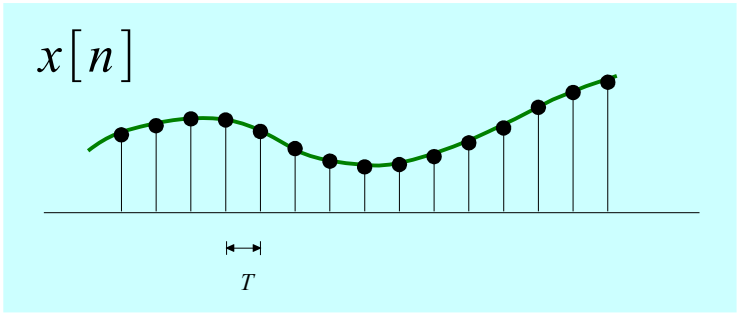
Sampling Time $T = \frac{1}{f_s}$

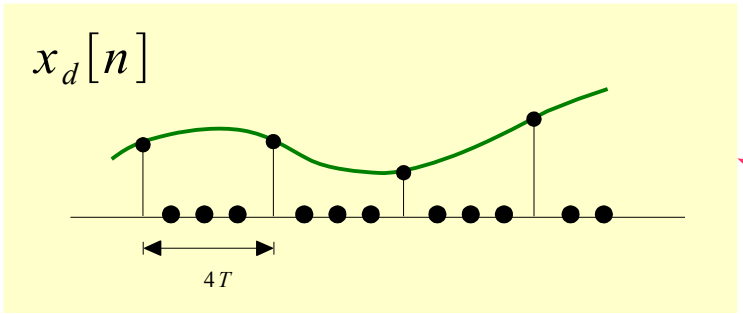
Sampling Frequency $f'_s = \frac{1}{4} f_s$

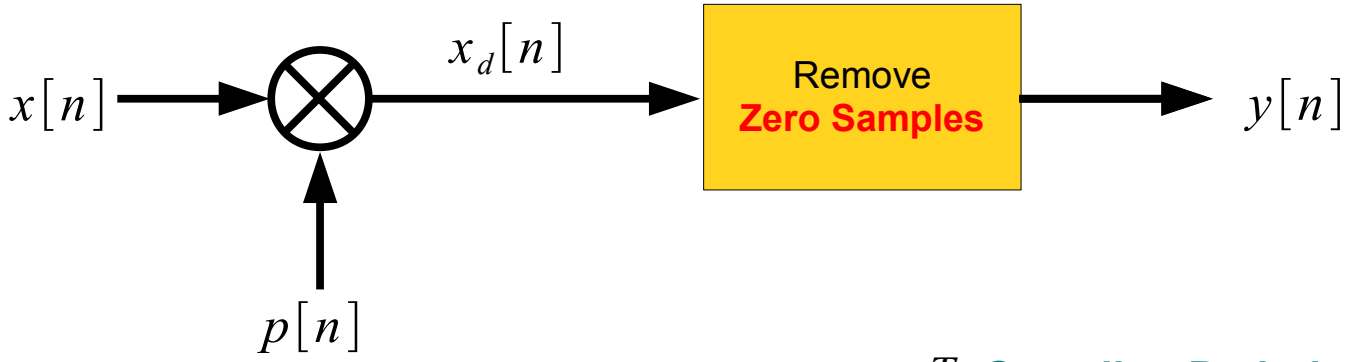
Sampling Time $T' = \frac{4}{f_s}$



Time Sequence

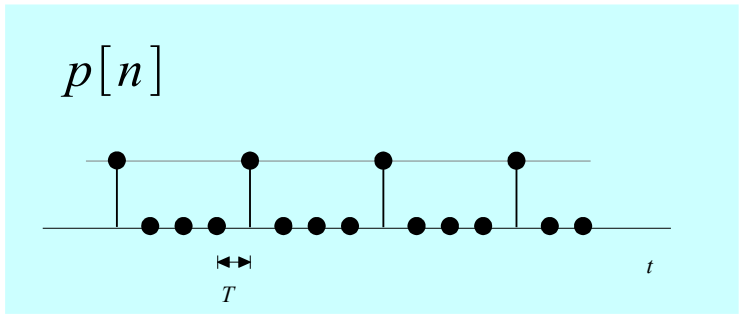


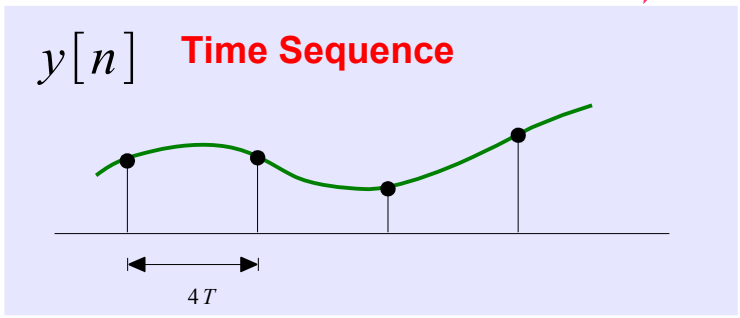




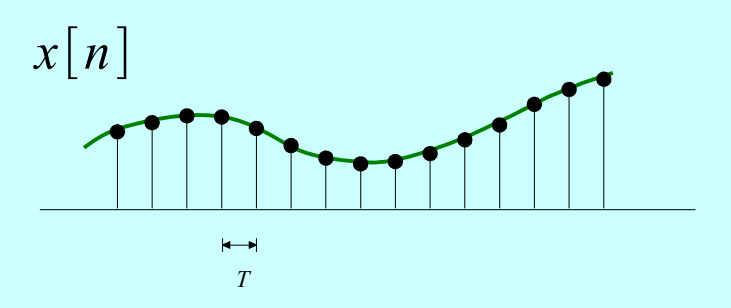
Ideal Sampling

T **Sampling Period**





Normalized Radian Frequency

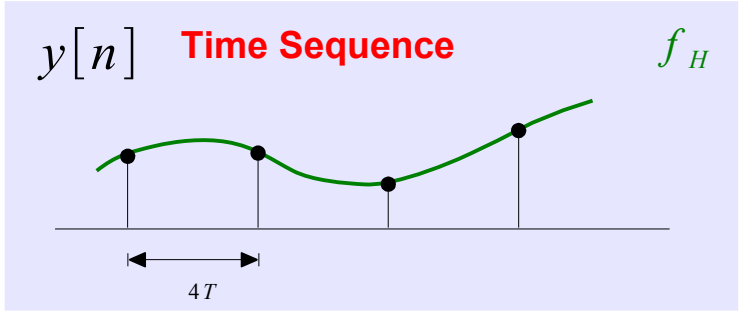


$$\hat{\omega} = \omega \cdot T_s = \frac{\omega}{1/T_s}$$

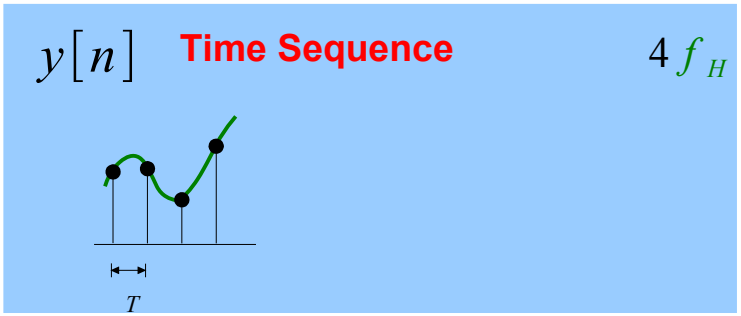
$$\hat{\omega} = \frac{\omega}{f_s} = 2\pi \frac{f}{f_s}$$

Normalized to f_s

Normalized Radian Frequency



The Same Time Sequence

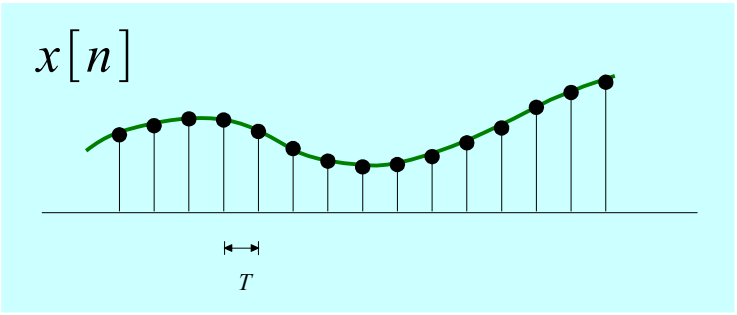


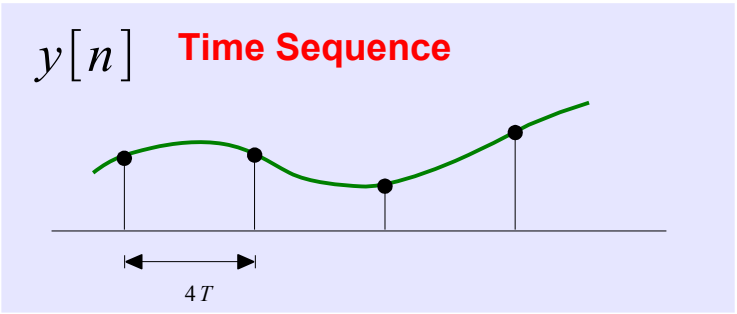
The Same Normalized Radian Frequency

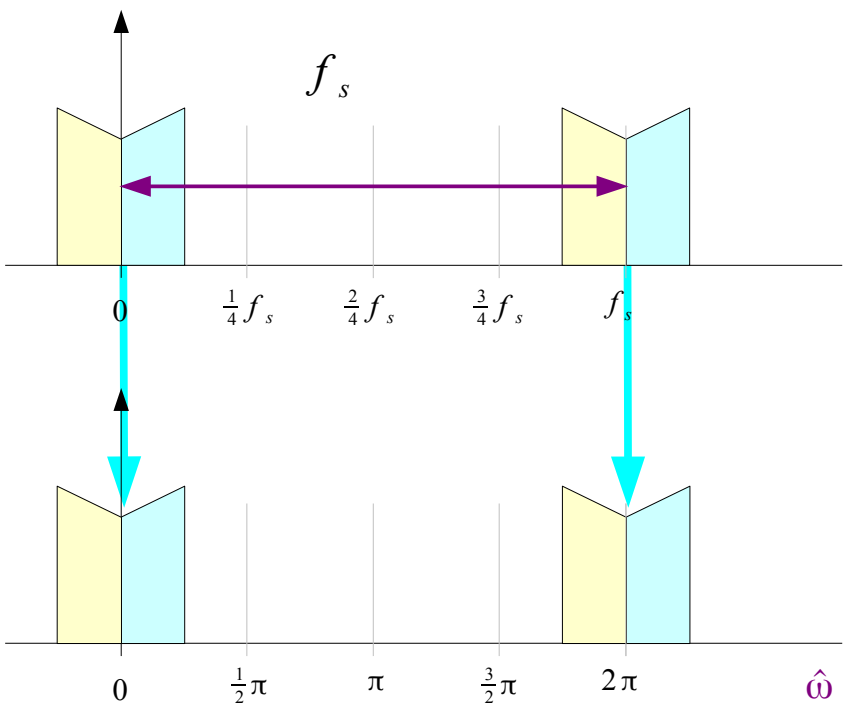
The Highest Frequency: f_H, $4 f_H$

$$\frac{f_H}{1/4T} = f_H \cdot 4T \qquad \frac{4 f_H}{1/T} = f_H \cdot 4T$$

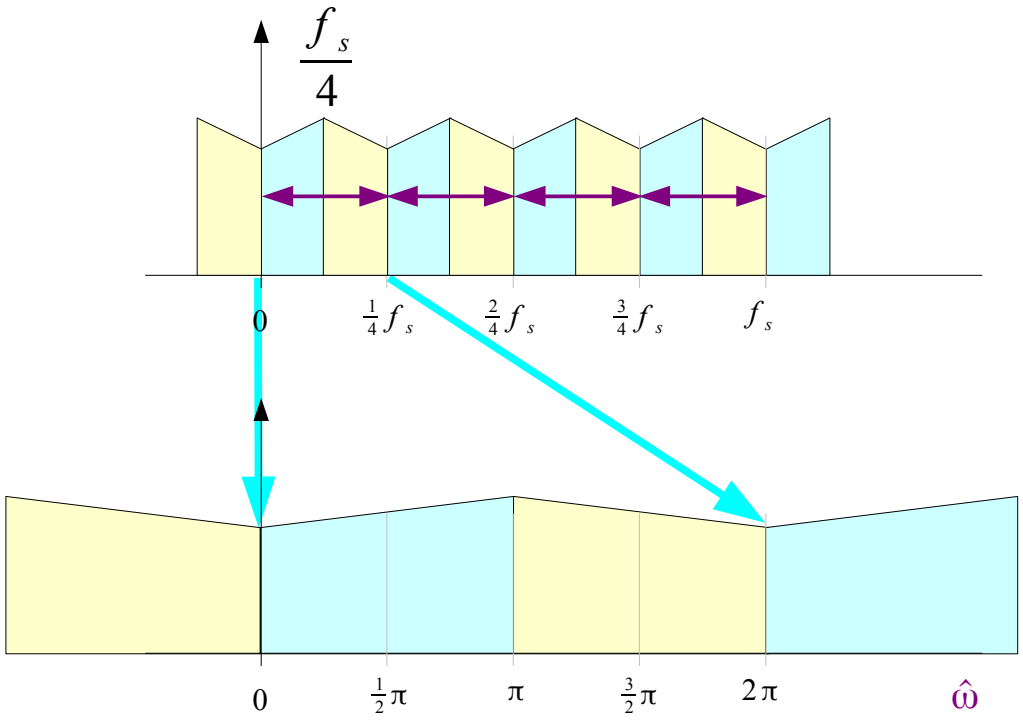
Time Sequence





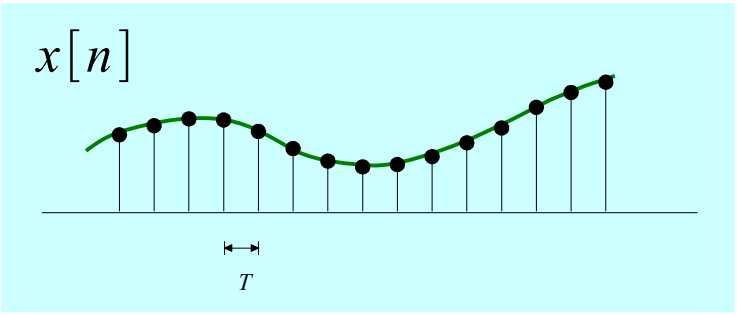


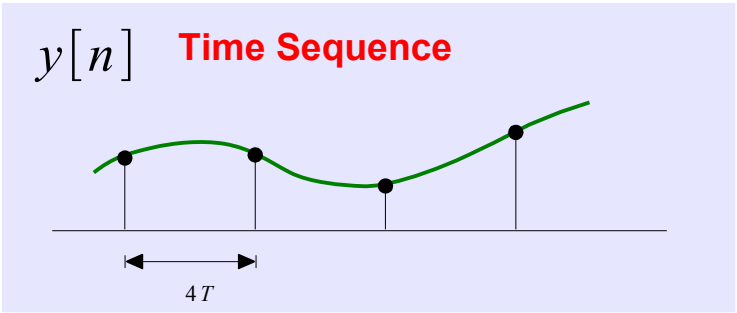
Normalized Radian Frequency



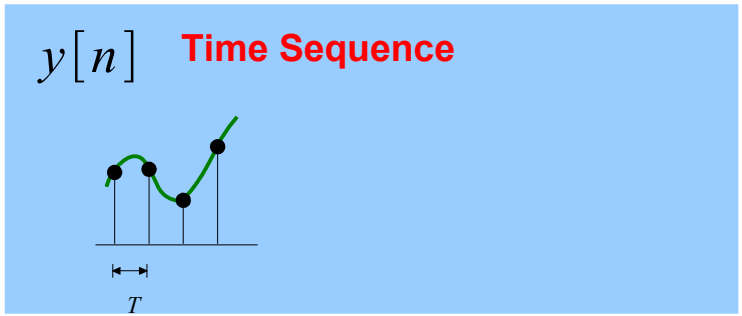
Normalized Radian Frequency

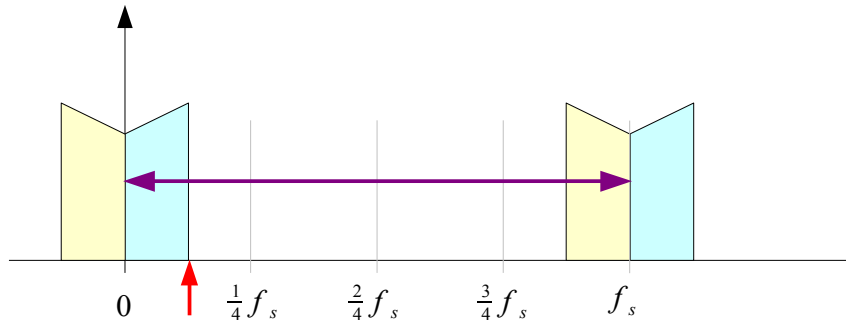
Time Sequence Spectrum in Linear Frequency

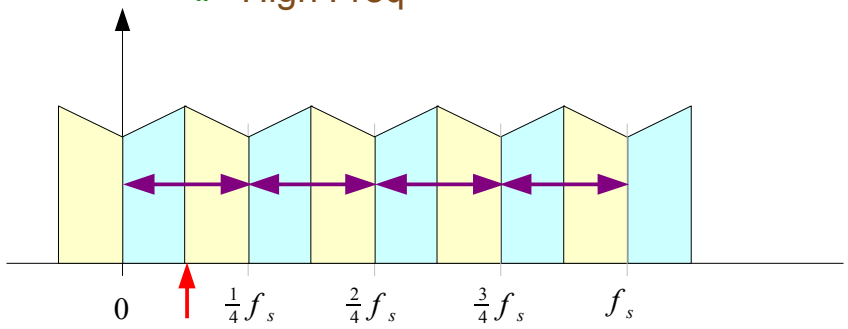


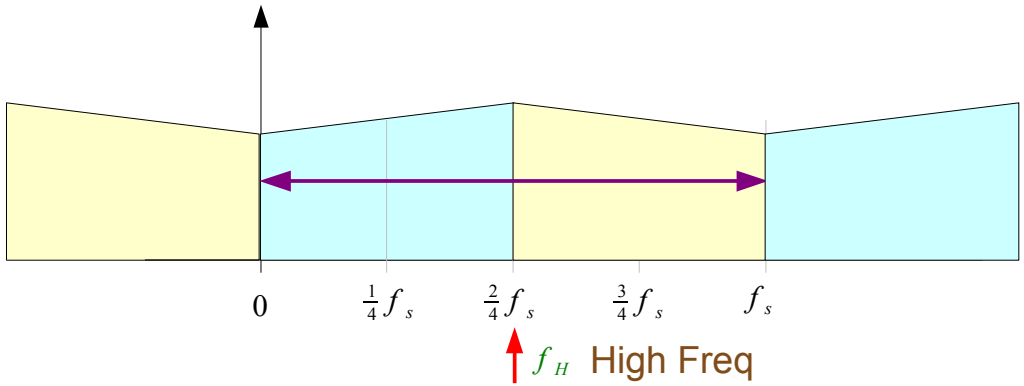


= The Same Time Sequence

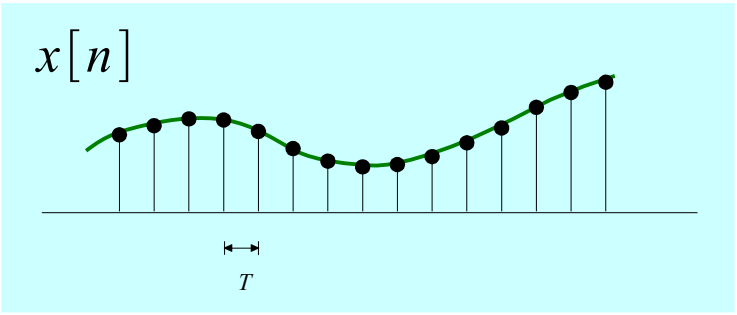


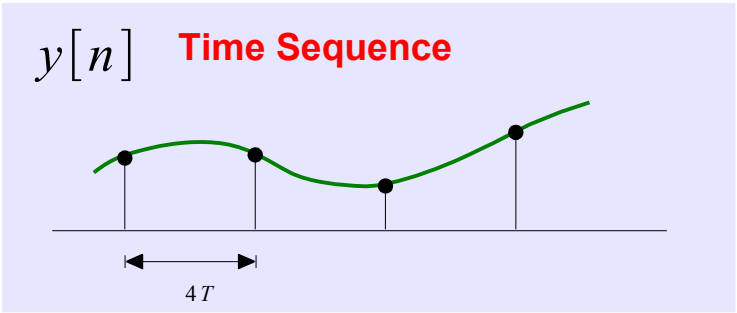




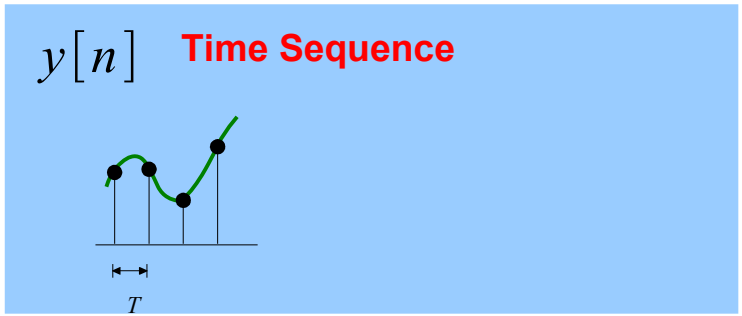


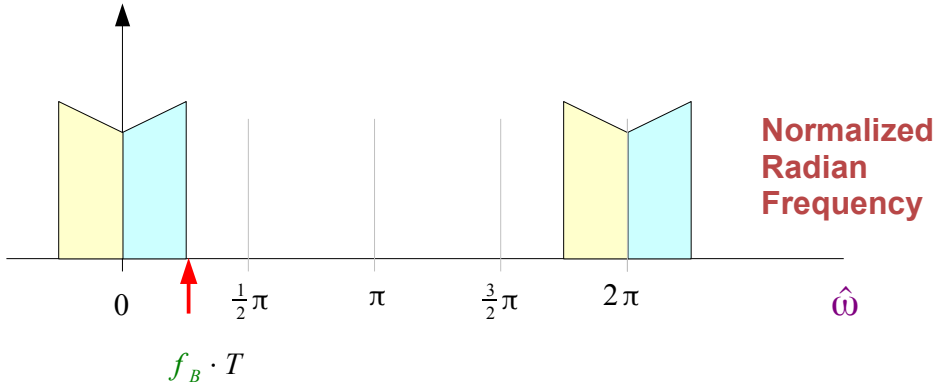
Time Sequence Spectrum in Normalized Frequency

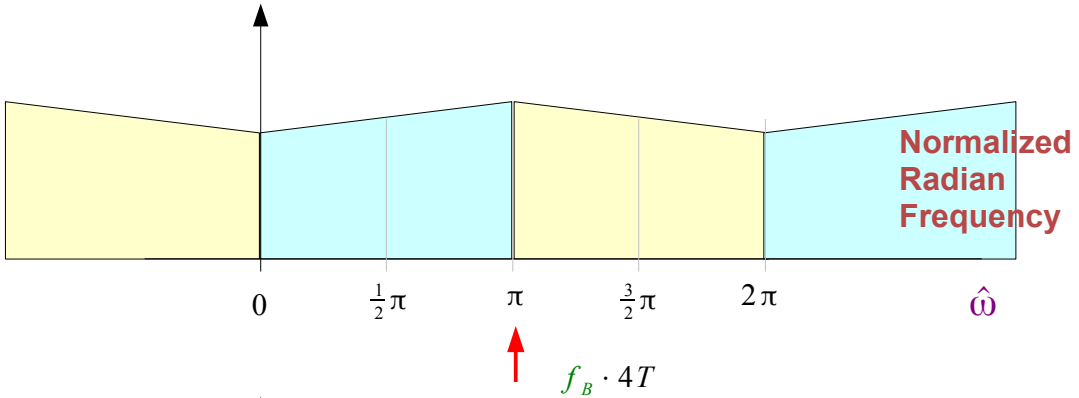


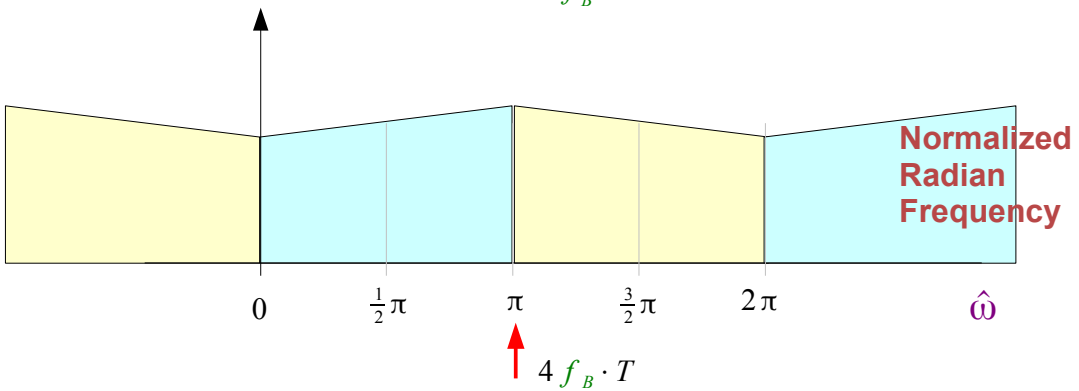


II The Same Time Sequence

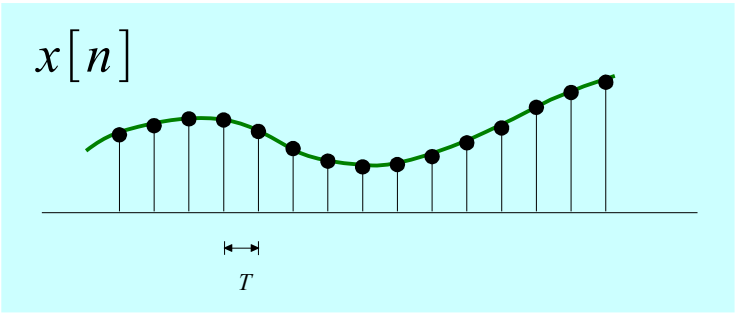


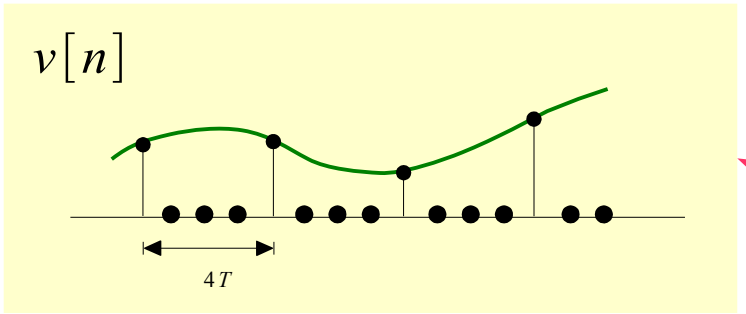




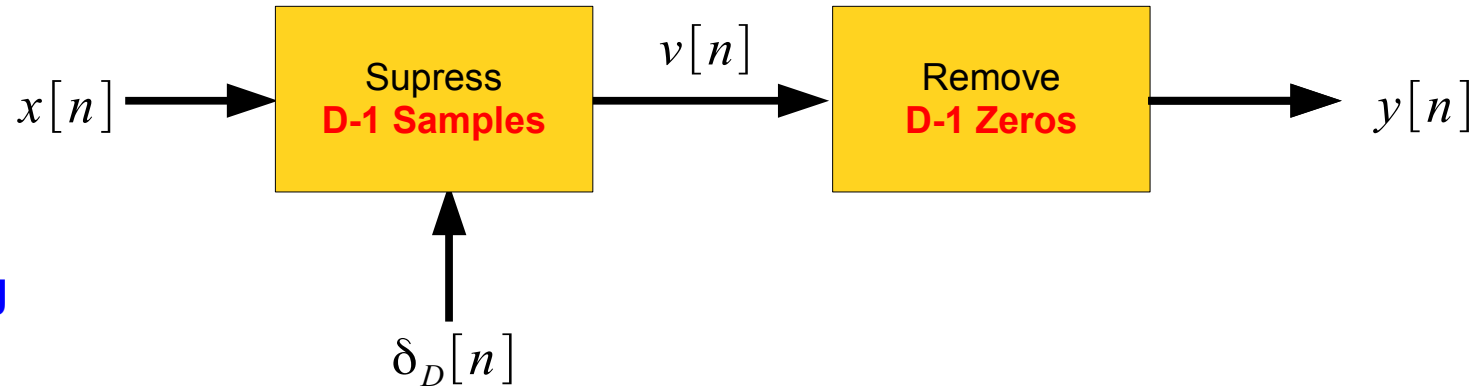


Z-Transform and Down Sampling

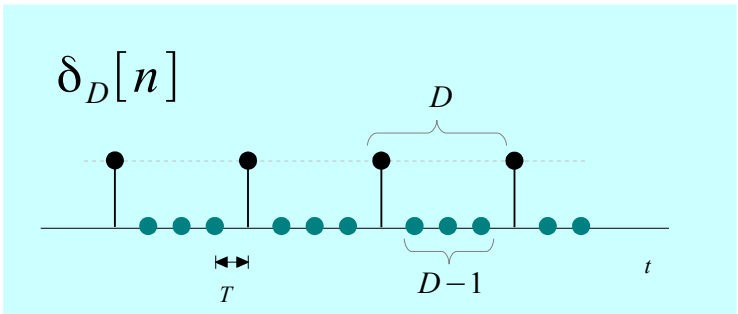


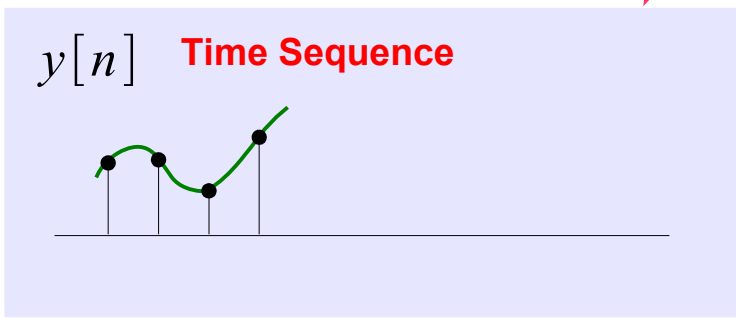


T **Sampling Period**

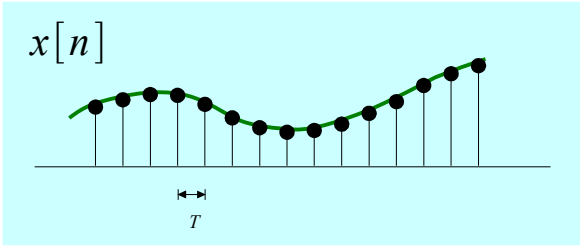


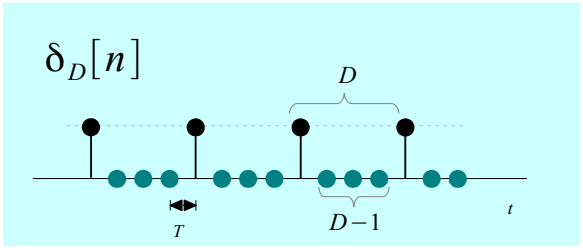
Ideal Sampling

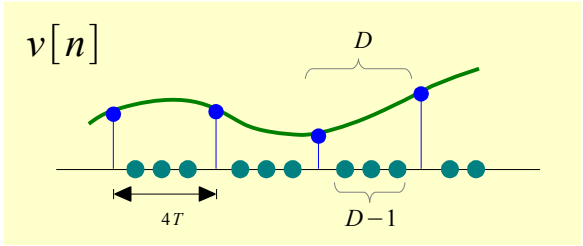




Suppressing D-1 Samples





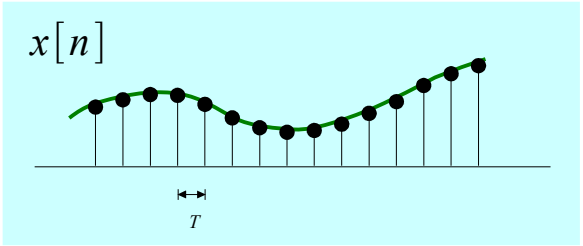


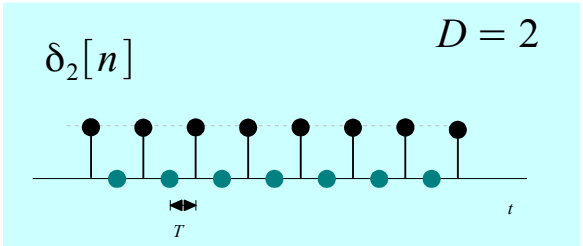
$$\delta_D[n] = \begin{cases} 1 & \text{if } \mathbf{mod}(n, D) = 0 \\ 0 & \text{otherwise} \end{cases} \qquad v[n] = \delta_D[n]x[n]$$

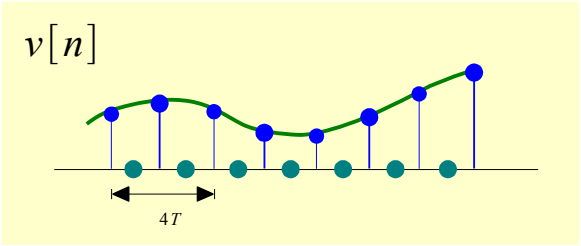
$$V[z] = \cdots + v[0]z^{0} + v[D]z^{-D} + v[2D]z^{-2D} + \cdots$$

$$V[z] = \sum_{n=-\infty}^{+\infty} v[n]z^{-n} = \sum_{m=-\infty}^{+\infty} v[mD]z^{-Dm} = F(z^{D})$$

Example When D=2 (1)







$$x[n] = e^{j\omega n}$$

$$\begin{aligned}\delta_2[n] &= \tfrac{1}{2}\left(1+(-1)^n\right)\\ &= \tfrac{1}{2}\left(1+e^{-j\pi n}\right)\\ &\left(e^{-j\pi} = -1\right)\end{aligned}$$

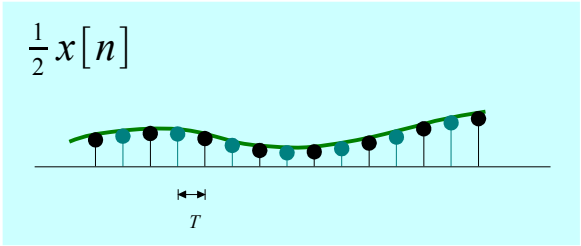
$$\begin{aligned}v[n] &= \tfrac{1}{2}x[n] + \tfrac{1}{2}e^{-j\pi n}x[n]\\ &= \tfrac{1}{2}e^{j\omega n} + \tfrac{1}{2}e^{-j\pi n}e^{j\omega n}\\ &= \tfrac{1}{2}e^{j\omega n} + \tfrac{1}{2}e^{+j(\omega-\pi)n}\end{aligned}$$

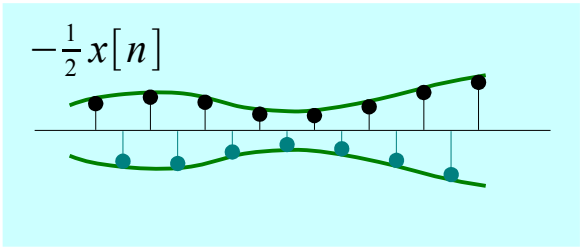
$$V(z) = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\left(x[n]z^{-n} + x[n](-z)^{-n}\right) = \frac{1}{2}X(z) + \frac{1}{2}X(-z)$$

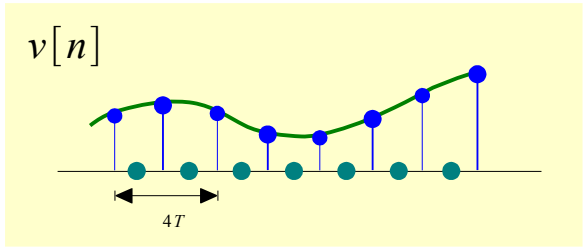
$$V(e^{j\omega}) = \frac{1}{2}X(e^{j\omega}) + \frac{1}{2}X(e^{-j\pi}e^{j\omega})$$

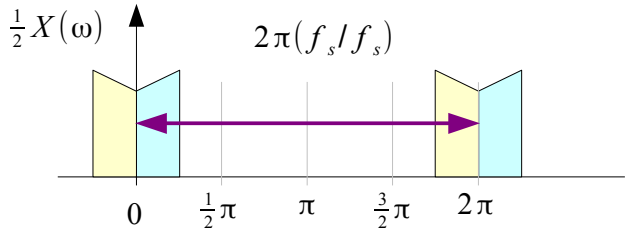
$$V(\omega) = \frac{1}{2}X(\omega) + \frac{1}{2}X(\omega-\pi)$$

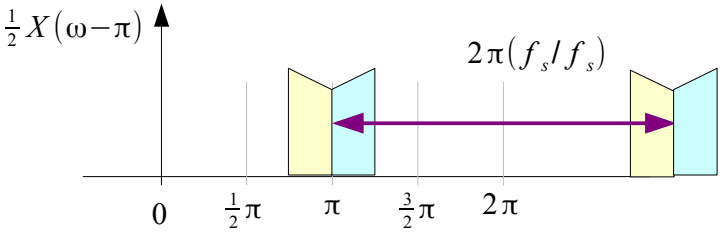
Example When D=2 (2)

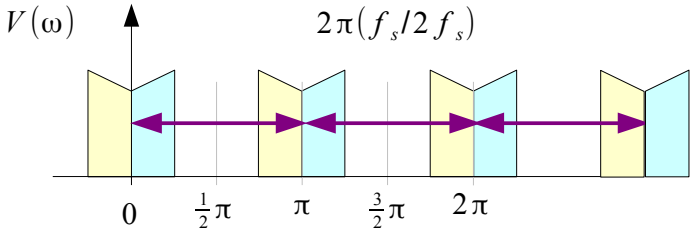












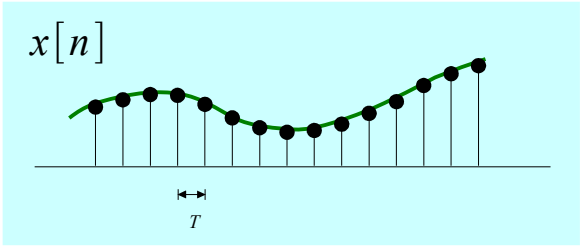
$$v[n] = \frac{1}{2}x[n] + \frac{1}{2}e^{-j\pi n}x[n]$$

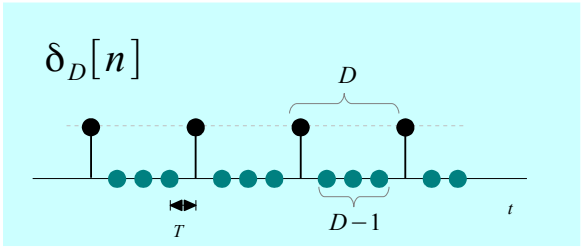
$$V(z) = \frac{1}{2}X(z) + \frac{1}{2}X(-z)$$

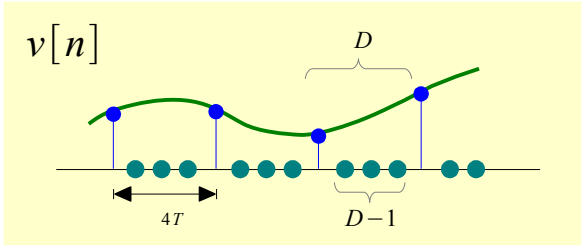
$$V(e^{j\omega}) = \frac{1}{2}X(e^{j\omega}) + \frac{1}{2}X(e^{-j\pi}e^{j\omega})$$

$$V(\omega) = \frac{1}{2}X(\omega) + \frac{1}{2}X(\omega - \pi)$$

Z-Transform & Suppressing D-1 Samples (1)







$$\delta_D[n] = \frac{1}{D}\sum_{k=0}^{D-1} e^{-j2\pi k n/D} = \begin{cases} \frac{1}{D} D = 1 & \text{if } \mathbf{mod}(n, D) = 0 \\ \frac{1}{D}\frac{1-e^{-j2\pi n}}{1-e^{-j2\pi n/D}} = 0 & \text{otherwise} \end{cases}$$

$$v[n] = \delta_D[n]x[n]$$

$$v[n] = \delta_D[n]x[n] = \frac{1}{D}\sum_{k=0}^{D-1} x[n]e^{-j2\pi k n/D}$$

$$Z\{x[n]e^{-j\omega_k n}\} = \sum_{n=-\infty}^{+\infty} x[n](e^{j\omega_k}\cdot z)^{-n} = X(e^{j\omega_k}\cdot z)$$

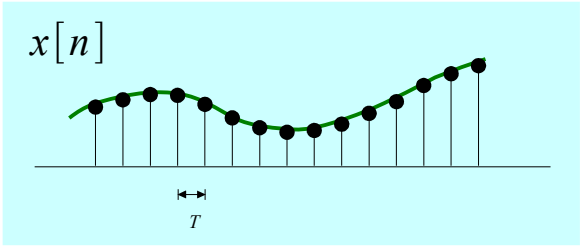
$$V(z) = \frac{1}{D}\sum_{k=0}^{D-1} X(e^{-j2\pi k/D}\cdot z)$$

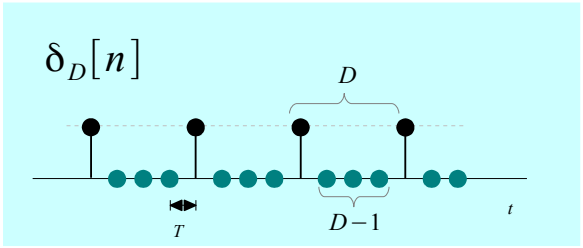
$$\omega_k = 2\pi k/D$$

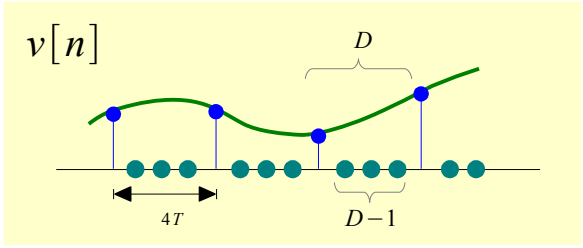
$$e^{-j2\pi k/D}\cdot z = e^{-j2\pi k/D} e^{j\omega} = e^{j(\omega - 2\pi k/D)}$$

$$V(\omega) = \frac{1}{D}\sum_{k=0}^{D-1} X(\omega - 2\pi k/D)$$

Z-Transform & Suppressing D-1 Samples (2)

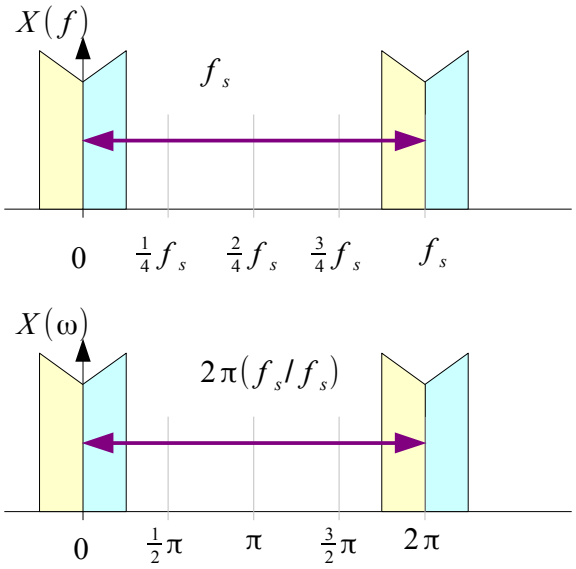


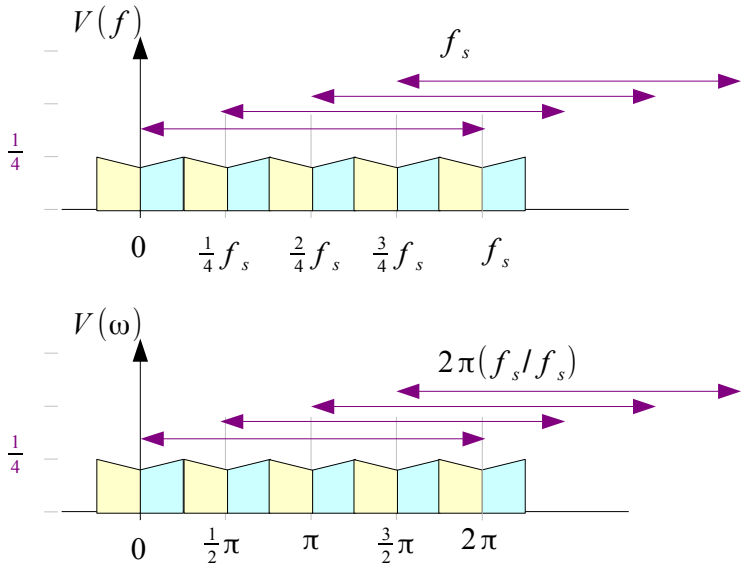




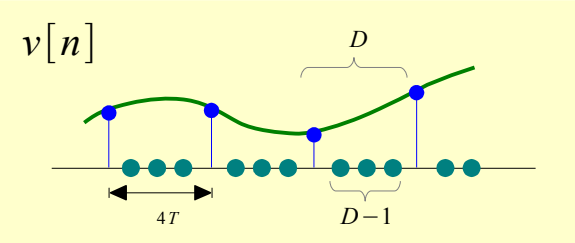
$$V(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{-j2\pi k/D}\cdot z\right)$$

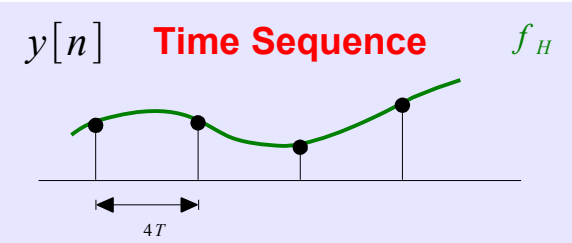
$$V(\omega) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\omega - 2\pi k/D\right)$$

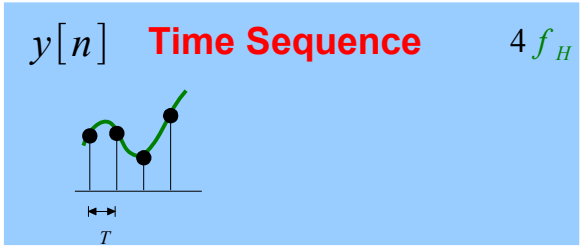




Z-Transform & Removing D-1 Zeros (1)







$$Y(z) = \cdots + y[0] + y[1]z^{-1} + y[2]z^{-2} + \cdots + y[n]z^{-n} + \cdots$$
$$= \cdots + v[0] + v[D]z^{-1} + v[2D]z^{-2} + \cdots + v[nD]z^{-n} + \cdots$$

$$Y(z) = V(z^{1/D})$$

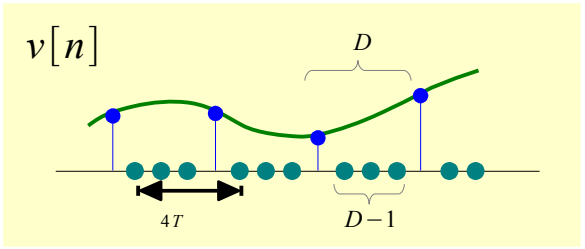
$$V(z) = \frac{1}{D}\sum_{k=0}^{D-1} X(e^{-j2\pi k/D}\cdot z)$$

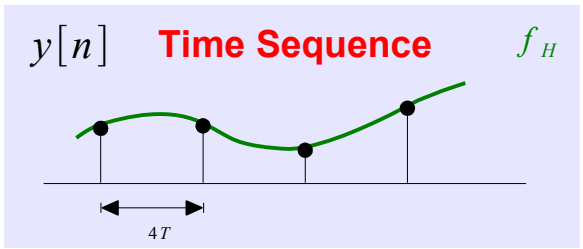
$$Y(z) = \frac{1}{D}\sum_{k=0}^{D-1} X(e^{-j2\pi k/D}\cdot z^{1/D})$$

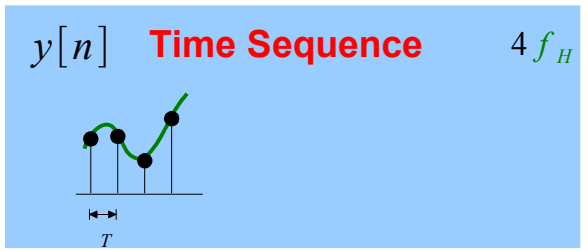
$$V(\omega) = \frac{1}{D}\sum_{k=0}^{D-1} X(\omega - 2\pi k/D)$$

$$V(\omega) = \frac{1}{D}\sum_{k=0}^{D-1} X(\omega/D - 2\pi k/D)$$

Z-Transform & Removing D-1 Zeros (2)





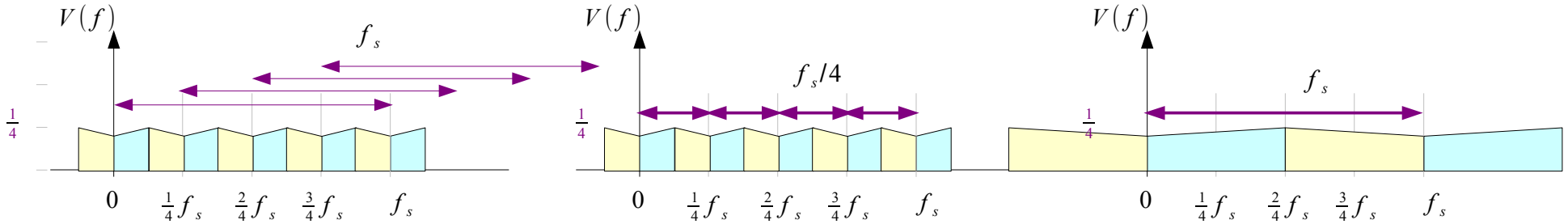


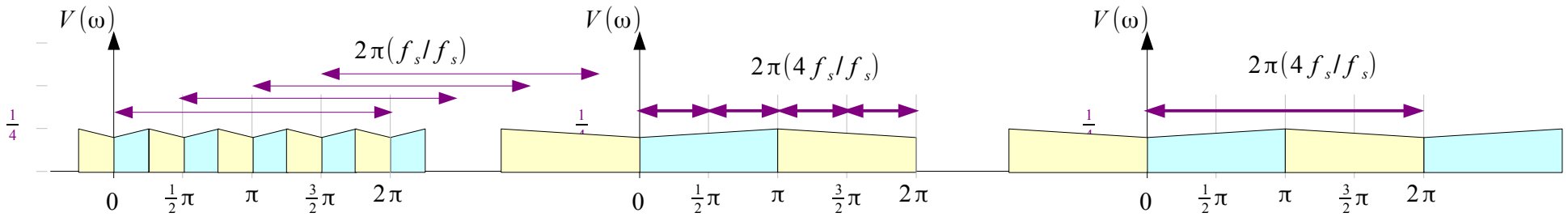
$$V(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{-j2\pi k/D}\cdot z\right)$$

$$Y(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{-j2\pi k/D}\cdot z^{1/D}\right)$$

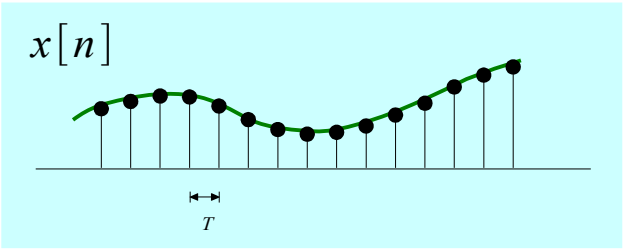
$$V(\omega) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\omega - 2\pi k/D\right)$$

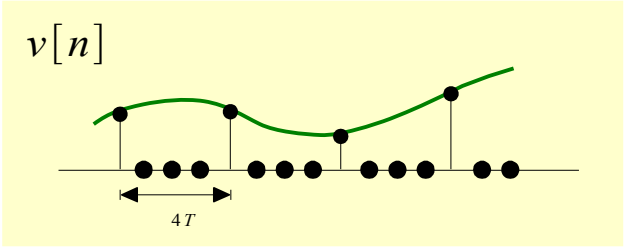
$$V(\omega) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\omega/D - 2\pi k/D\right)$$

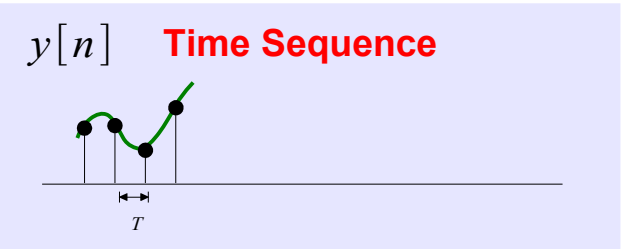




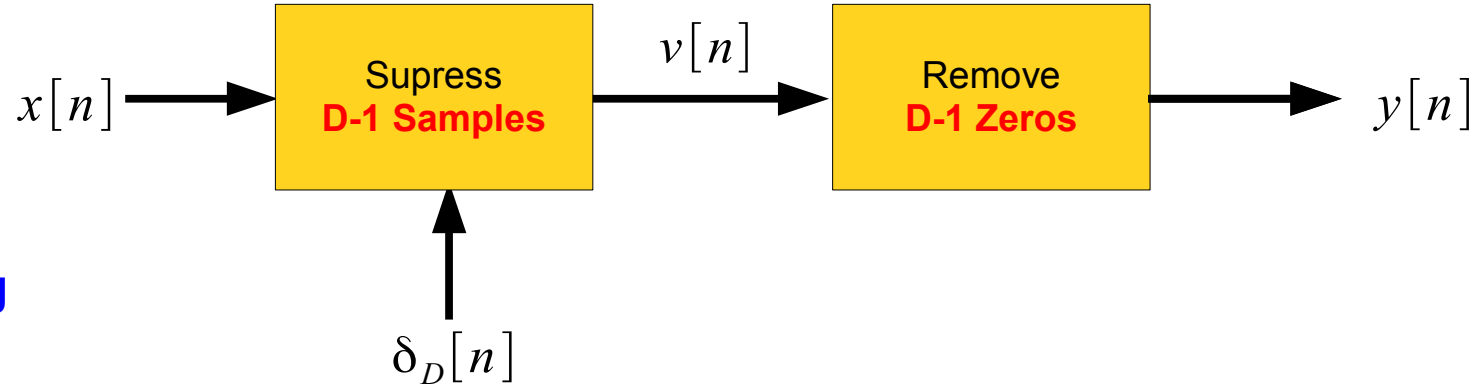
Down Sampling







T **Sampling Period**



Ideal Sampling

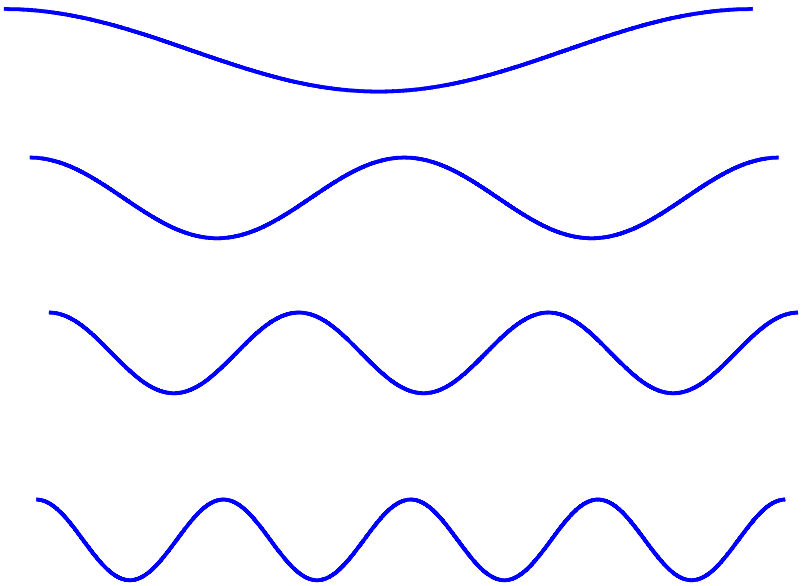
Amplitude Axis Scaling
1 / D

Time Axis Scaling
1 / D

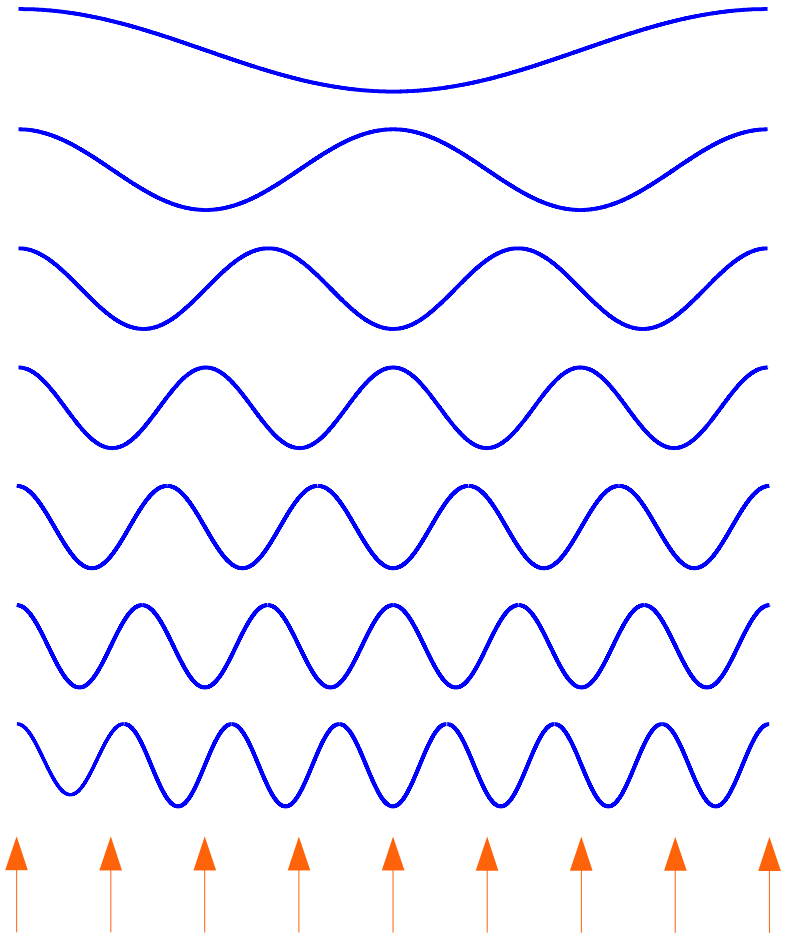
Spectrum Replication
f_s / D

Stretching Spectrum
ω / D

Measuring Rotation Rate



Signals with Harmonic Frequencies (1)



1 Hz
1 cycle / sec

$$\cos(1 \cdot 2\pi t) = \frac{e^{+j(1 \cdot 2\pi)t} + e^{-j(1 \cdot 2\pi)t}}{2}$$

2 Hz
2 cycles / sec

$$\cos(2 \cdot 2\pi t) = \frac{e^{+j(2 \cdot 2\pi)t} + e^{-j(2 \cdot 2\pi)t}}{2}$$

3 Hz
3 cycles / sec

$$\cos(3 \cdot 2\pi t) = \frac{e^{+j(3 \cdot 2\pi)t} + e^{-j(3 \cdot 2\pi)t}}{2}$$

4 Hz
4 cycles / sec

$$\cos(4 \cdot 2\pi t) = \frac{e^{+j(4 \cdot 2\pi)t} + e^{-j(4 \cdot 2\pi)t}}{2}$$

5 Hz
5 cycles / sec

$$\cos(5 \cdot 2\pi t) = \frac{e^{+j(5 \cdot 2\pi)t} + e^{-j(5 \cdot 2\pi)t}}{2}$$

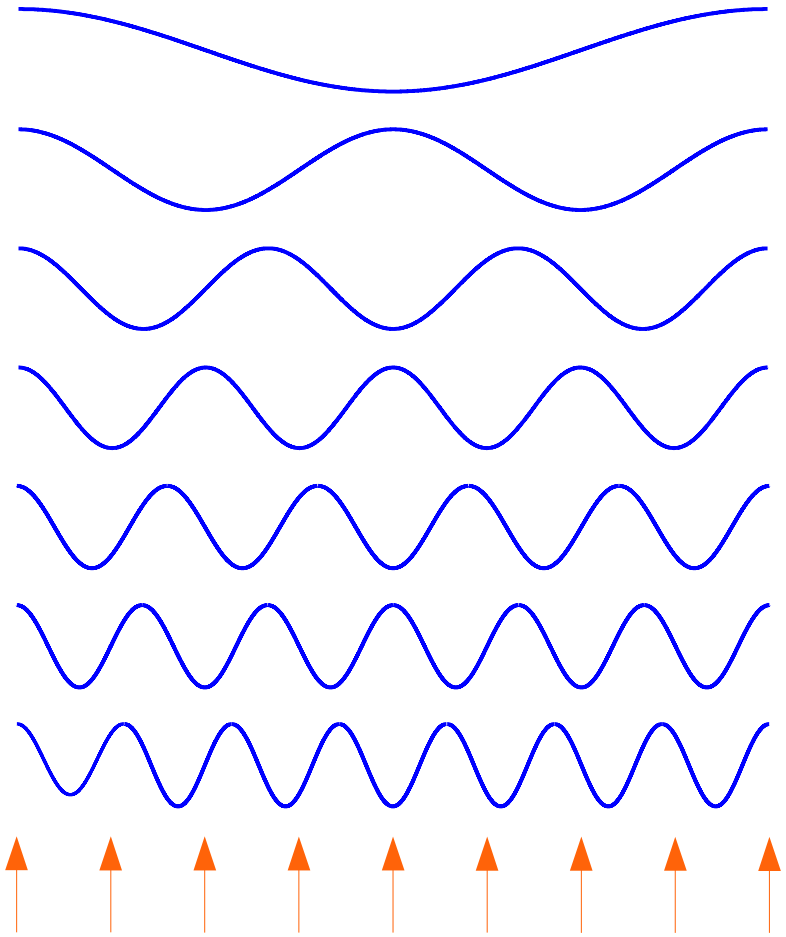
6 Hz
6 cycles / sec

$$\cos(6 \cdot 2\pi t) = \frac{e^{+j(6 \cdot 2\pi)t} + e^{-j(6 \cdot 2\pi)t}}{2}$$

7 Hz
7 cycles / sec

$$\cos(7 \cdot 2\pi t) = \frac{e^{+j(7 \cdot 2\pi)t} + e^{-j(7 \cdot 2\pi)t}}{2}$$

Signals with Harmonic Frequencies (2)



1 Hz
1 cycle / sec

2 Hz
2 cycles / sec

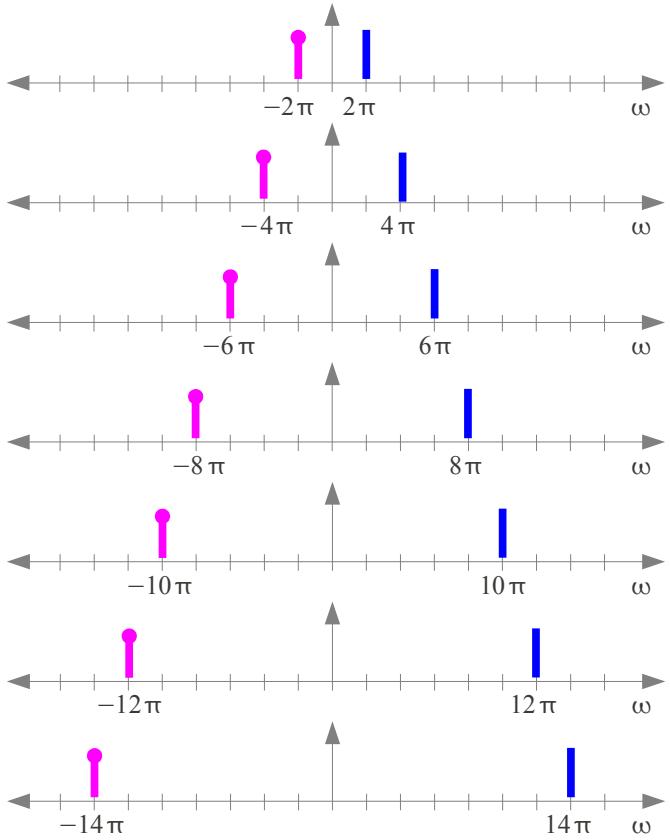
3 Hz
3 cycles / sec

4 Hz
4 cycles / sec

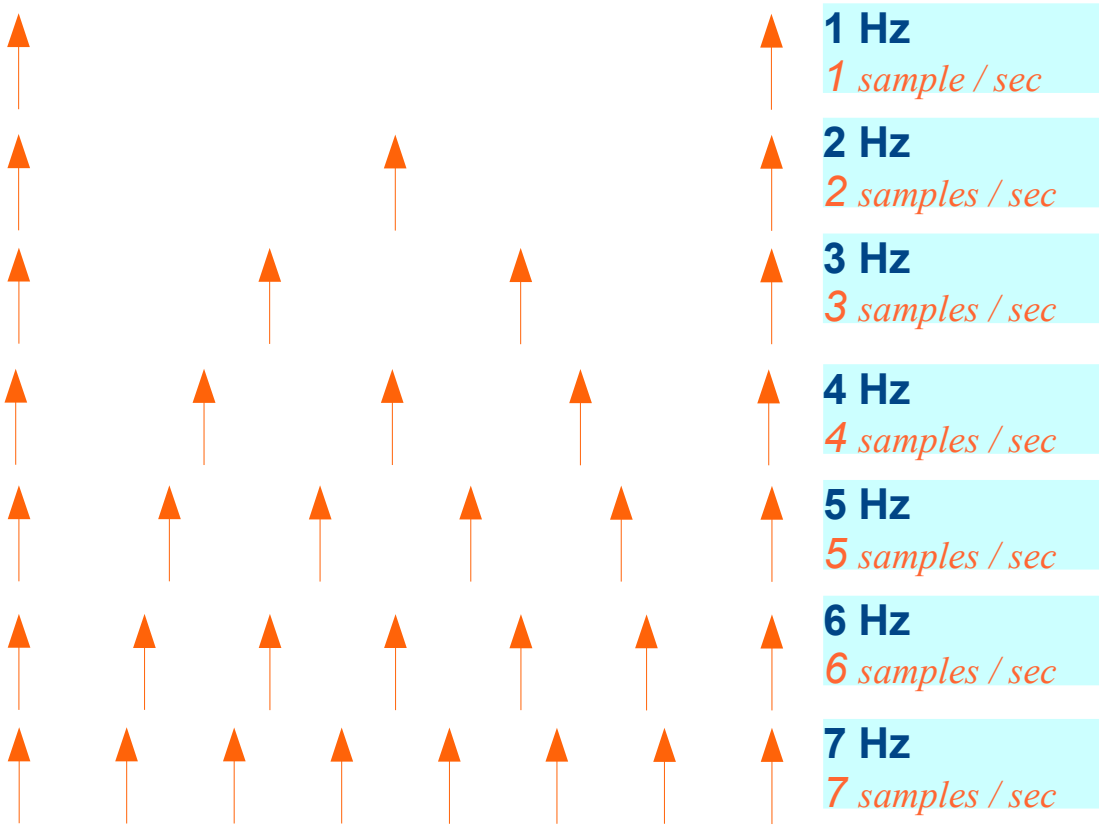
5 Hz
5 cycles / sec

6 Hz
6 cycles / sec

7 Hz
7 cycles / sec



Sampling Frequency



1 Hz
1 sample / sec

2 Hz
2 samples / sec

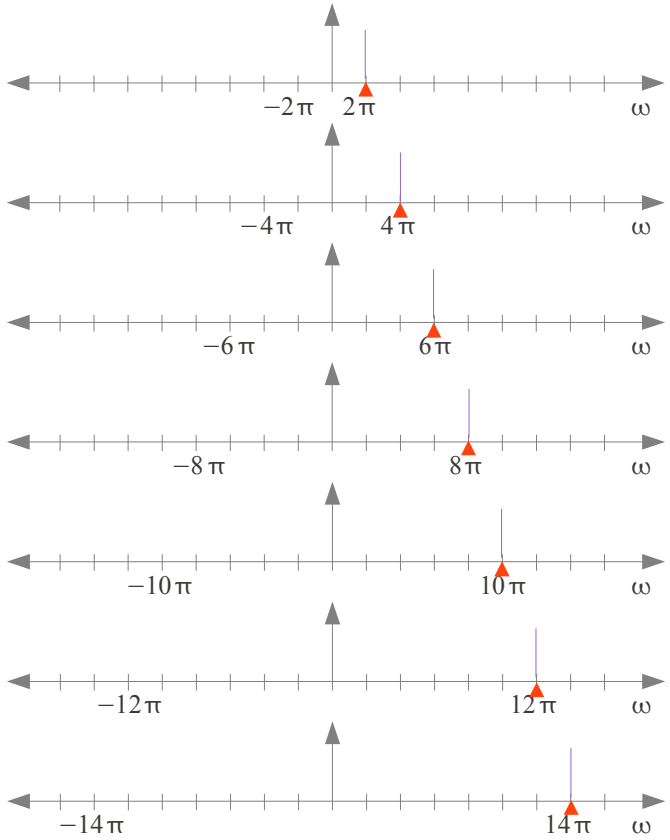
3 Hz
3 samples / sec

4 Hz
4 samples / sec

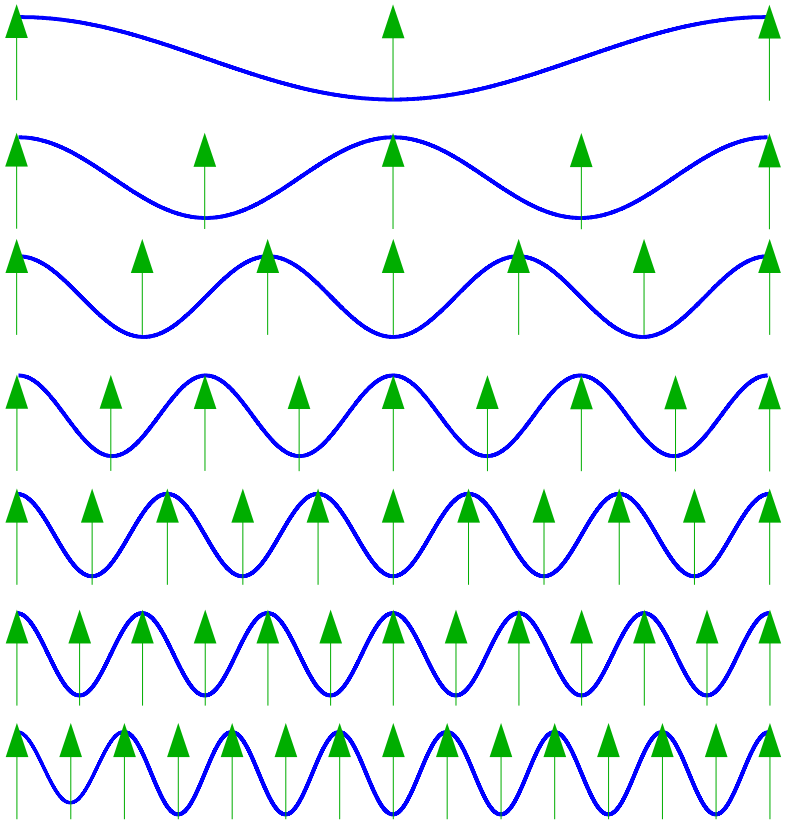
5 Hz
5 samples / sec

6 Hz
6 samples / sec

7 Hz
7 samples / sec

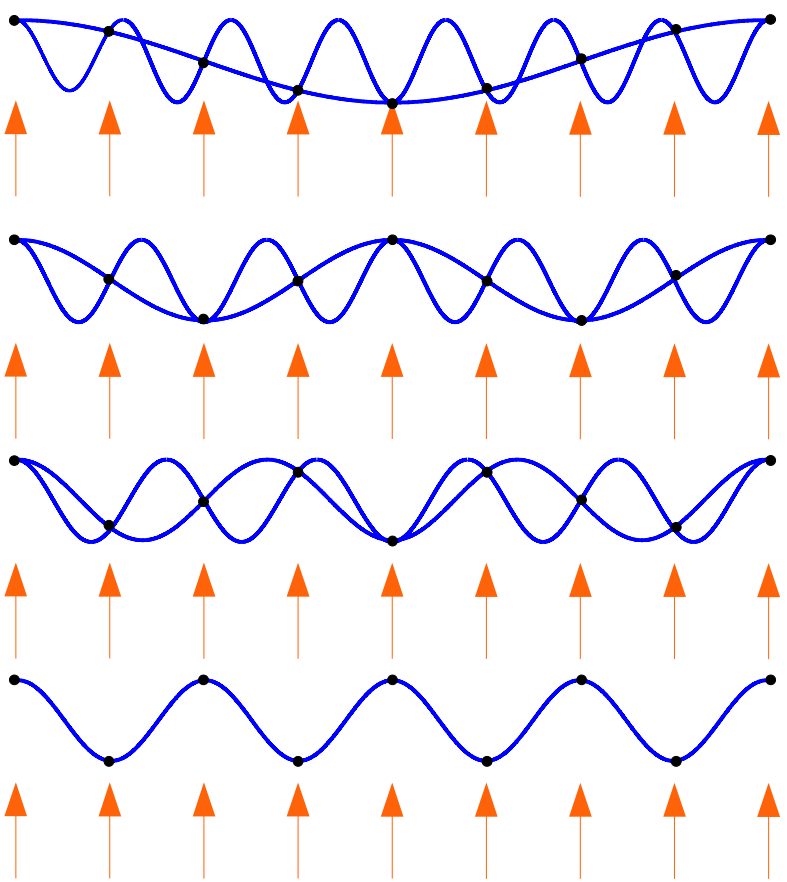


Nyquist Frequency



1 Hz *1 cycle / sec*	*2x1 sample / sec*
2 Hz *2 cycles / sec*	*2x2 samples / sec*
3 Hz *3 cycles / sec*	*2x3 samples / sec*
4 Hz *4 cycles / sec*	*2x4 samples / sec*
5 Hz *5 cycles / sec*	*2x5 samples / sec*
6 Hz *6 cycles / sec*	*2x6 samples / sec*
7 Hz *7 cycles / sec*	*2x7 samples / sec*

Aliasing



1 Hz
7 Hz

2x4 *samples / sec*

2 Hz
6 Hz

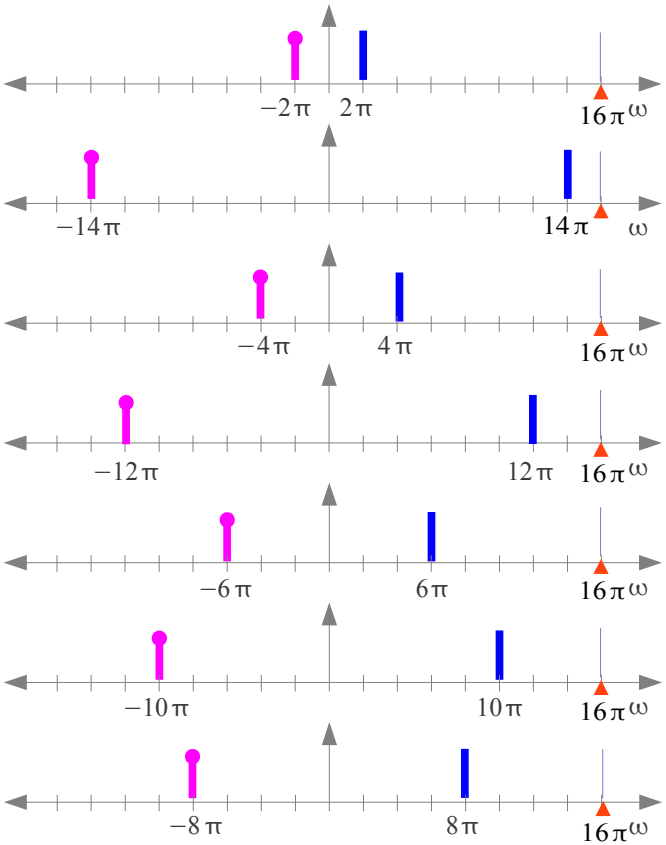
2x4 *samples / sec*

3 Hz
5 Hz

2x4 *samples / sec*

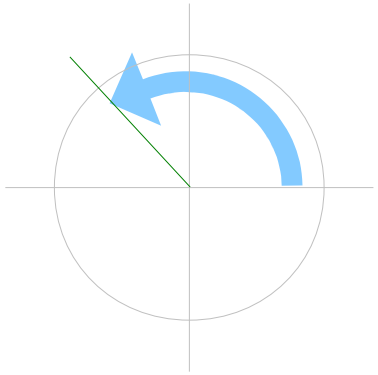
4 Hz

2x4 *samples / sec*



Sampling

$$\omega_s = 2\pi f_s \ (rad/sec)$$

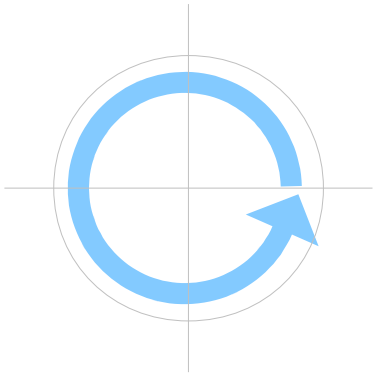


$$\omega_1 = 2\pi f_1 \qquad \omega_2 = 2\pi f_2$$

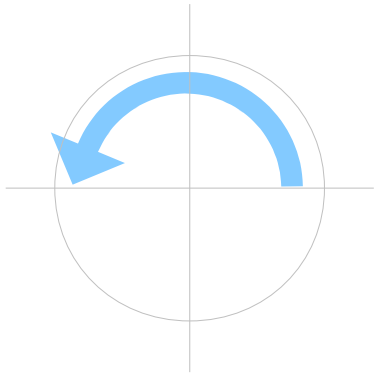
$$\omega_1 = \frac{\omega_s}{2} \ (rad/sec) \qquad \omega_2 = -\frac{\omega_s}{2} \ (rad/sec)$$

$$f_1 = \frac{f_s}{2} \ (rad/sec) \qquad f_2 = -\frac{f_s}{2} \ (rad/sec)$$

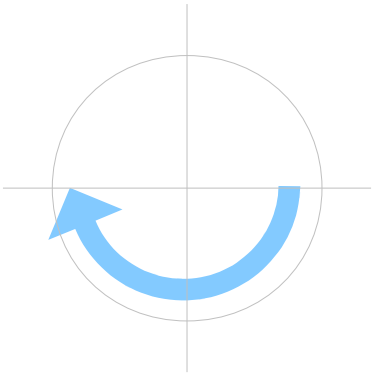
$2\pi \ (rad) \ / \ T_s \ (sec)$



$\pi \ (rad) \ / \ T_s \ (sec)$

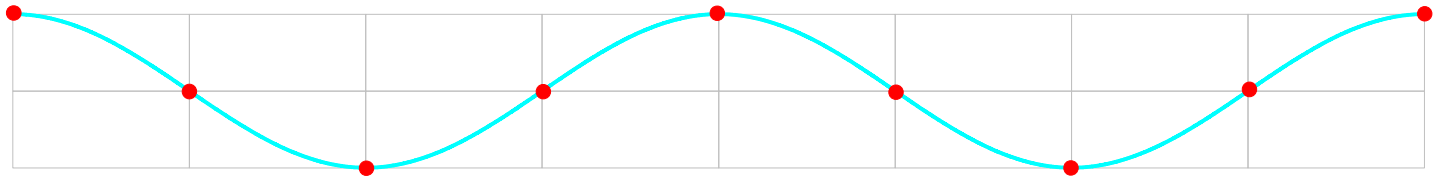


$-\pi \ (rad) \ / \ T_s \ (sec)$

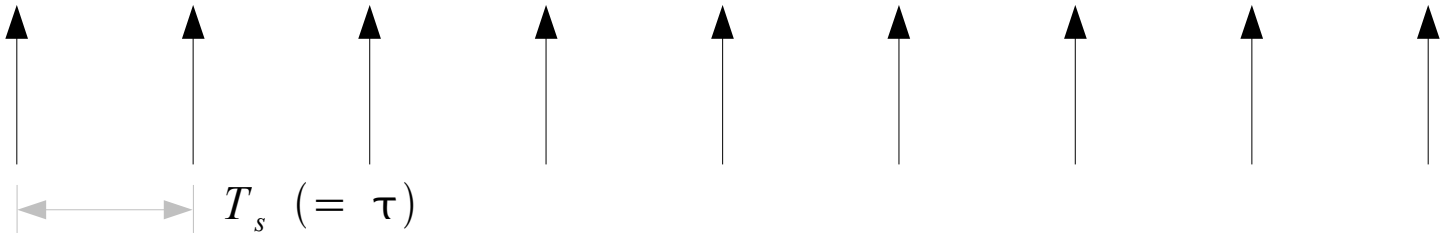


Sampling

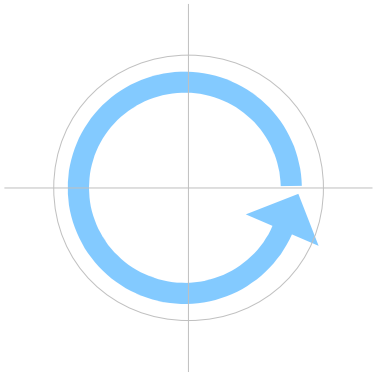
$\omega_1 = 2\pi f_1 \ (rad/sec)$



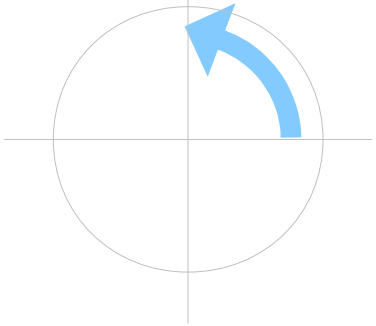
$\omega_s = 2\pi f_s \ (rad/sec)$



$2\pi \ (rad) \ / \ T_s \ (sec)$



$\frac{\pi}{2} \ (rad) \ / \ T_s \ (sec)$



For the period of T_s

Angular displacement $\frac{\pi}{2} \ (rad)$

$$\hat{\omega} = \omega \cdot T_s \ (rad)$$

$$= 2\pi f_1 \cdot T_s \ (rad)$$

$$= 2\pi \frac{f_s}{4} \cdot T_s \ (rad)$$

$$= \frac{\pi}{2} \ (rad)$$

Angular Frequencies in Sampling

continuous-time signals

Signal Frequency

$$f_0 = \frac{1}{T_0}$$

Signal Angular Frequency

$$\omega_0 = 2\pi f_0 \ (rad/sec)$$

sampling sequence

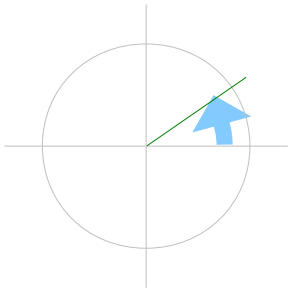
Sampling Frequency

$$f_s = \frac{1}{T_s}$$

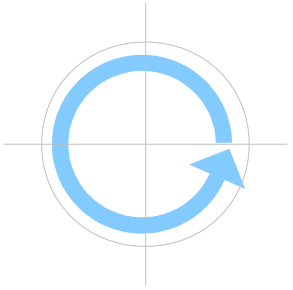
Sampling Angular Frequency

$$\omega_s = 2\pi f_s \ (rad/sec)$$

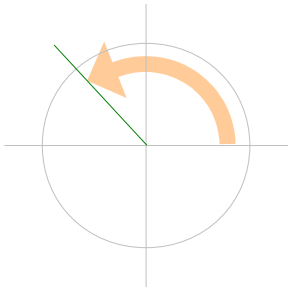
For 1 second $\omega_0 = 2\pi f_0 \ (rad/sec)$



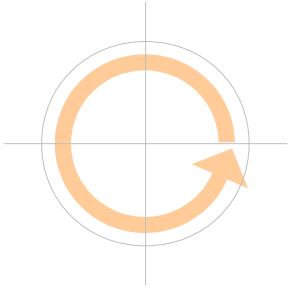
For 1 revolution $2\pi \ (rad) = T_0 \ (sec)$



For 1 second $\omega_s = 2\pi f_s \ (rad/sec)$



For 1 revolution $2\pi \ (rad) = T_s \ (sec)$



References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A "graphical interpretation" of the DFT and FFT, by Steve Mann
[4] R. Cristi, "Modern Digital Signal Processing"