

Vector Functions (1A)

- Vector Functions
- Motion
- Curvature

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Vector Valued Functions

Set of points (x, y, z)

Parametric functions $\left\{ \begin{array}{l} x = f(t) \\ y = g(t) \\ z = h(t) \end{array} \right. \Rightarrow (f(t), g(t), h(t))$
parameter t

Vector Valued Function

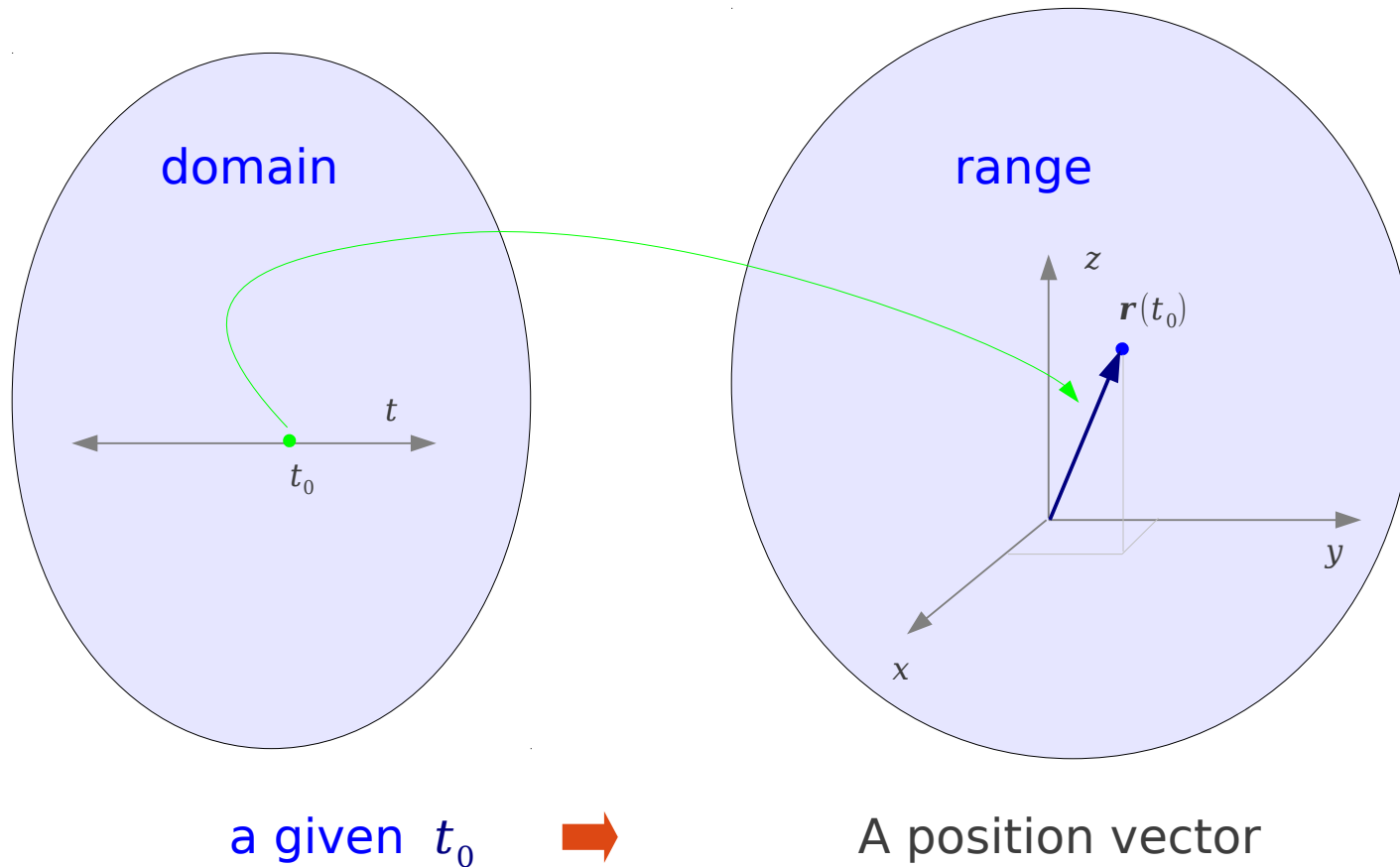
a given t \Rightarrow A point in 3-d space $\xrightarrow{\text{Position Vector}}$

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$
$$= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Vector Valued Functions (1)

Vector Valued Function

$$\begin{aligned}\mathbf{r}(t) &= \langle f(t), g(t), h(t) \rangle \\ &= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}\end{aligned}$$

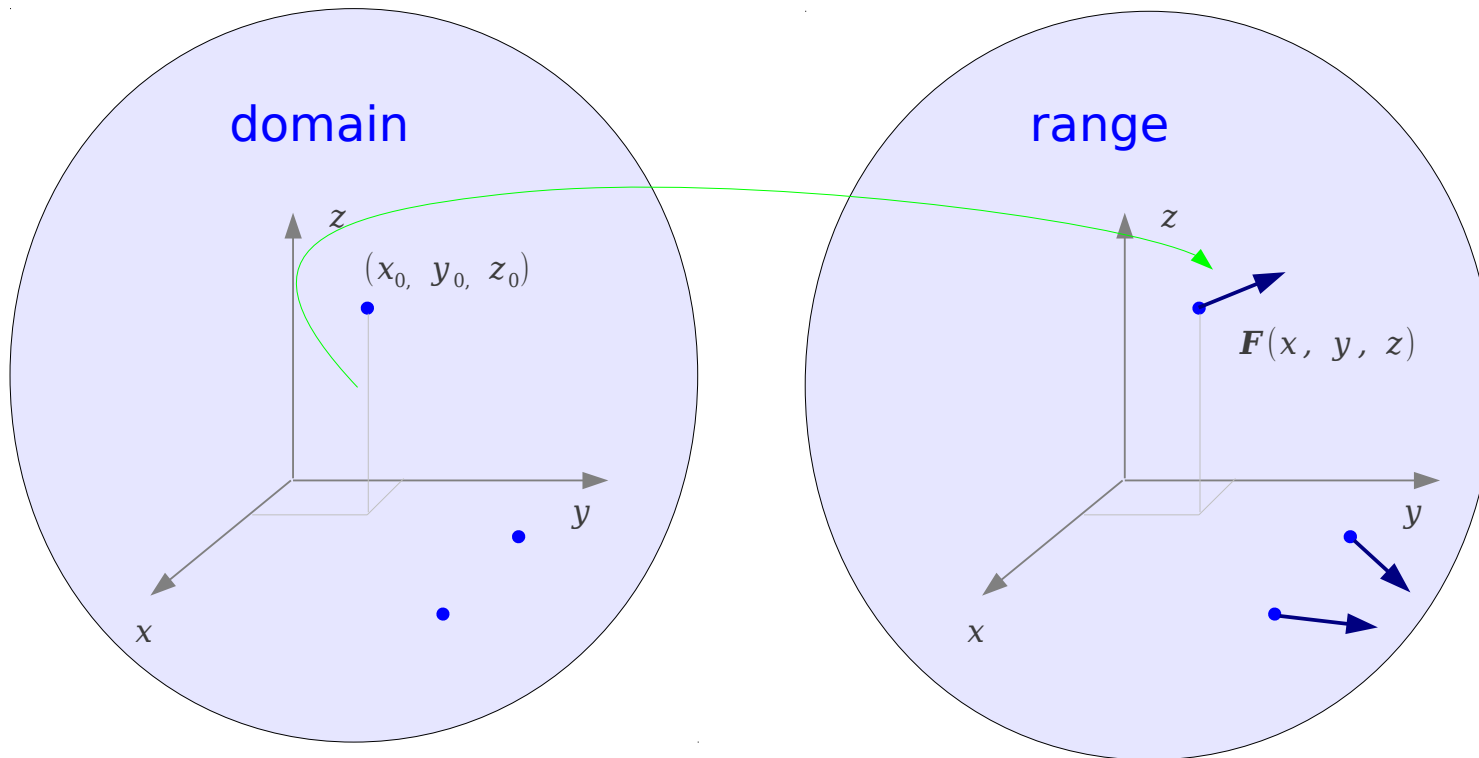


Vector Valued Functions (2)

Vector Field

(x, y, z)

$F(x, y, z)$



a given point in a 3-d space

(x_0, y_0, z_0)



A vector

(u_0, v_0, w_0)

Limit of a Vector Function

Vector Valued Function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$

Limit of a Vector Valued Function

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle$$

Limit of a Vector Valued Function

$\mathbf{r}(a)$ is defined

$\lim_{t \rightarrow a} \mathbf{r}(t)$ exists

$$\mathbf{r}(a) = \lim_{t \rightarrow a} \mathbf{r}(t)$$

Derivative of a Vector Function

Vector Valued Function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$

Derivative of a Vector Valued Function

$$\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

$\mathbf{r}(t)$ Position Vector $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$

$\mathbf{r}'(t)$ Tangent Vector $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

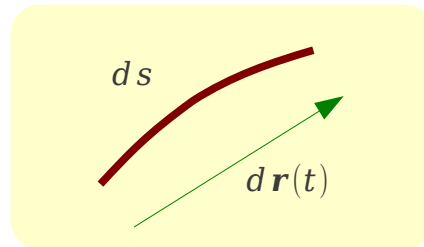
Arc Length (1)

Vector Valued Function

Derivative of a Vector Valued Function

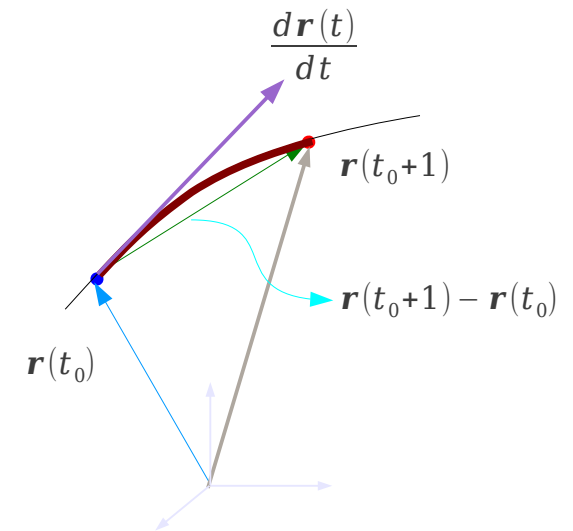
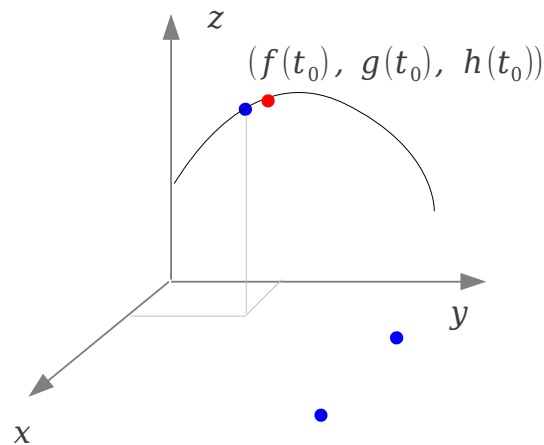
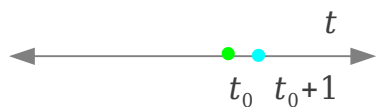
$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$



$$\mathbf{r}(t_0) = (f(t_0), g(t_0), h(t_0))$$

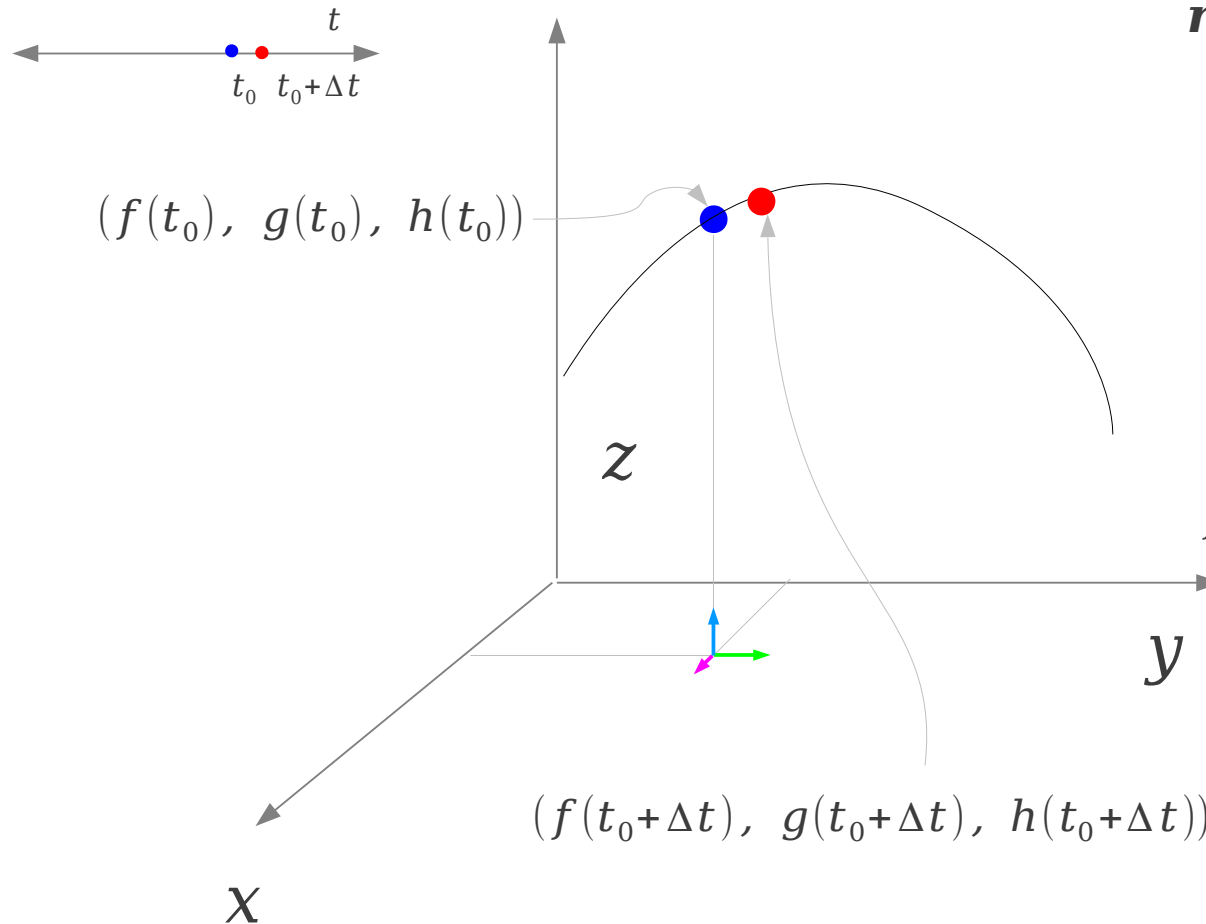
$$\mathbf{r}(t_0+1) = (f(t_0+1), g(t_0+1), h(t_0+1))$$



Arc Length (2)

Vector Valued Function

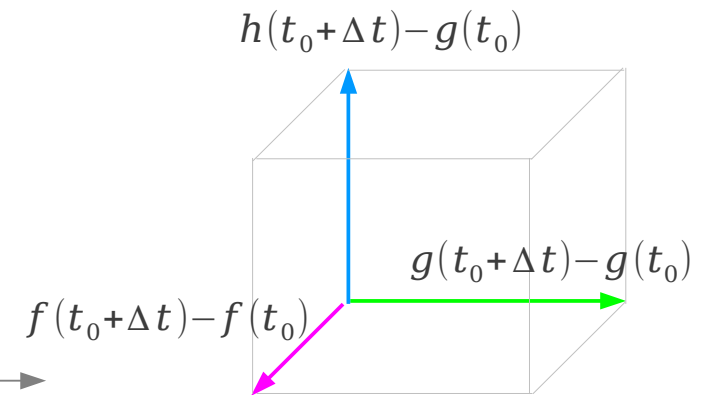
Derivative of a Vector Valued Function



$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

$$\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$



Arc Length (3)

Vector Valued Function

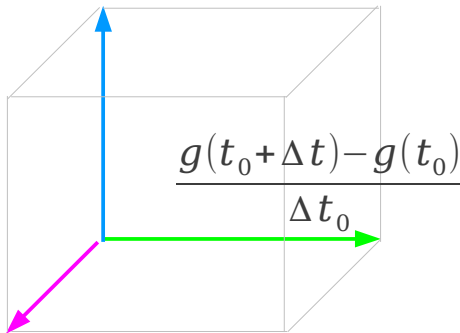
Derivative of a Vector Valued Function

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

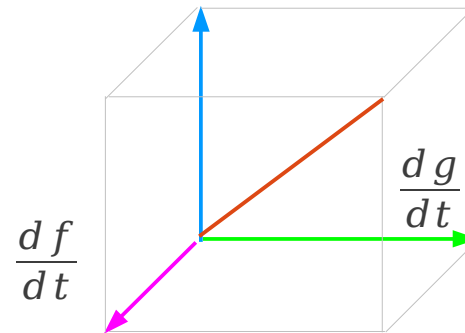
$$\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

$$\frac{h(t_0 + \Delta t) - h(t_0)}{\Delta t_0}$$



$$\frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t_0}$$

$$\frac{dh}{dt}$$



$$\sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} = \|\mathbf{r}'(t)\|$$

Arc Length (4)

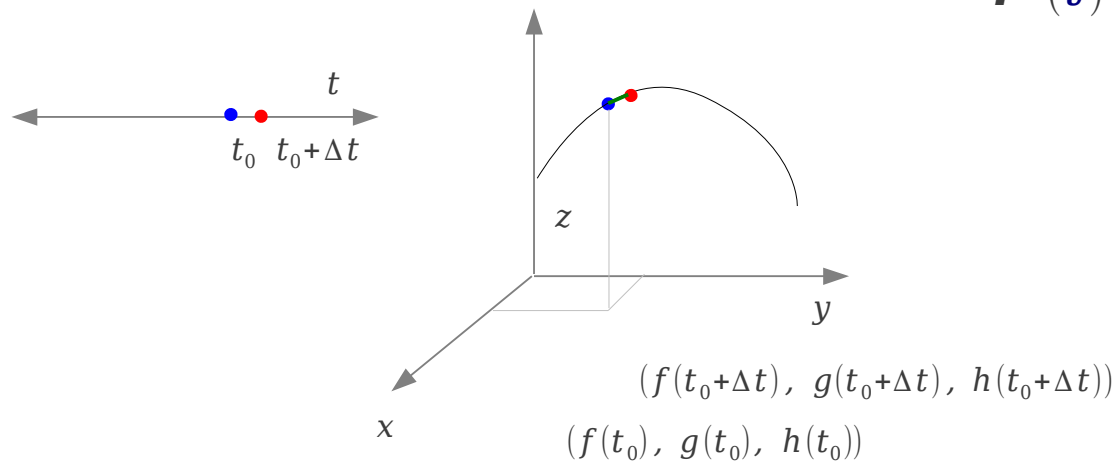
Vector Valued Function

Derivative of a Vector Valued Function

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

$$\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$



Length of a Smooth Curve

$$\begin{aligned} s &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt \\ &= \int_a^b \|\mathbf{r}'(t)\| dt \end{aligned}$$

Arc Length as a Parameter

Length of a Smooth Curve

$$\begin{aligned} s &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt \\ &= \int_a^b \|\mathbf{r}'(t)\| dt \end{aligned}$$

Directed Distance from $P(t_0)$

$$s(t) = \int_{t_0}^t |\mathbf{v}(\tau)| d\tau$$

s increases in the direction of increasing t



Arc Length Parameter

$$= \int_{t_0}^t \|\mathbf{r}'(\tau)\| d\tau$$

$$= \int_{t_0}^t \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2 + [h'(\tau)]^2} d\tau$$

Speed, Velocity, Unit Tangent Vector

Arc Length Parameter

$$s(t) = \int_{t_0}^t |\mathbf{v}(\tau)| d\tau$$

Speed is the absolute value of $\mathbf{v}(t)$

$$\frac{ds}{dt} = |\mathbf{v}(t)| \quad \frac{d}{dt}s(t) = \frac{d}{dt} \int_{t_0}^t |\mathbf{v}(\tau)| d\tau$$

Velocity

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}(t)$$

Unit Tangent Vector

$$\mathbf{T} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

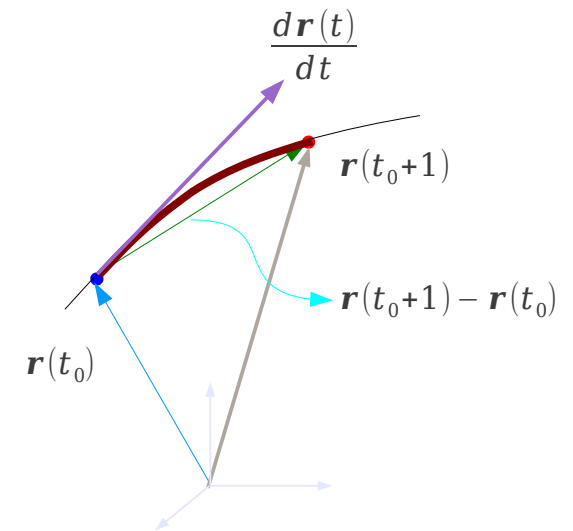
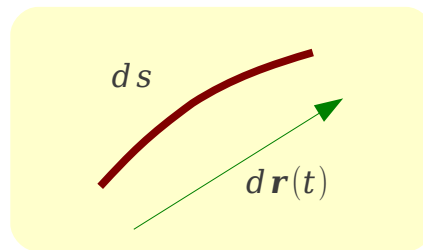
Unit Tangent Vector

$$s(t) = \int_{t_0}^t |\mathbf{v}(\tau)| d\tau \quad \frac{ds}{dt} = |\mathbf{v}(t)| \quad \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) \quad \mathbf{T} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

$$\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{|\mathbf{v}(t)|}$$

Unit Tangent Vector

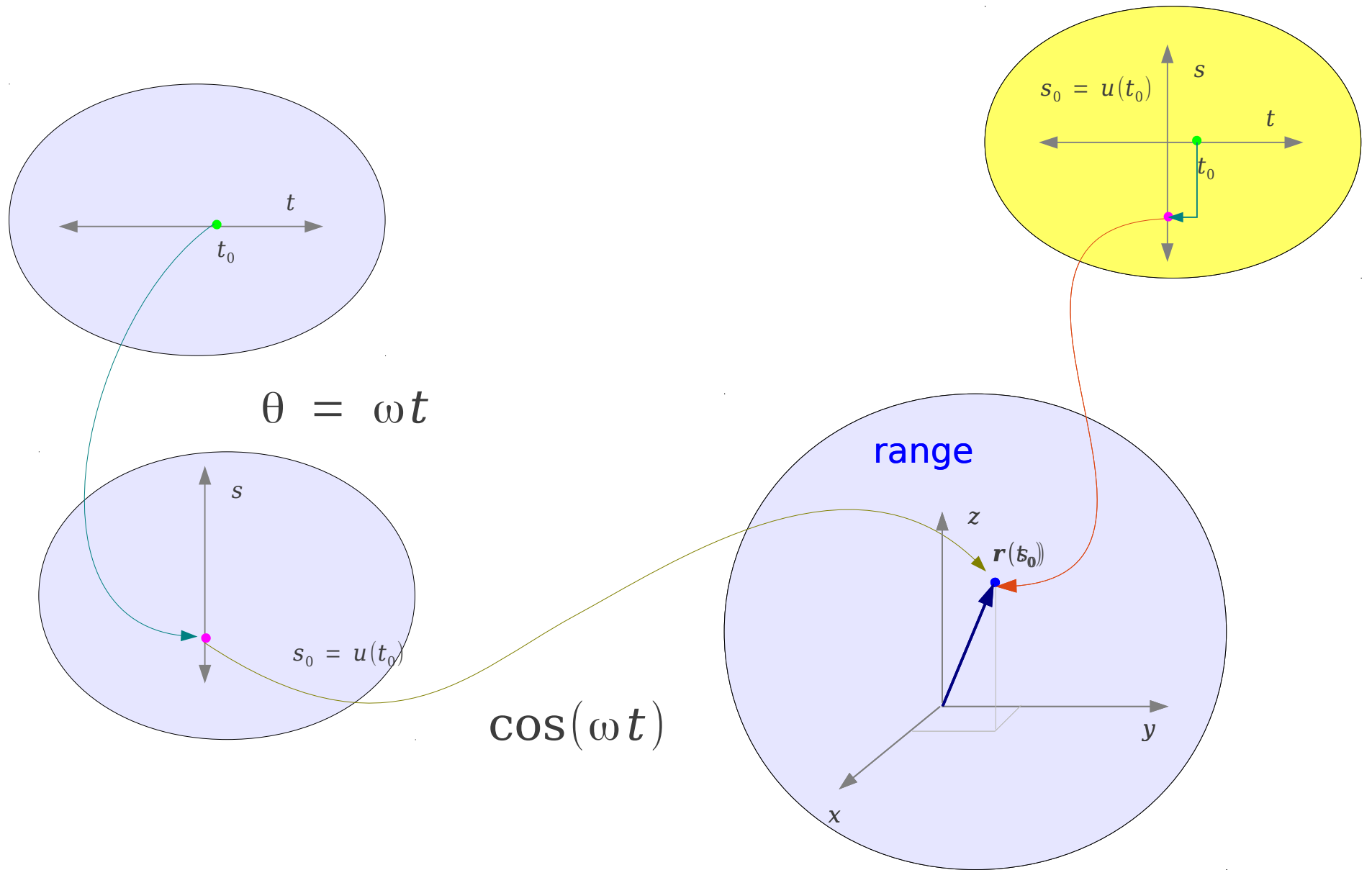
$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt} \frac{dt}{ds} = \mathbf{v}(t) \frac{1}{|\mathbf{v}(t)|} = \mathbf{T}$$



$$\mathbf{r}(t_0) = (f(t_0), g(t_0), h(t_0))$$

$$\mathbf{r}(t_0+1) = (f(t_0+1), g(t_0+1), h(t_0+1))$$

Composite Function



Chain Rule of a Vector Function (1)

Derivative of a Vector Valued Function

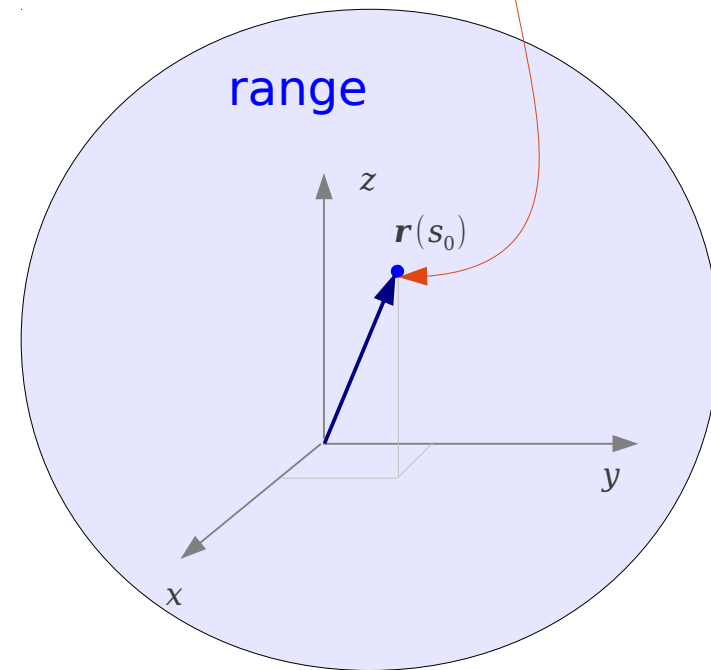
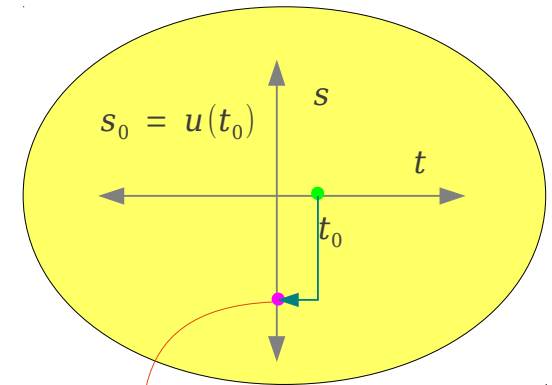
$$s = u(t)$$

$$\frac{ds}{dt} = \frac{du(t)}{dt} \Rightarrow u'(t)$$

$$\frac{dr}{dt} = \frac{dr}{ds} \frac{ds}{dt} = \mathbf{r}'(s) u'(t)$$

$$s(t) = \int_{t_0}^t |\mathbf{v}(\tau)| d\tau = \int_{t_0}^t \|\mathbf{r}'(\tau)\| d\tau$$
$$= \int_{t_0}^t \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2 + [h'(\tau)]^2} d\tau$$

$$\frac{ds}{dt} = |\mathbf{v}(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$



Chain Rule of a Vector Function (2)

Vector Valued Function

$$\mathbf{r}(s) = \langle f(s), g(s), h(s) \rangle$$

Scalar Function

$$s = u(t)$$

$$\mathbf{r}(u(t)) = \langle f(u(t)), g(u(t)), h(u(t)) \rangle$$

Derivative of a Vector Valued Function

$$s = u(t)$$

$$\frac{ds}{dt} = \frac{du(t)}{dt} \Rightarrow u'(t)$$

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \frac{ds}{dt} = \mathbf{r}'(s) u'(t)$$

Integration of a Vector Function

Vector Valued Function

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Limit of a Vector Valued Function

$$\int \mathbf{r}(t) dt = \langle \int f(t) dt, \int g(t) dt, \int h(t) dt \rangle$$

$$= \int f(t) dt \mathbf{i} + \int g(t) dt \mathbf{j} + \int h(t) dt \mathbf{k}$$

Displacement, Velocity, Acceleration

Displacement

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Velocity

$$\mathbf{v}(t) = \mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Acceleration

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = f''(t)\mathbf{i} + g''(t)\mathbf{j} + h''(t)\mathbf{k}$$

Speed

$$\|\mathbf{v}(t)\| = \left\| \frac{d\mathbf{r}(t)}{dt} \right\| = \|f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}\|$$

$$= \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Unit Tangent of a Vector Function (1)

Displacement

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Velocity

$$\mathbf{v}(t) = \mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Unit Tangent

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

$$\mathbf{r}'(t) \quad \rightarrow \quad \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \frac{ds}{dt}$$

Arc length

s

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt = \int_a^b \|\mathbf{r}'(t)\| dt$$

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|$$

Unit Tangent of a Vector Function (2)

Displacement

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Velocity

$$\mathbf{v}(t) = \mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Unit Tangent

direction

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

$$\frac{d\mathbf{r}}{ds} = \frac{\frac{d\mathbf{r}}{dt}}{\frac{ds}{dt}} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \mathbf{T}(t)$$

velocity · speed
· direction

speed

$$\mathbf{r}'(t) \rightarrow \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \frac{ds}{dt} \leftarrow \frac{ds}{dt} = \|\mathbf{r}'(t)\|$$

Curvature of a Vector Function (1)

Vector Valued Function

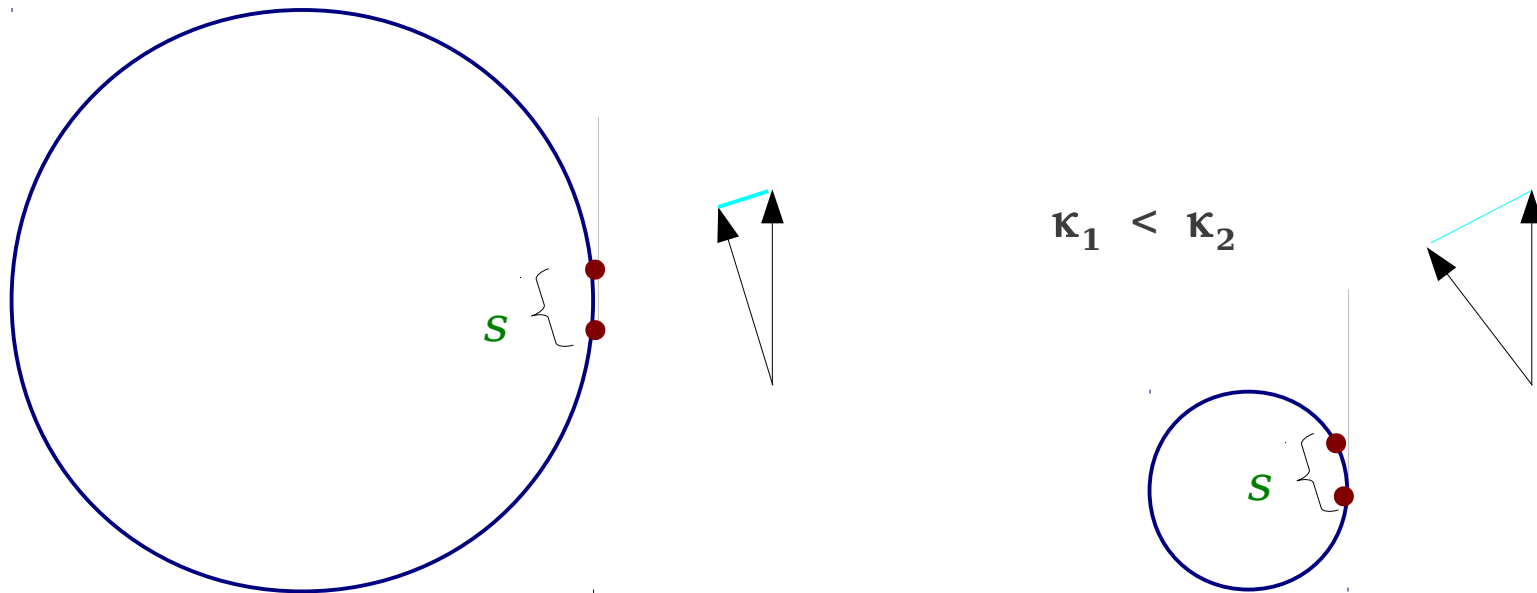
$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Unit Tangent

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{d\mathbf{r}}{ds}$$

Curvature

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$$



Curvature of a Vector Function (2)

Vector Valued Function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Unit Tangent

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{d\mathbf{r}}{ds}$$

Curvature

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$$

Arc length s

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt} \qquad \frac{ds}{dt} = \|\mathbf{r}'(t)\|$$

$$\frac{d\mathbf{T}}{ds} = \frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}} = \frac{\mathbf{T}'(t)}{\|\mathbf{r}'(t)\|} = \kappa(t)$$

Line Equations (2)

References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [4] D.G. Zill, “Advanced Engineering Mathematics”