

CORDIC Background (4A)

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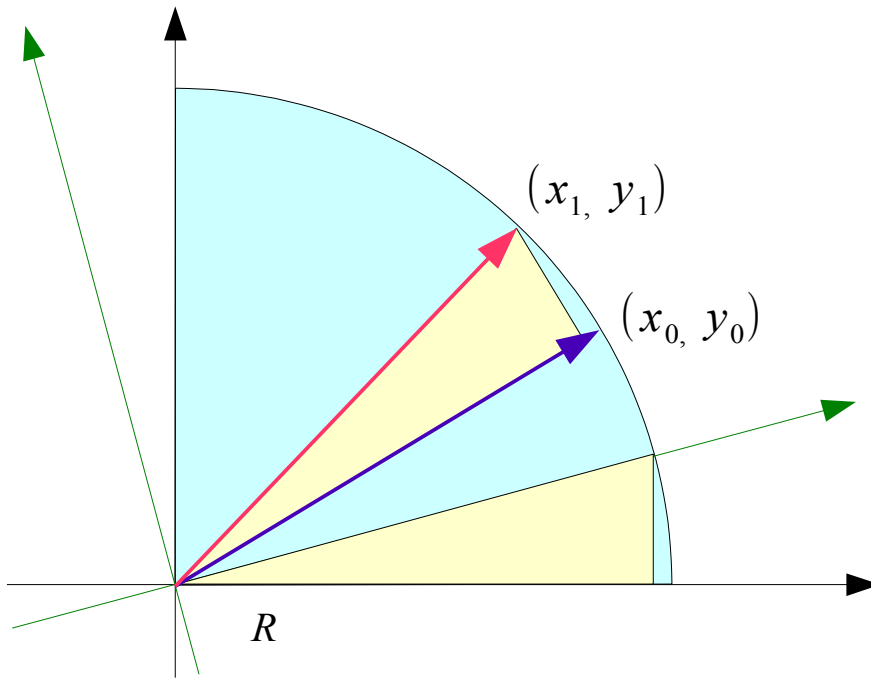
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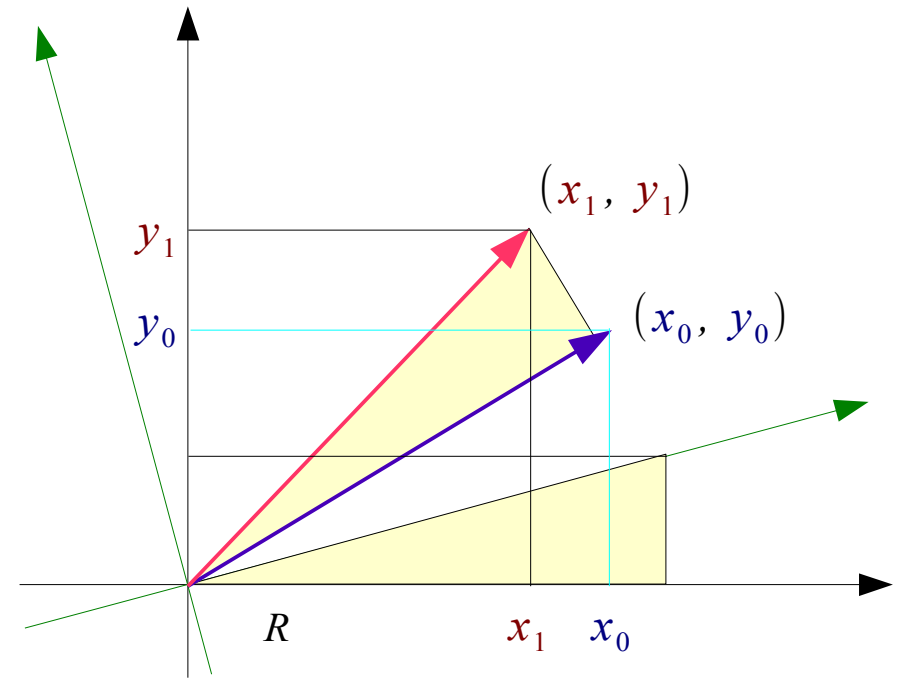
CORDIC Background

J. P. Deschamps, G. A. Bioul, G.D. Sutter, Synthesis of Arithmetic Circuits

Vector Rotation (1)

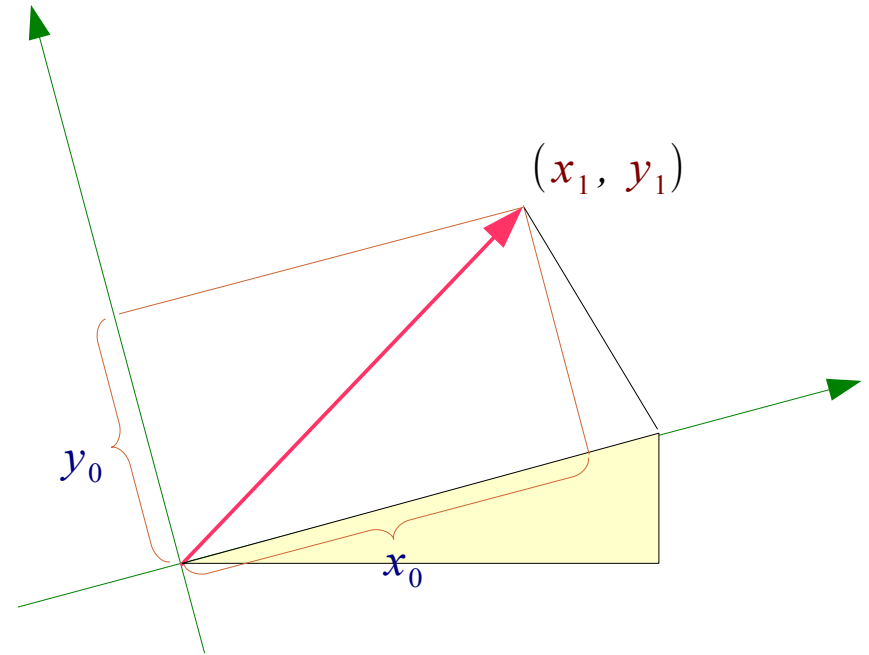
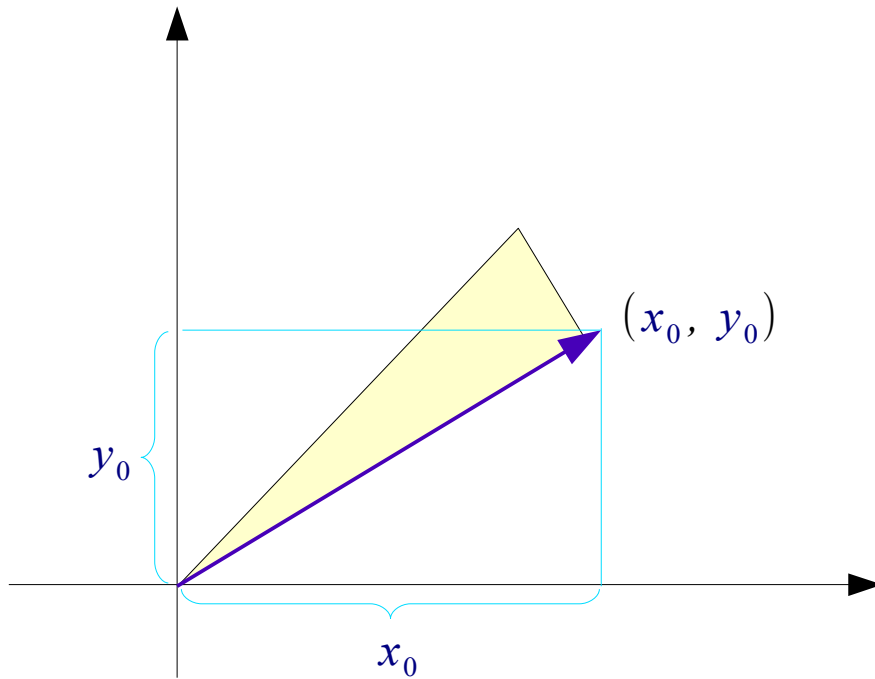


$$y_1 = x_0 \sin \alpha + y_0 \cos \alpha$$

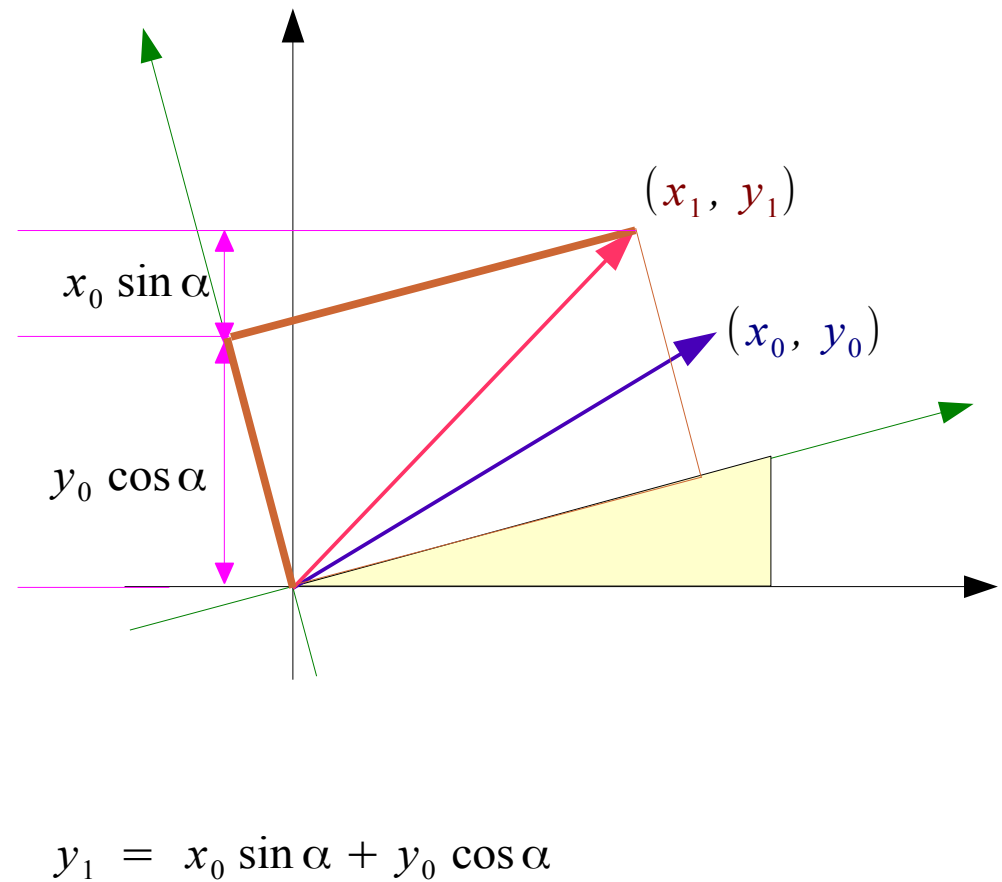
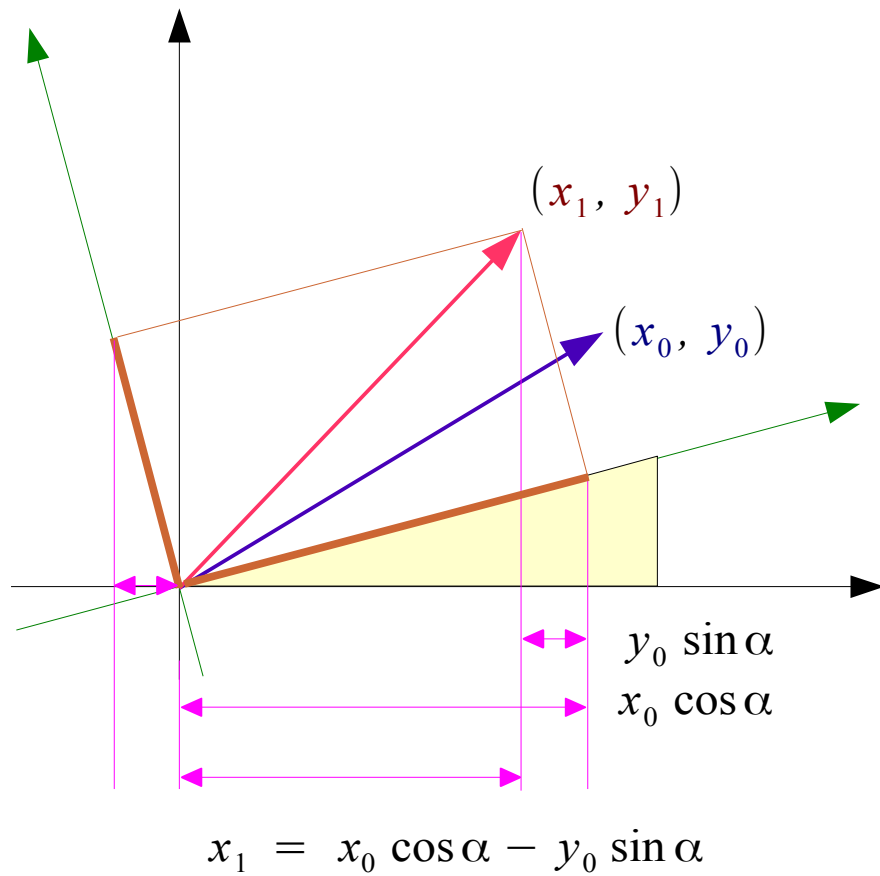


$$x_1 = x_0 \cos \alpha - y_0 \sin \alpha$$

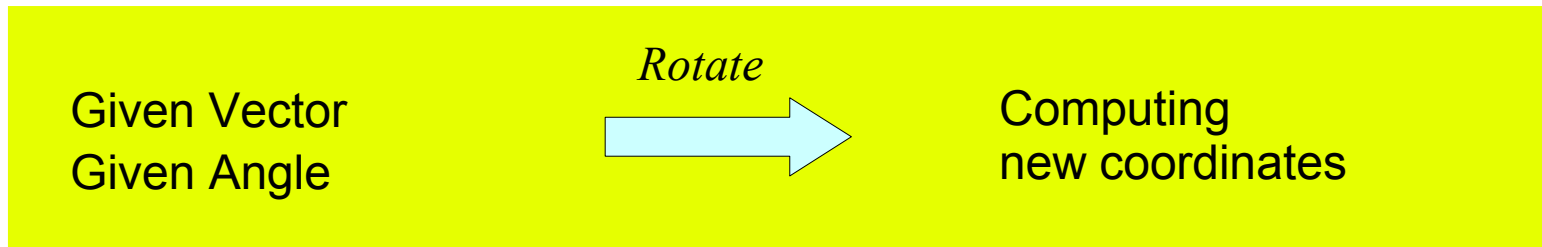
Vector Rotation (2)



Vector Rotation (3)

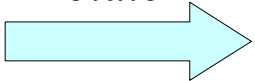


Iterative Rotation



Given Unit Vector
Given Angle α

Rotate




$$x = \cos \alpha$$

$$y = \sin \alpha$$

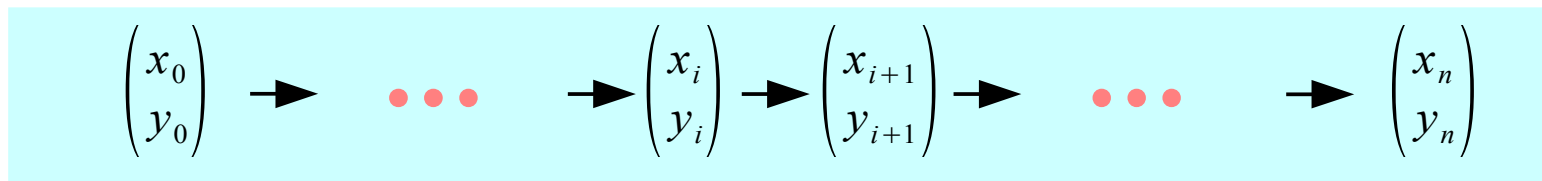
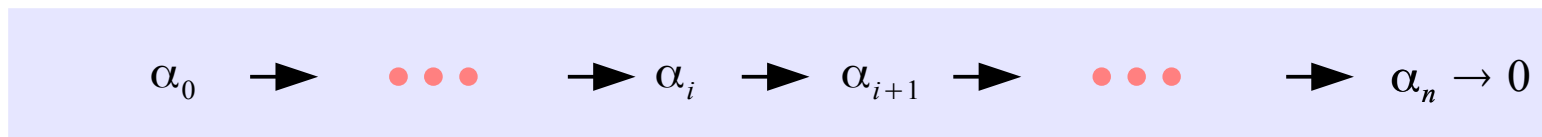
Given Vector (x_0, y_0)
Given Angle α

Rotate

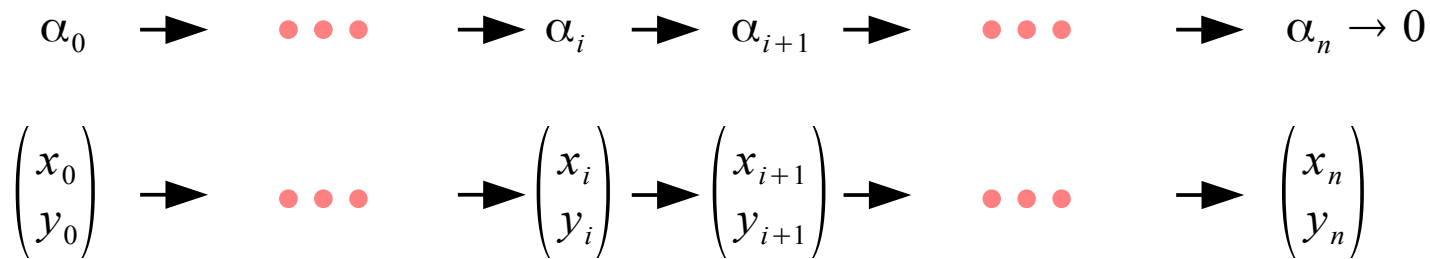


$$x_n = x_0 \cos \alpha - y_0 \sin \alpha$$

$$y_n = x_0 \sin \alpha + y_0 \cos \alpha$$



CORDIC Rotation



$$\begin{cases} x_{i+1} = x_i \cos \alpha_i - y_i \sin \alpha_i \\ y_{i+1} = x_i \sin \alpha_i + y_i \cos \alpha_i \end{cases}$$

$$\begin{cases} x_{i+1} = \cos \alpha_i (x_i - y_i \tan \alpha_i) \\ y_{i+1} = \cos \alpha_i (x_i \tan \alpha_i + y_i) \end{cases}$$

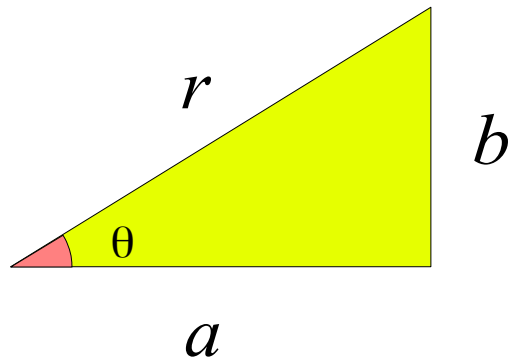
$$\begin{cases} x_{i+1} = \frac{1}{\sqrt{1 + \tan^2 \alpha_i}} (x_i - y_i \tan \alpha_i) \\ y_{i+1} = \frac{1}{\sqrt{1 + \tan^2 \alpha_i}} (x_i \tan \alpha_i + y_i) \end{cases}$$



Pseudo-rotation

$$\begin{cases} x'_{i+1} = (x_i - y_i \tan \alpha_i) \\ y'_{i+1} = (x_i \tan \alpha_i + y_i) \end{cases}$$

COS θ



$$\cos \theta = \frac{a}{r}$$

$$\sin \theta = \frac{b}{r}$$

$$\tan \theta = \frac{b}{a}$$

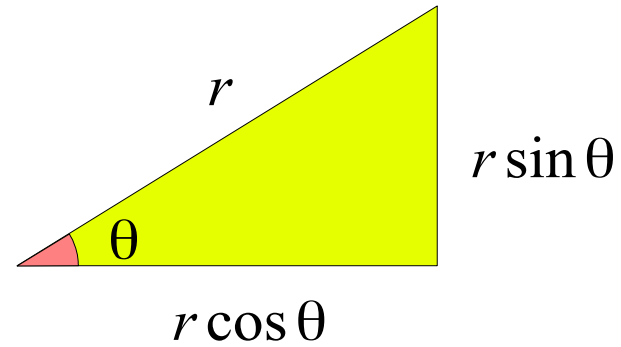
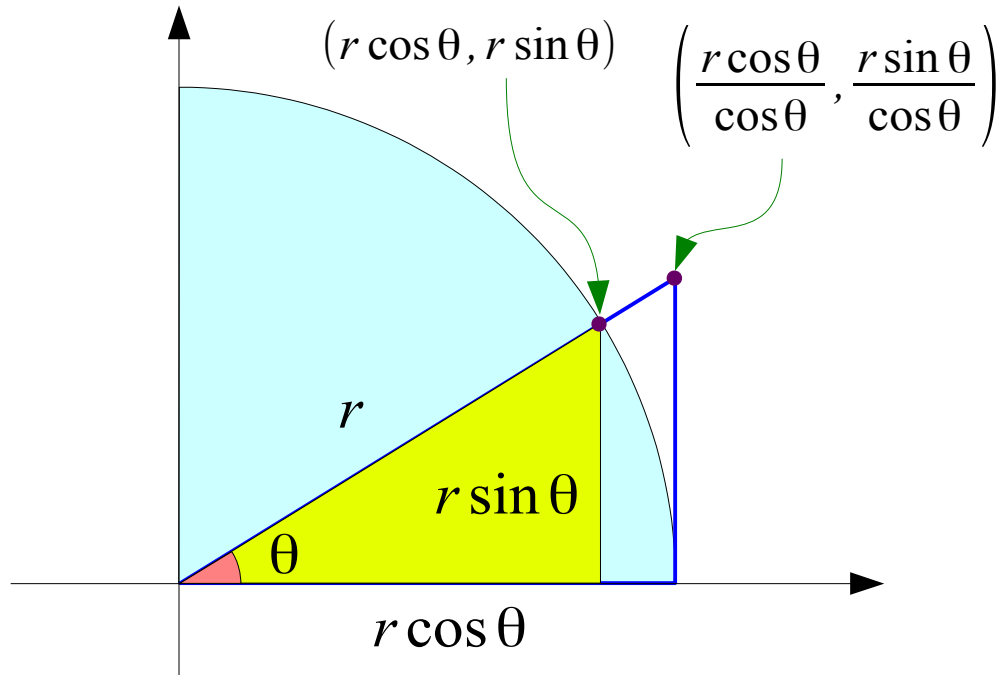
$$r = \sqrt{a^2 + b^2}$$

$$\cos \theta = \frac{a}{r} = \frac{a}{\sqrt{a^2 + b^2}}$$

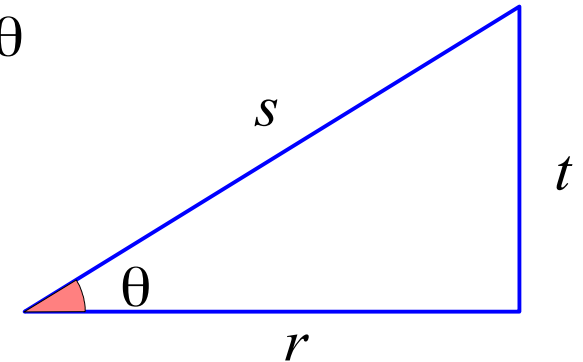
$$= \frac{1}{\sqrt{1 + (b/a)^2}}$$

$$= \frac{1}{\sqrt{1 + \tan^2 \theta}}$$

Pseudo-rotation – factor of $1/\cos \theta$



$r \cos \theta$



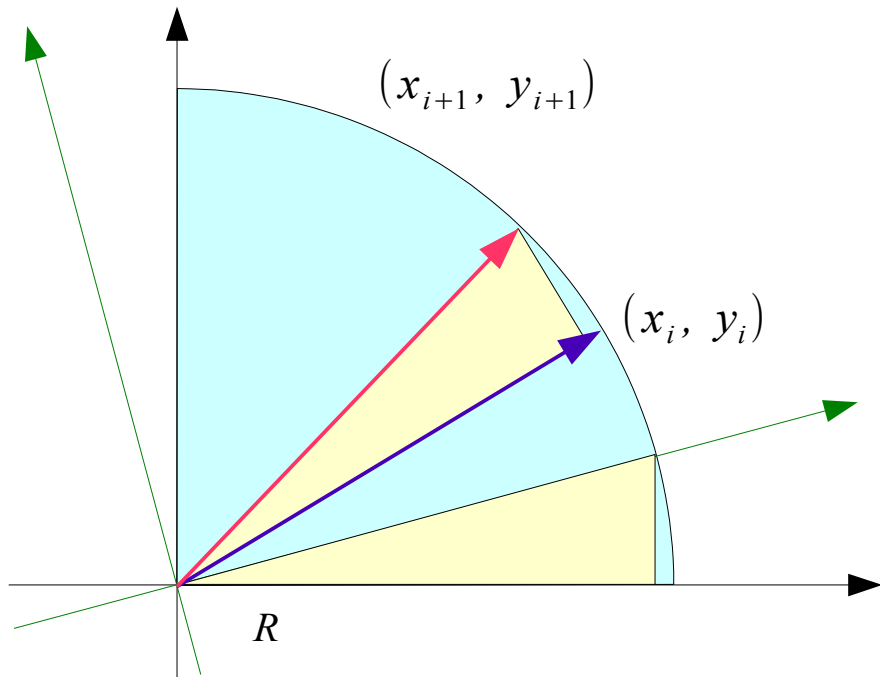
$$r : r \cos \theta = s : r$$

$$r \cos \theta : r \sin \theta = r : t$$

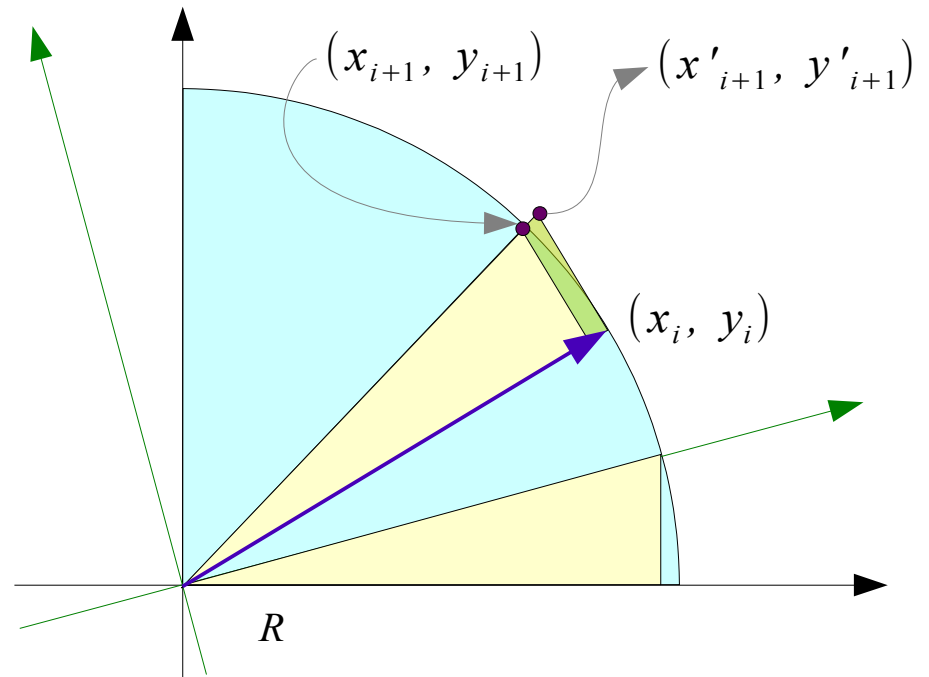
$$s = \frac{r}{\cos \theta}$$

$$s = \frac{r \sin \theta}{\cos \theta}$$

Pseudo-rotation (1)

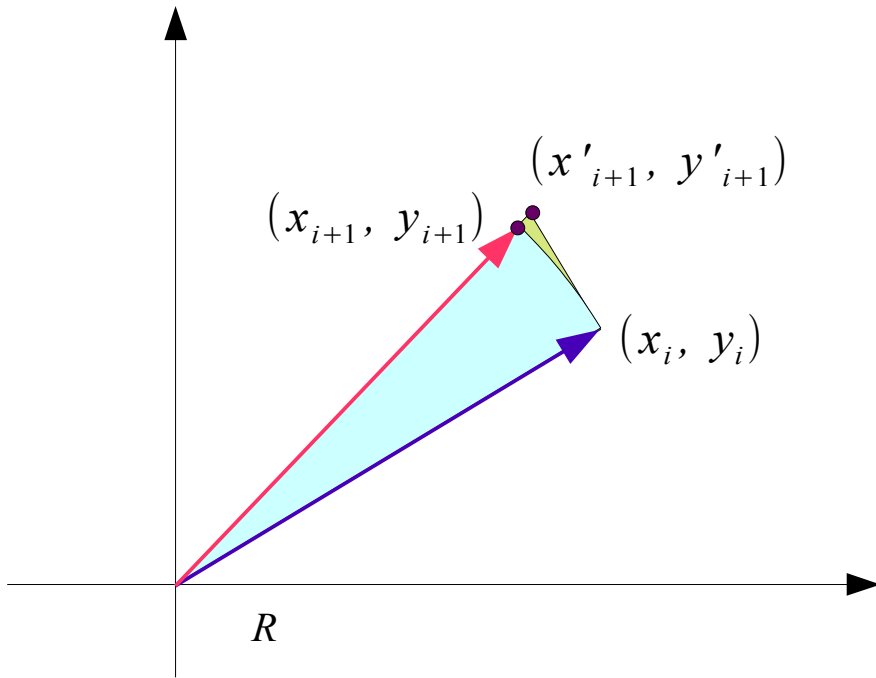


$$\begin{cases} x_{i+1} = \cos \alpha_i (x_i - y_i \tan \alpha_i) \\ y_{i+1} = \cos \alpha_i (x_i \tan \alpha_i + y_i) \end{cases}$$

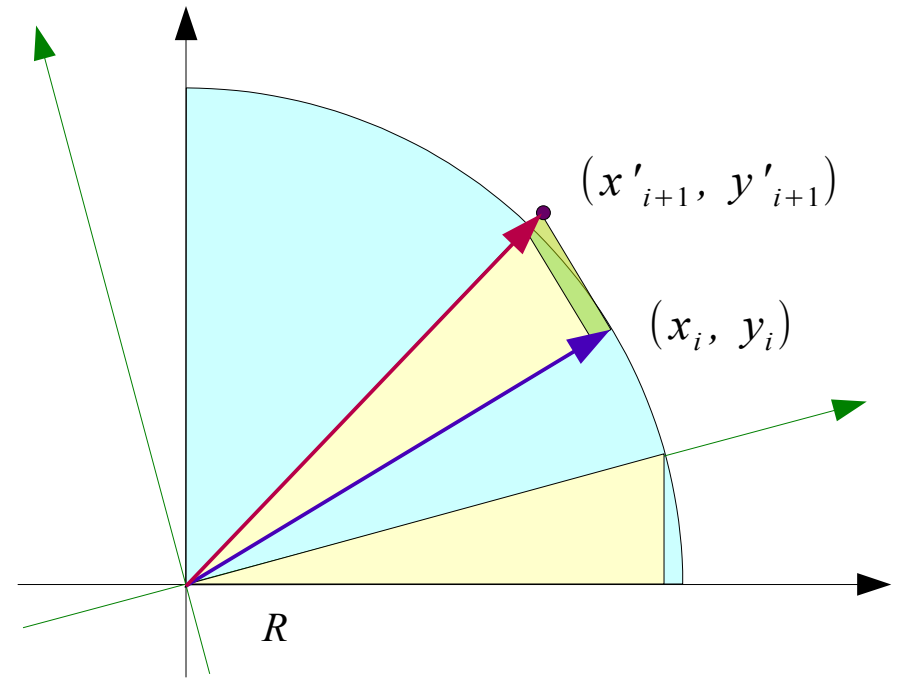


$$\begin{cases} x'_{i+1} = (x_i - y_i \tan \alpha_i) \\ y'_{i+1} = (x_i \tan \alpha_i + y_i) \end{cases}$$

Pseudo-rotation (2)

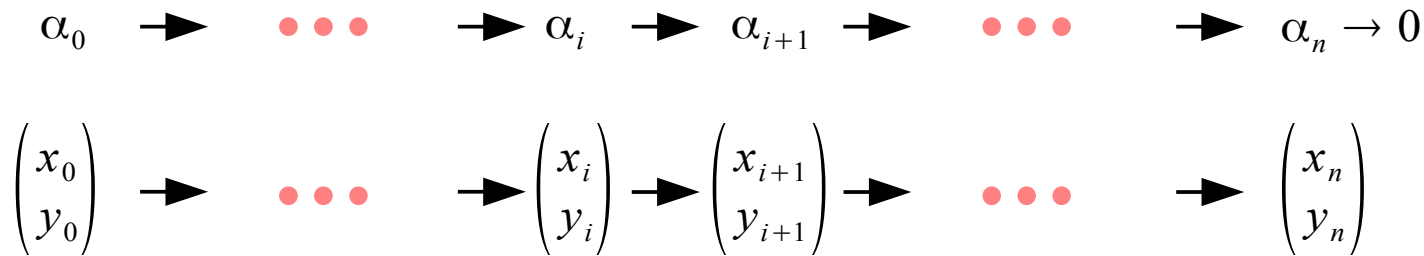


$$\begin{aligned}x'_{i+1} &= x_{i+1} / \cos \alpha_i \\y'_{i+1} &= y_{i+1} / \cos \alpha_i\end{aligned}$$



$$\begin{aligned}x'_{i+1} &> x_{i+1} \\y'_{i+1} &> y_{i+1}\end{aligned}$$

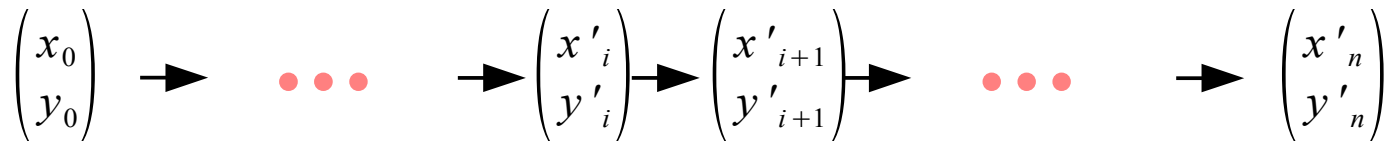
CORDIC Iteration Equations (1)



$$x_{i+1} = x_i \cos \alpha_i - y_i \sin \alpha_i = \cos \alpha_i (x_i - y_i \tan \alpha_i) = (1/\sqrt{1 + \tan^2 \alpha_i}) (x_i - y_i \tan \alpha_i)$$

$$y_{i+1} = x_i \sin \alpha_i + y_i \cos \alpha_i = \cos \alpha_i (x_i \tan \alpha_i + y_i) = (1/\sqrt{1 + \tan^2 \alpha_i}) (x_i \tan \alpha_i + y_i)$$

Pseudo-rotation



$$x'_{i+1} = (x'_i - y'_i \tan \alpha_i) \sqrt{1 + \tan^2 \alpha_i} = (x'_i \cos \alpha_i - y'_i \sin \alpha_i) \sqrt{1 + \tan^2 \alpha_i}$$

$$y'_{i+1} = (x'_i \tan \alpha_i + y'_i) \sqrt{1 + \tan^2 \alpha_i} = (x'_i \sin \alpha_i + y'_i \cos \alpha_i) \sqrt{1 + \tan^2 \alpha_i}$$

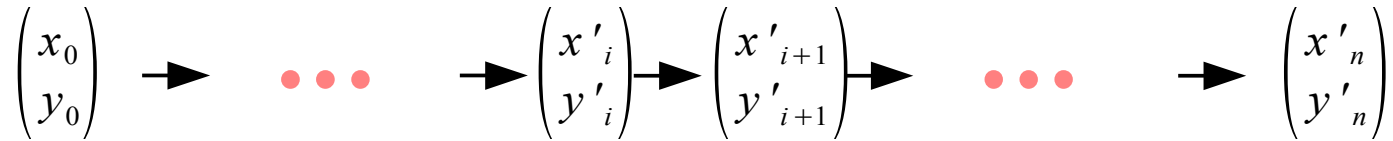
$$x'_n = \{x_0 \cos(\sum \alpha_i) - y_0 \sin(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\}$$

$$y'_n = \{x_0 \sin(\sum \alpha_i) + y_0 \cos(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\}$$

$$\alpha_n = \alpha - \sum \alpha_i$$

CORDIC Iteration Equations (2)

Pseudo-rotation



$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i \tan \alpha_i) &= (x'_i \cos \alpha_i - y'_i \sin \alpha_i) \sqrt{1 + \tan^2 \alpha_i} \\ y'_{i+1} &= (x'_i \tan \alpha_i + y'_i) &= (x'_i \sin \alpha_i + y'_i \cos \alpha_i) \sqrt{1 + \tan^2 \alpha_i} \end{aligned}$$

$$x'_n = \{x_0 \cos(\sum \alpha_i) - y_0 \sin(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\}$$

$$y'_n = \{x_0 \sin(\sum \alpha_i) + y_0 \cos(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\}$$

$$\alpha_n = \alpha - \sum \alpha_i$$

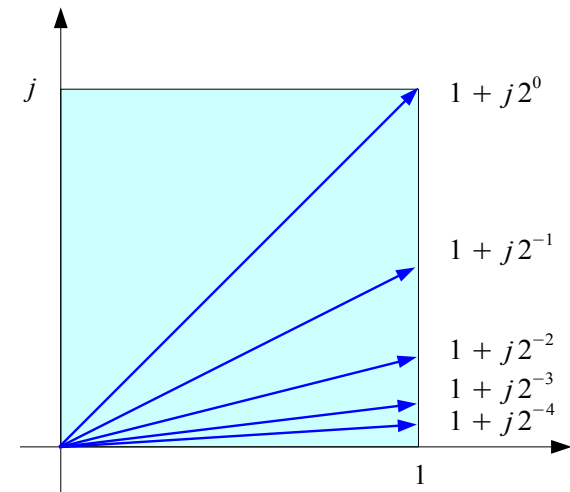
Choose α_i such that $\tan \alpha_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

$$\tan \alpha_i = \sigma_i 2^{-i} \quad \sigma_i \in \{+1, -1\}$$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

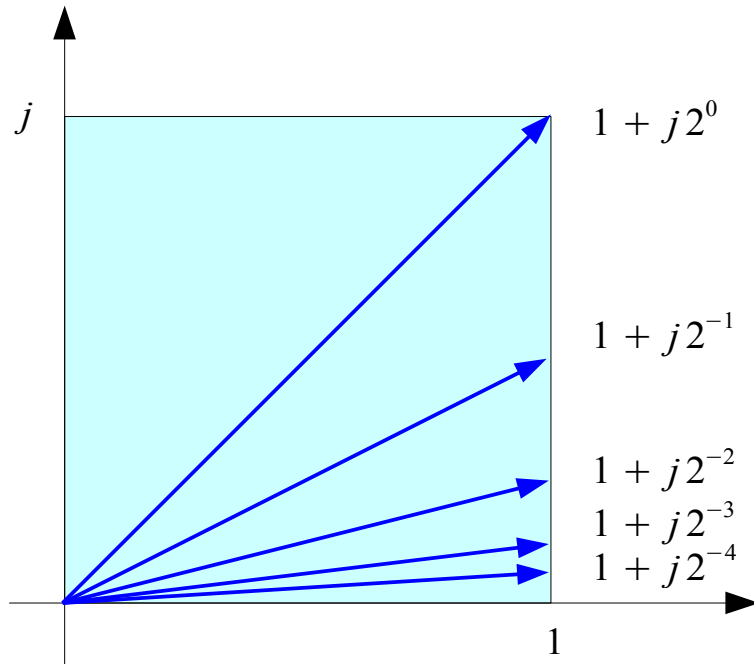
$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan(\sigma_i 2^{-i})$$



CORDIC Iteration Equations (2)

Choose α_i such that $\tan \alpha_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$
 $\tan \alpha_i = \sigma_i 2^{-i} \quad \sigma_i \in \{+1, -1\}$



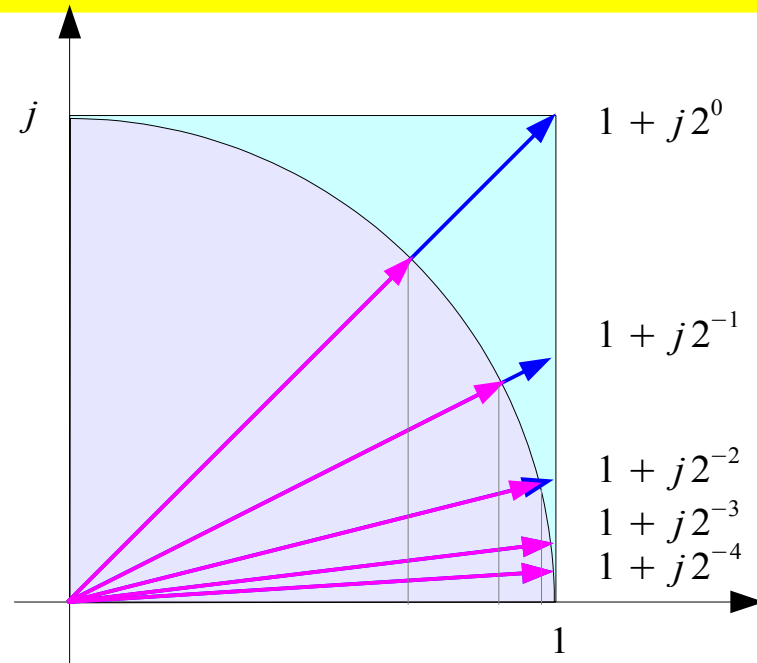
$$\tan \alpha_i = \pm 2^{-i} \quad \cos \alpha_i = \frac{+1}{\sqrt{1 + 2^{-2i}}}$$

$$\sin \alpha_i = \frac{\pm 2^{-i}}{\sqrt{1 + 2^{-2i}}}$$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan(\sigma_i 2^{-i})$$



$$\begin{pmatrix} +\cos \alpha_i & -\sin \alpha_i \\ +\sin \alpha_i & +\cos \alpha_i \end{pmatrix} = \frac{1}{\sqrt{1 + 2^{-2i}}} \begin{pmatrix} +1 & \mp 2^{-i} \\ \pm 2^{-i} & +1 \end{pmatrix}$$

CORDIC Iteration Equations (3)

Pseudo-rotation

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \dots \rightarrow \begin{pmatrix} x'_i \\ y'_i \end{pmatrix} \rightarrow \begin{pmatrix} x'_{i+1} \\ y'_{i+1} \end{pmatrix} \rightarrow \dots \rightarrow \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}$$

$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i \tan \alpha_i) = (x'_i \cos \alpha_i - y'_i \sin \alpha_i) \sqrt{1 + \tan^2 \alpha_i} \\ y'_{i+1} &= (x'_i \tan \alpha_i + y'_i) = (x'_i \sin \alpha_i + y'_i \cos \alpha_i) \sqrt{1 + \tan^2 \alpha_i} \end{aligned}$$

$$x'_n = \{x_0 \cos(\sum \alpha_i) - y_0 \sin(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\}$$

$$y'_n = \{x_0 \sin(\sum \alpha_i) + y_0 \cos(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\}$$

$$\alpha_n = \alpha - \sum \alpha_i$$

Choose α_i such that $\tan \alpha_i = \sigma_i 2^{-i}$ $\sigma_i \in \{+1, -1\}$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_n = K(x_0 \cos \alpha - y_0 \sin \alpha)$$

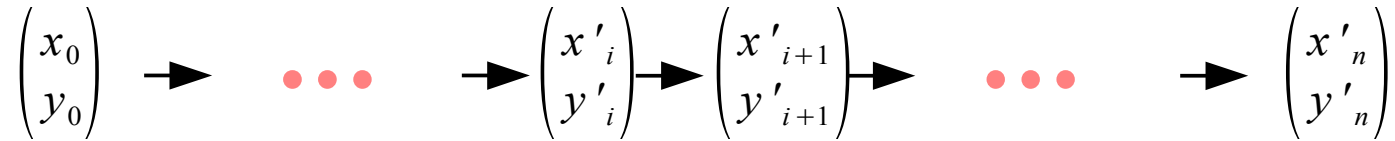
$$y'_n = K(x_0 \sin \alpha + y_0 \cos \alpha)$$

$$\alpha_n = \alpha - \sum \tan^{-1}(\sigma_i 2^{-i})$$

$$K = \prod \{\sqrt{1 + \tan^2 \alpha_i}\}$$

CORDIC Iteration Equations (4)

Pseudo-rotation



Choose α_i such that $\tan \alpha_i = \sigma_i 2^{-i}$ $\sigma_i \in \{+1, -1\}$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_n = K(x_0 \cos \alpha - y_0 \sin \alpha)$$

$$y'_n = K(x_0 \sin \alpha + y_0 \cos \alpha)$$

$$\alpha_n = \alpha - \sum \tan^{-1}(\sigma_i 2^{-i})$$

pre-compute $K = \prod \{\sqrt{1 + \tan^2 \alpha_i}\} = 1.647$

set $x_0 = 1/K = 0.607$

$$y_0 = 0$$

then $x'_n = K(1/K \cos \alpha - 0 \cdot \sin \alpha) = \cos \alpha$

$$y'_n = K(1/K \sin \alpha + 0 \cdot \cos \alpha) = \sin \alpha$$

References

- [1] <http://en.wikipedia.org/>
- [2] CORDIC FAQ, www.dspguru.com
- [3] R. Andraka, A survey of CORDIC algorithms for FPGA based computers
- [4] J. S. Walther, A Unified Algorithm for Elementary Functions
- [5] J. P. Deschamps, G. A. Bioul, G.D. Sutter, Synthesis of Arithmetic Circuits