

# Sampler Spectra (8B)

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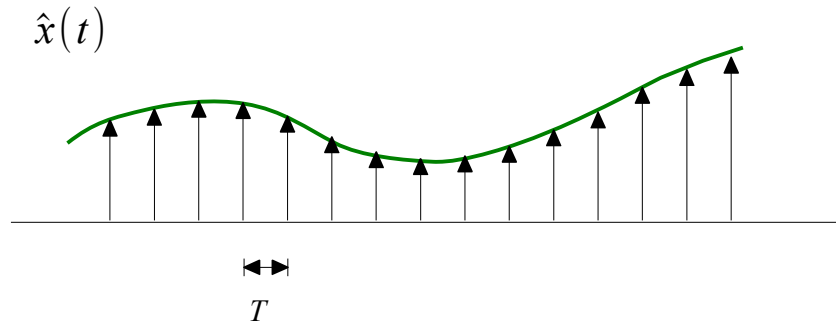
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# Sampler

## Ideal Sampling

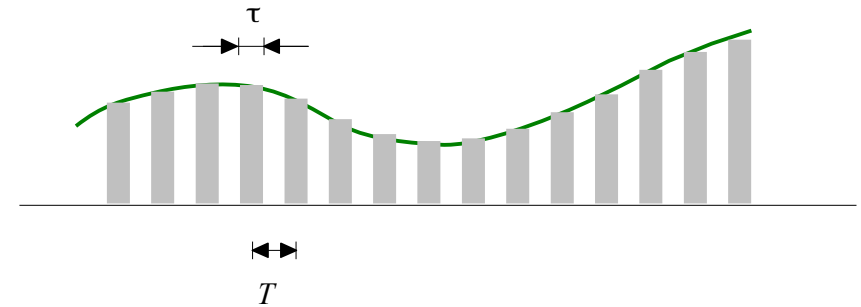


$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

CTFT

$$\hat{X}(f) = \int_{-\infty}^{+\infty} \hat{x}(t) e^{-j2\pi ft} dt$$

## Practical Sampling

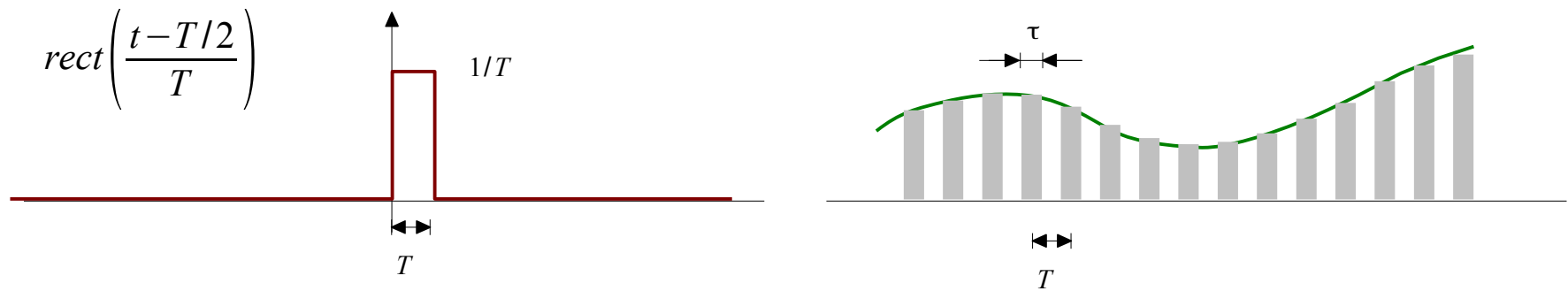


$$\hat{x}(t) \approx \sum_{n=-\infty}^{+\infty} x(nT) p(t-nT)$$

CTFT

?

# Zero Order Hold (ZOH)



$$x_{ZOH}(t) = \sum_{n=-\infty}^{+\infty} x[n] \cdot rect\left(\frac{t-T/2-nT}{T}\right)$$

# Square Wave CTFS (1)

## Continuous Time Fourier Series

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt \quad \longleftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(t) e^{-jk\omega_0 t} dt$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t) e^{-jk\omega_0 t} dt$$

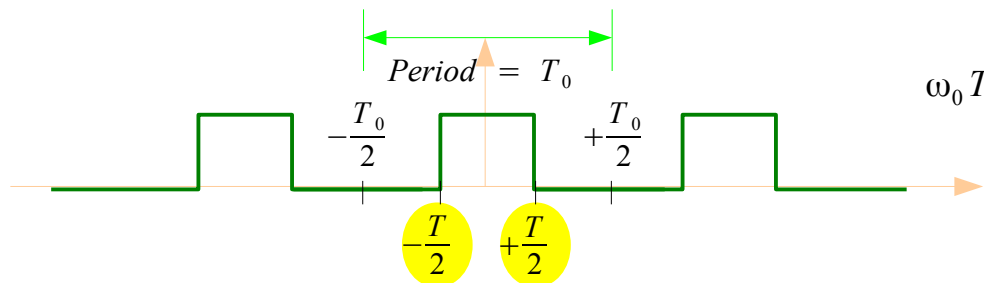
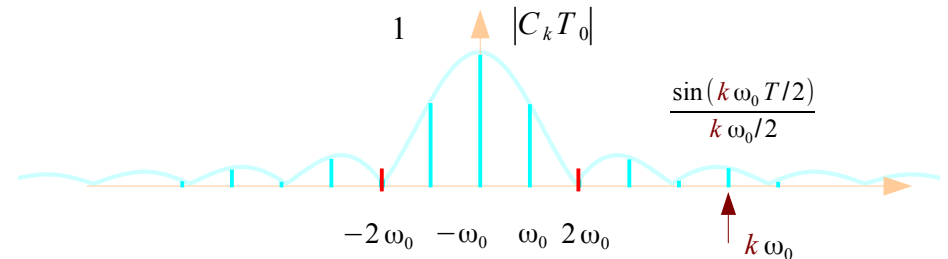
$$= \int_{-T_0/2}^{+T_0/2} e^{-jk\omega_0 t} dt = \left[ \frac{-1}{jk\omega_0} e^{-jk\omega_0 t} \right]_{-T_0/2}^{+T_0/2}$$

$$= \frac{e^{-jk\omega_0 T_0/2} - e^{+jk\omega_0 T_0/2}}{jk\omega_0} = \frac{\sin(k\omega_0 T_0/2)}{k\omega_0/2}$$

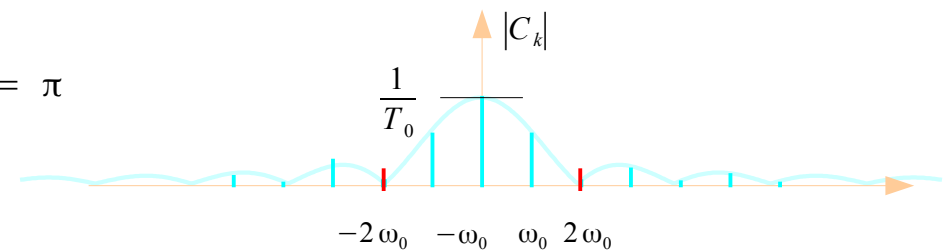
$$\omega_0 = \frac{2\pi}{T_0}$$

Fundamental Frequency

$$\omega_0 T_0 = 2\pi \quad \Rightarrow \quad T/T_0 = 1/2 \quad \Rightarrow \quad \omega_0 T = \pi$$



$$\omega_0 T = \pi$$



# Square Wave CTFS (2)

$$C_k = \frac{1}{T_0} \cdot \frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{1}{T_0} \cdot \frac{\sin(T k \omega_0/2)}{k\omega_0/2}$$

$\omega_0 = \frac{2\pi}{T_0}$

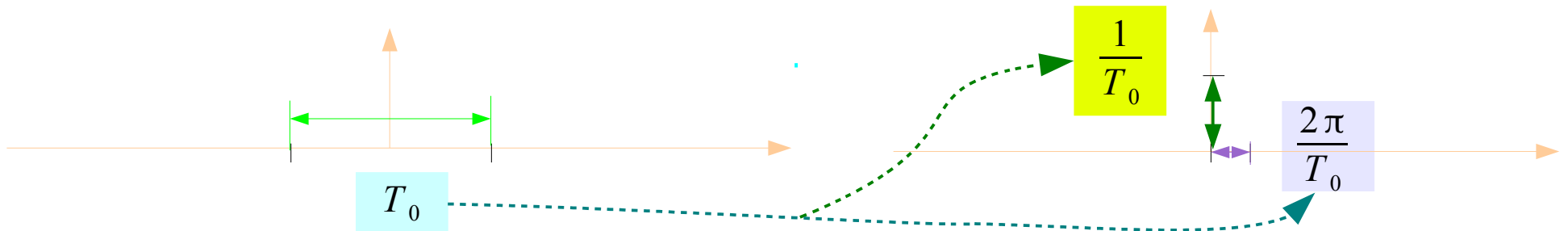
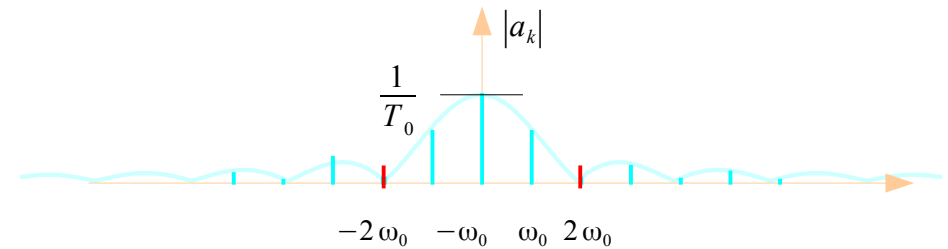
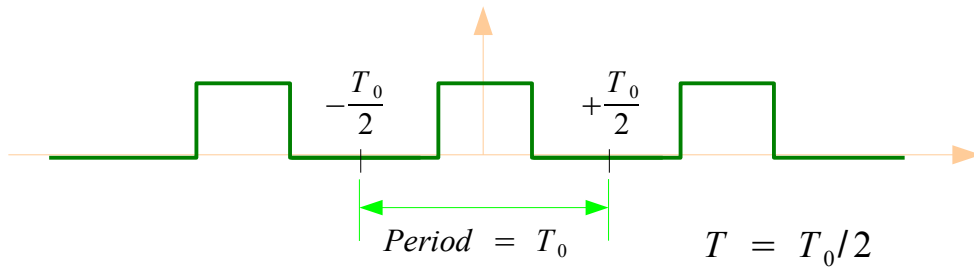
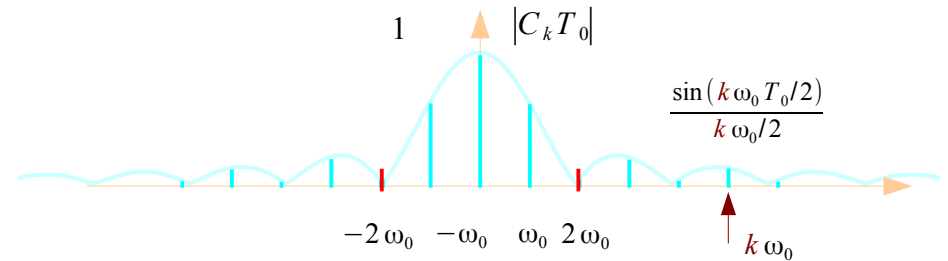
Fundamental Frequency

$$\sin(k\omega_0 T/2) = 0 \quad \rightarrow \quad T/T_0 = 1/2$$

$$\sin(k\omega_0 T_0/2/2) = 0 \quad \rightarrow \quad \sin(k\pi/2) = 0$$

$$C_k = 0 \quad k = \pm 2, \pm 4, \pm 6, \dots$$

$$\rightarrow \quad \omega = \pm 2\omega_0, \pm 4\omega_0, \pm 6\omega_0, \dots$$



# CTFT of a Rect(t/T) function (1)

## Continuous Time Fourier Transform

## Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

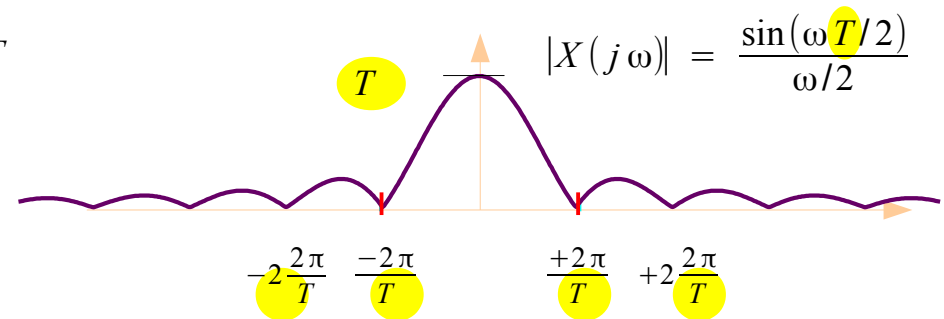
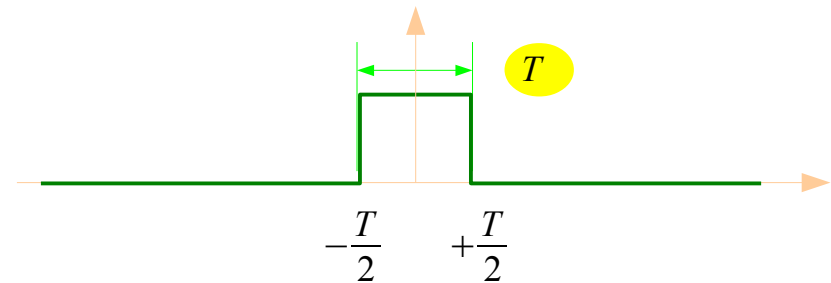
$$\begin{aligned} X(j\omega) &= \int_{-T/2}^{+T/2} e^{-j\omega t} dt \\ &= \left[ \frac{-1}{j\omega} e^{-j\omega t} \right]_{-T/2}^{+T/2} = -\frac{e^{-j\omega T/2} - e^{+j\omega T/2}}{j\omega} \\ &= \frac{\sin(\omega T/2)}{\omega/2} \end{aligned}$$

$$X(j0) = \lim_{\omega \rightarrow 0} \frac{\sin(\omega T/2)}{\omega/2} = \lim_{\omega \rightarrow 0} \frac{T \cos(\omega T/2)}{2 \cdot 1/2} = T$$

$$\sin(\omega T/2) = 0 \quad \rightarrow \quad \omega T/2 = \pi k$$

$$\rightarrow \quad \omega = \frac{2\pi k}{T}$$

$$\rightarrow \quad \omega = \pm \frac{2\pi}{T}, \pm 2 \frac{2\pi}{T}, \pm 3 \frac{2\pi}{T}, \dots$$



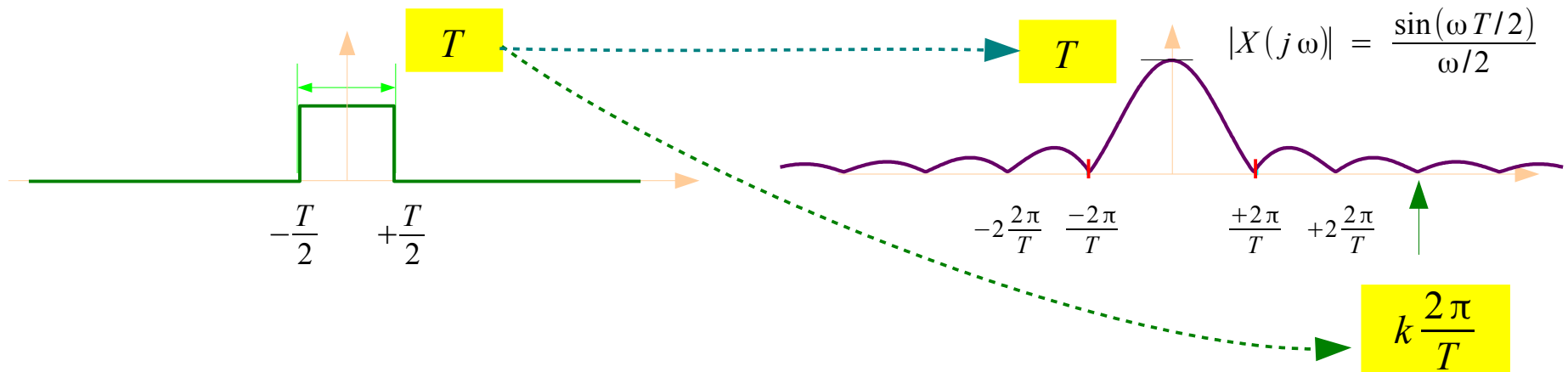
# CTFT of a Rect(t/T) function (2)

## Continuous Time Fourier Transform

Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

$$X(j\omega) = \int_{-T/2}^{+T/2} e^{-j\omega t} dt = \frac{\sin(\omega T/2)}{\omega/2}$$



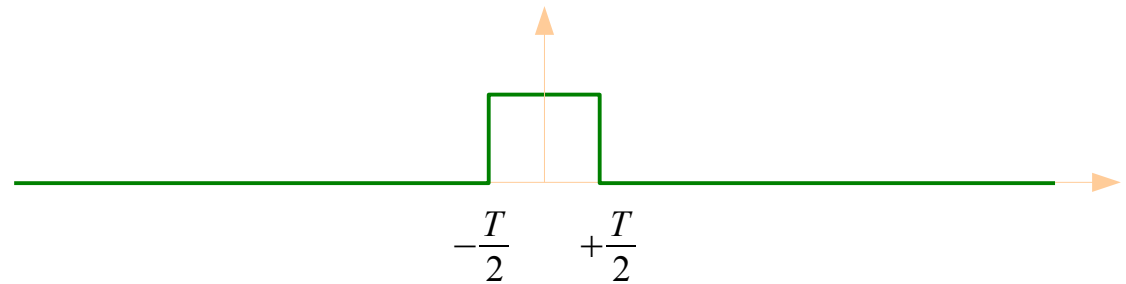


# CTFT and CTFS

## Continuous Time Fourier Transform

## Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

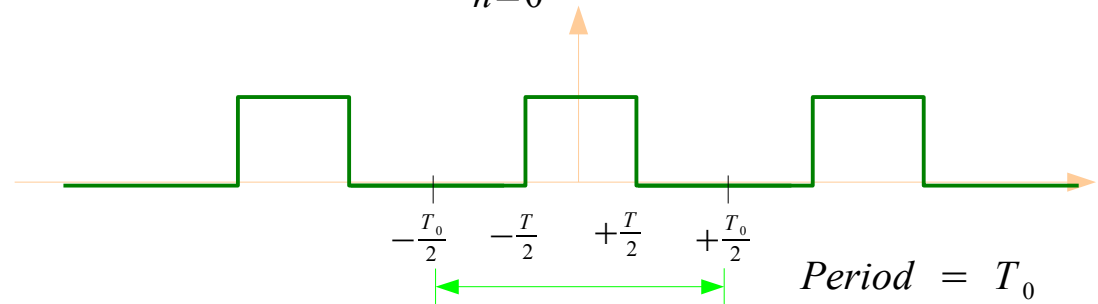


## Continuous Time Fourier Series

## Periodic Continuous Time Signal

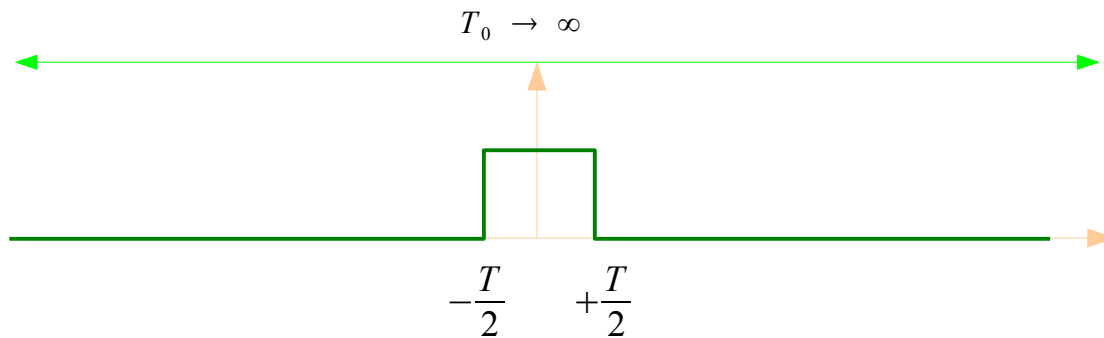
$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt \quad \longleftrightarrow \quad x(t) = \sum_{n=-\infty}^{\infty} C_k e^{+jk\omega_0 t}$$

Period =  $T_0$

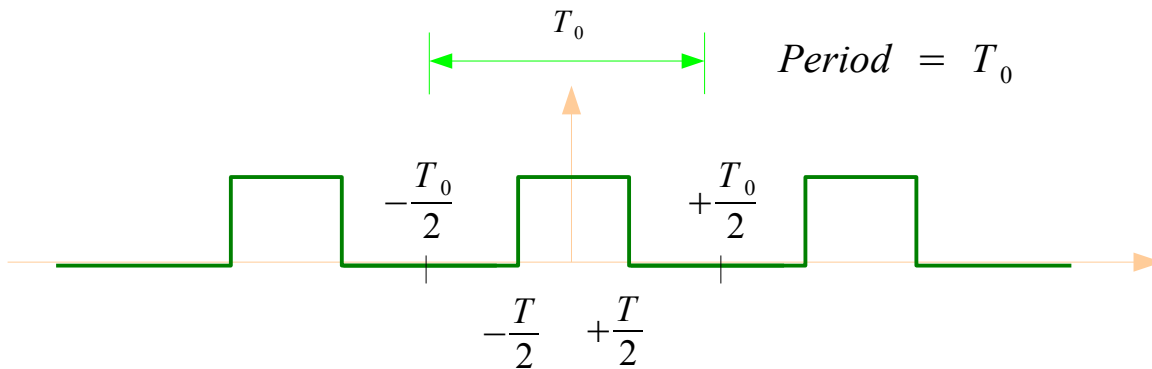


# CTFT ← CTFS

## Aperiodic Continuous Time Signal Continuous Time Fourier Transform



## Periodic Continuous Time Signal Continuous Time Fourier Series



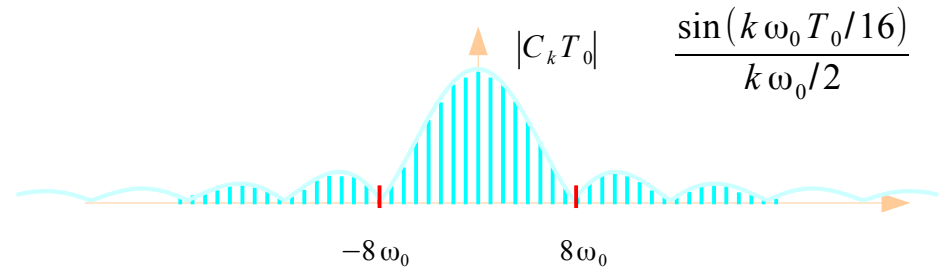
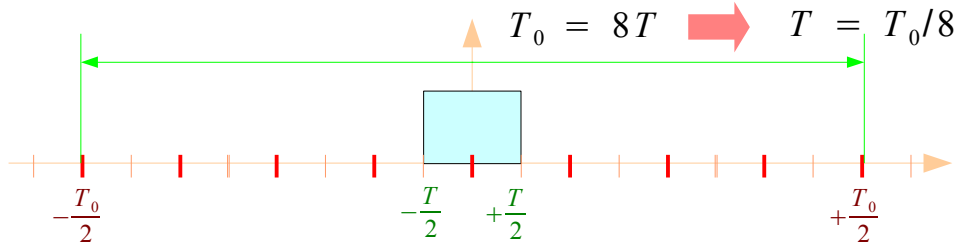
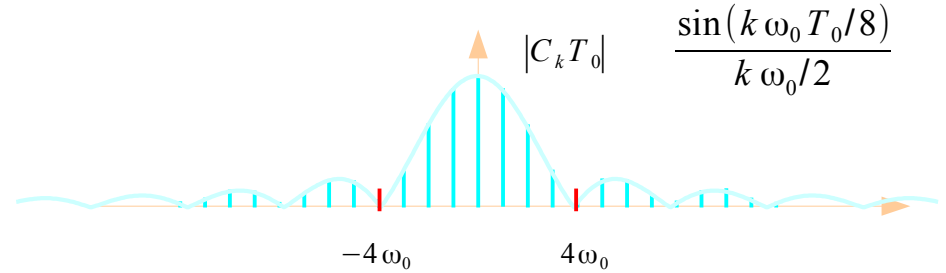
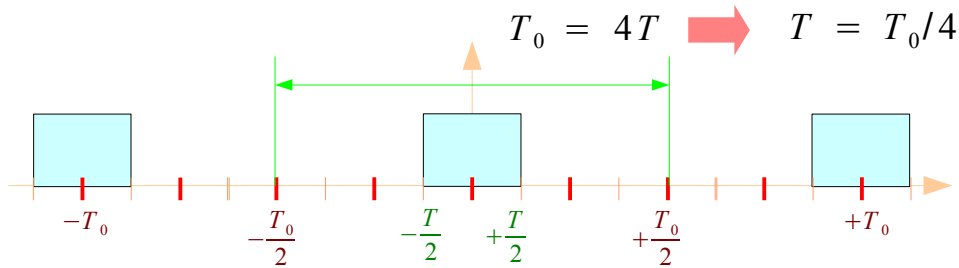
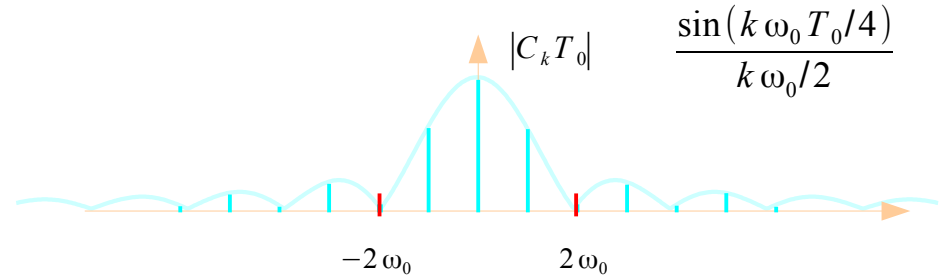
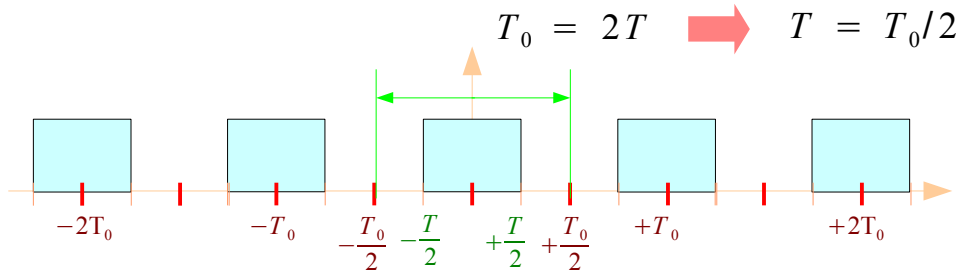
$$x(t)$$

$$\text{As } T_0 \rightarrow \infty, \\ x_{T_0}(t) \rightarrow x(t)$$

$$\omega_0 = \frac{2\pi}{T_0} \rightarrow 0$$

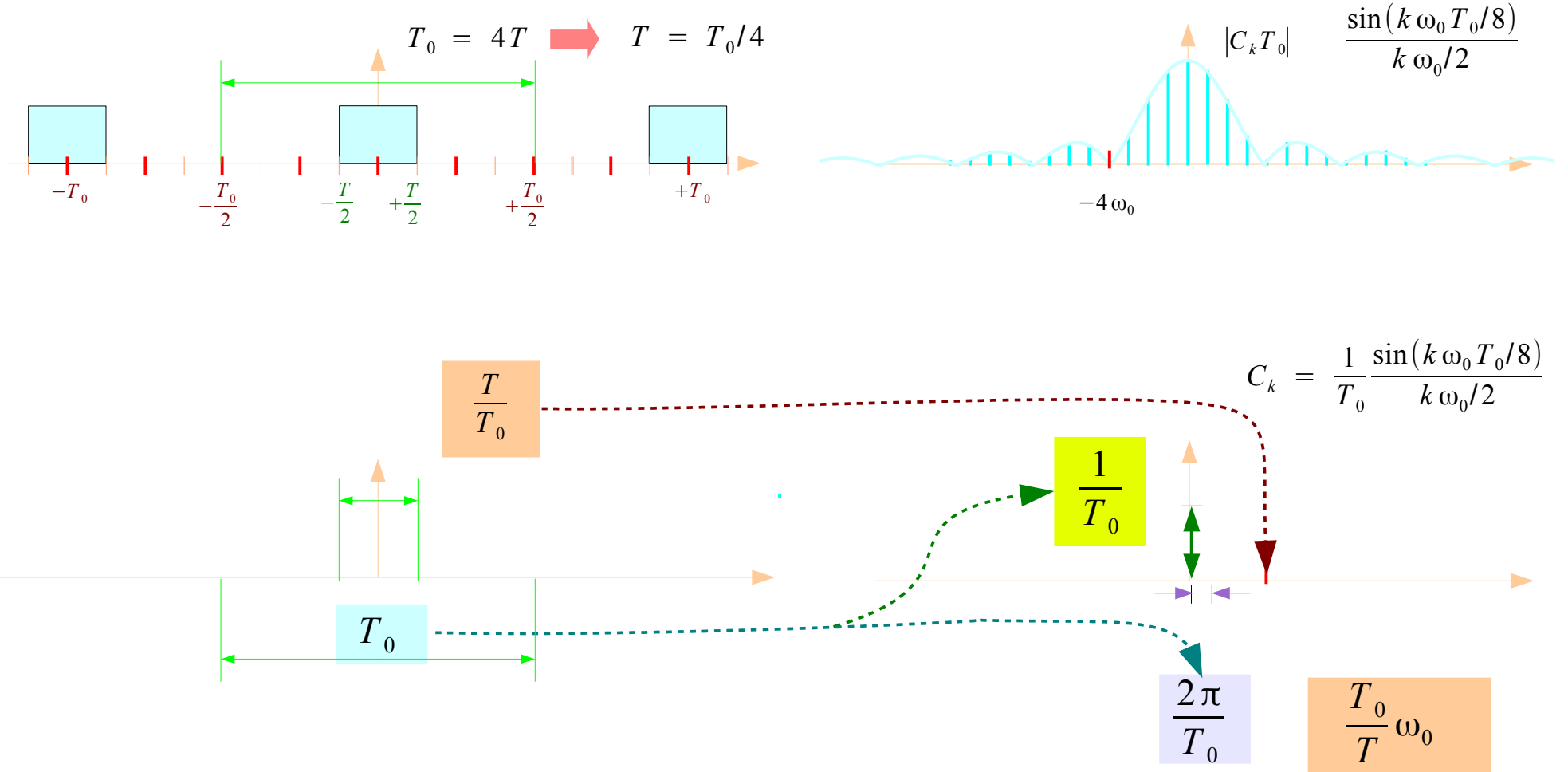
$$x_{T_0}(t) = \sum_{n=-\infty}^{+\infty} x(t - nT_0)$$

# CTFT and CTFS as $T_0 \rightarrow \infty$ (1)

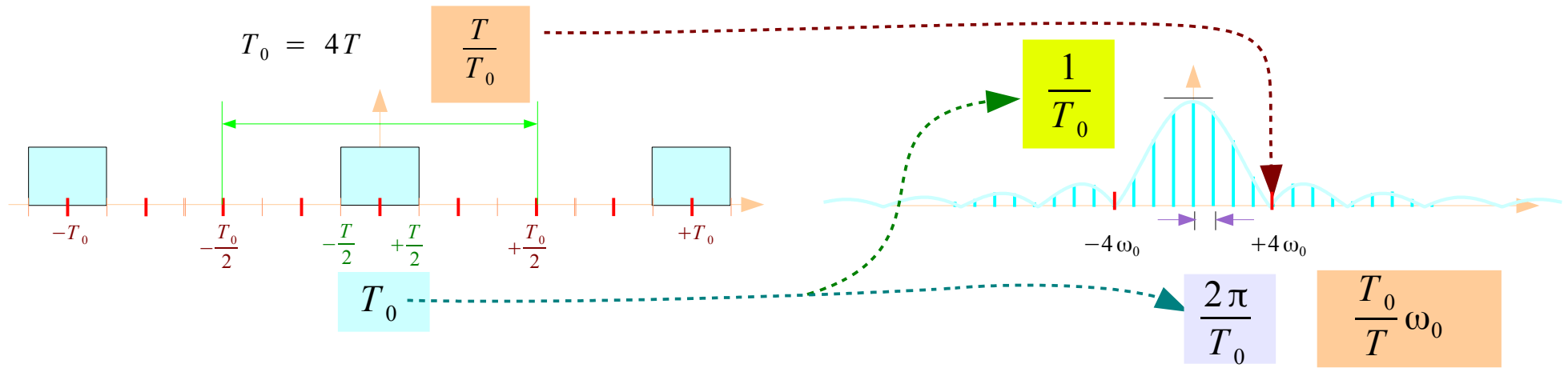


$$X(j\omega) = \lim_{k \omega_0 \rightarrow \omega} \frac{\sin(k \omega_0 T/2)}{k \omega_0/2} = \frac{\sin(\omega T/2)}{\omega/2}$$

# CTFT and CTFS as $T_0 \rightarrow \infty$ (2)



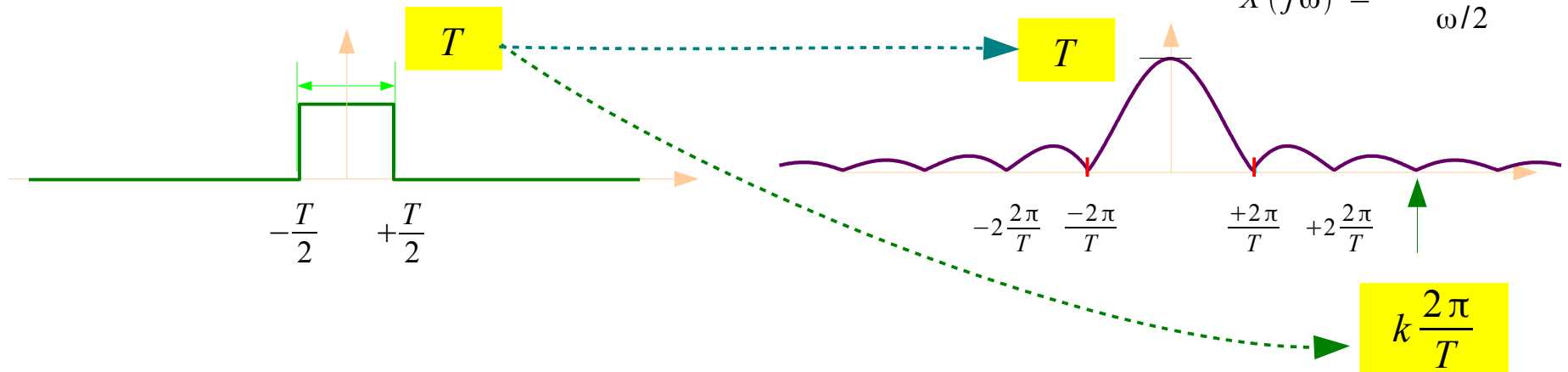
# CTFT of a Rect(t/T) function (3)



$$X(j\omega) = \lim_{k\omega_0 \rightarrow \omega} \frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{\sin(\omega T/2)}{\omega/2}$$

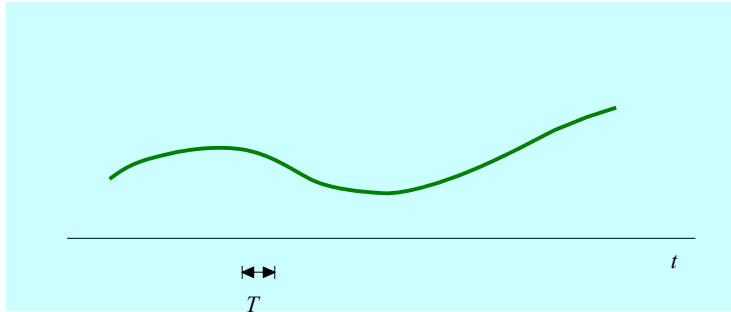
$$C_k = \frac{1}{T_0} \frac{\sin(k\omega_0 T_0/8)}{k\omega_0/2}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

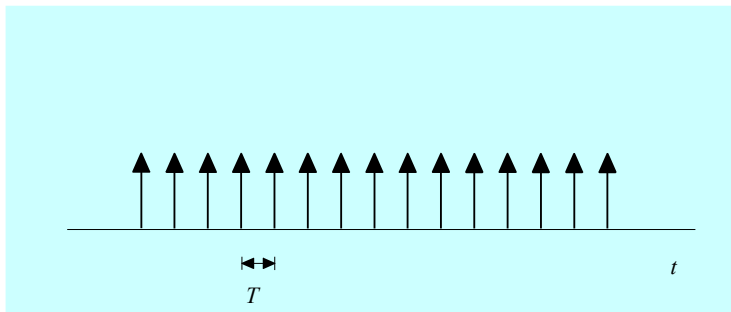


# Sampling (1)

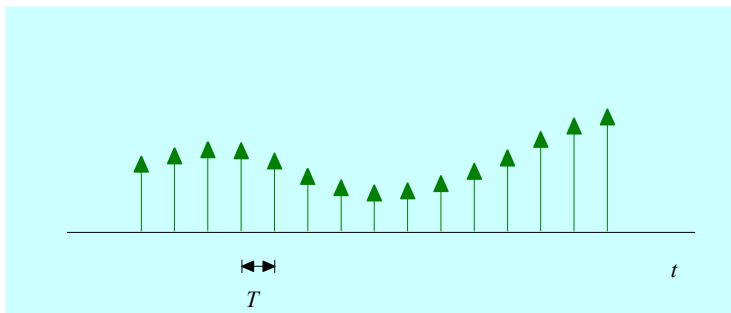
## Ideal Sampling



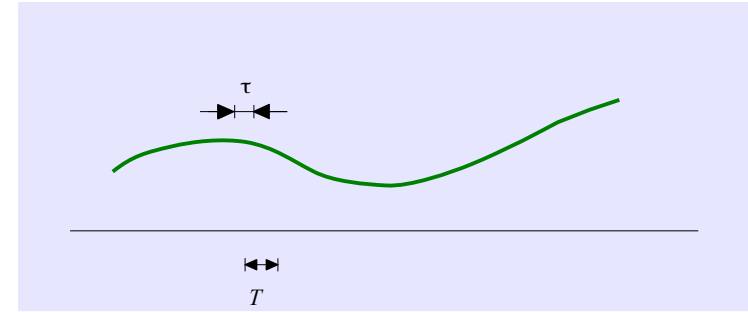
X



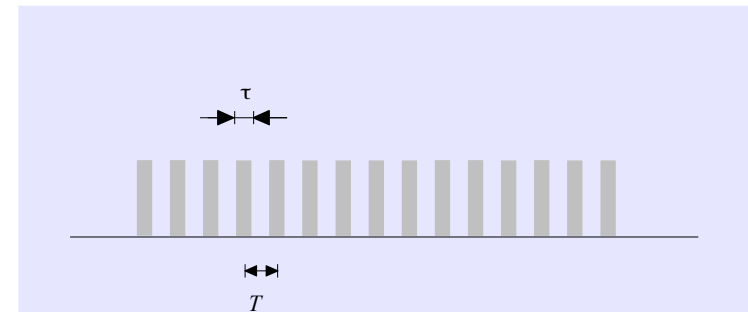
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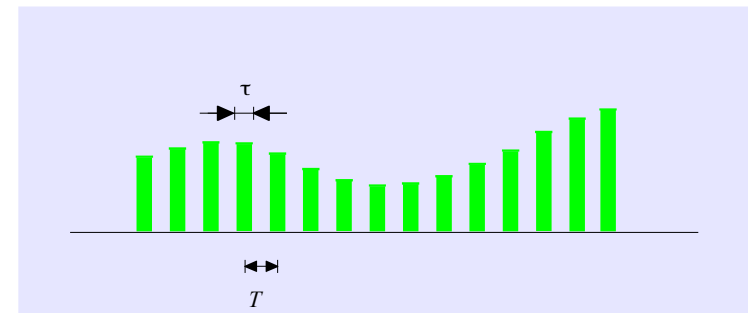
## Practical Sampling



X

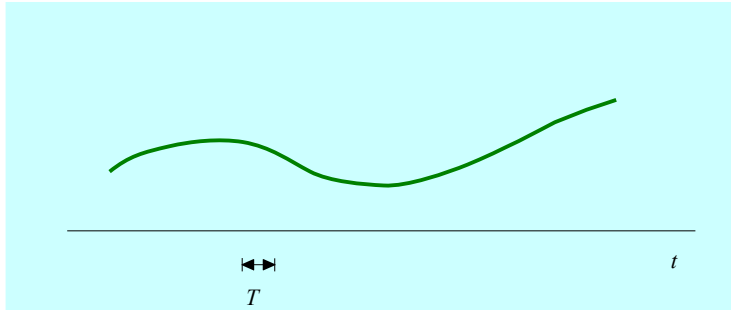


||



# Sampling (2)

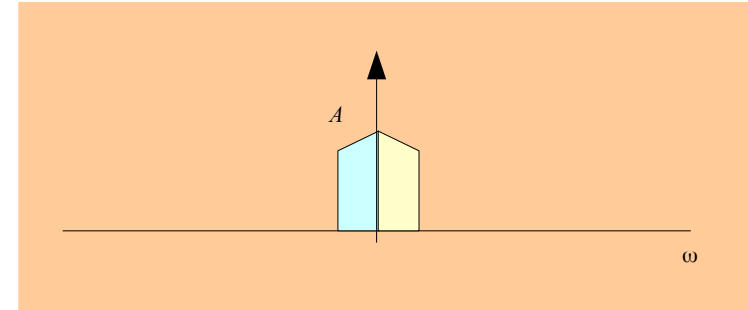
## Ideal Sampling



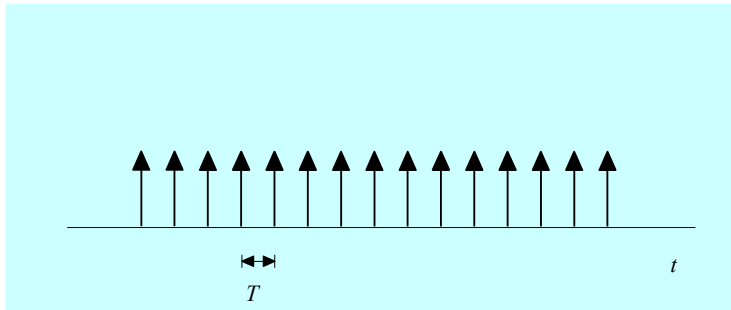
CTFT



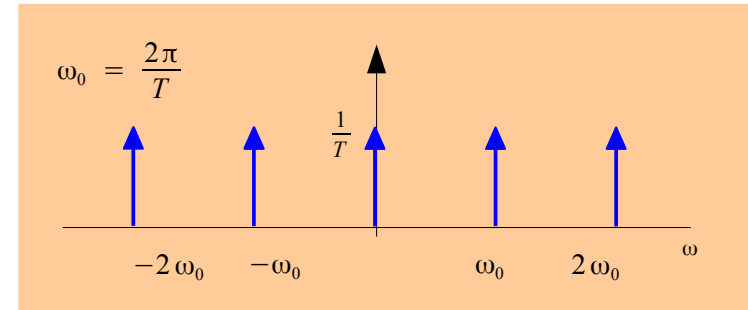
## Frequency Domain



X

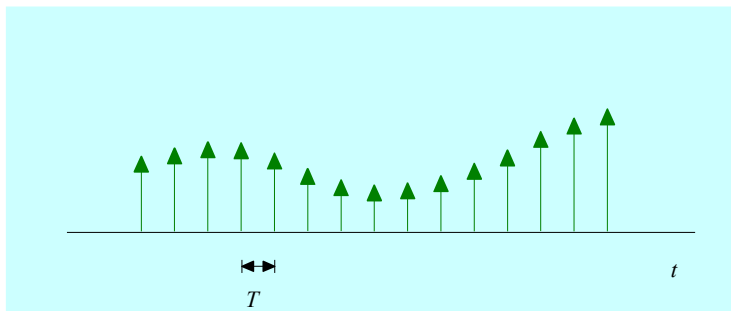


CTFT

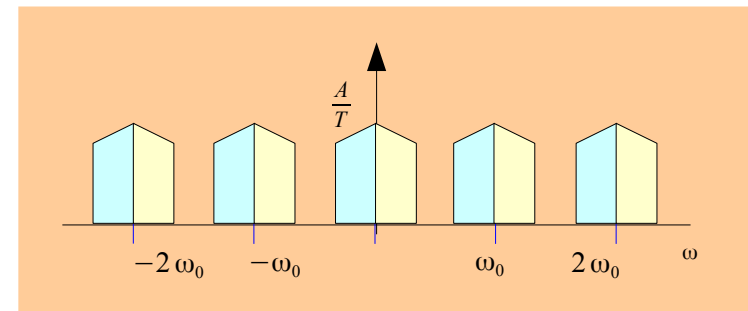


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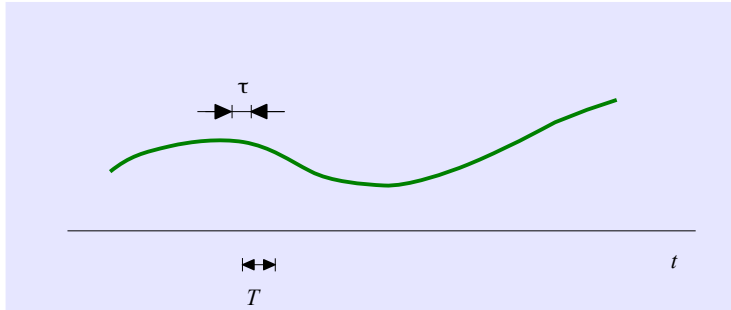
CTFT



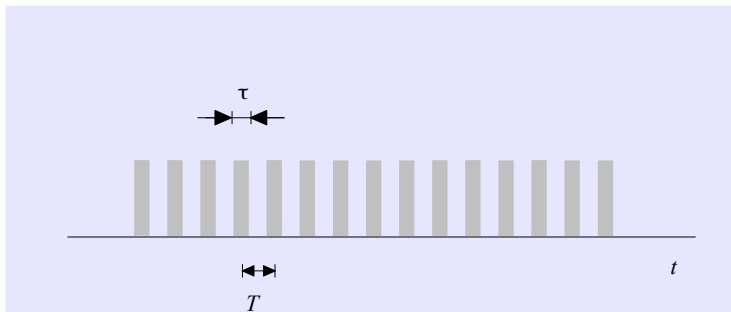
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# Sampling (3)

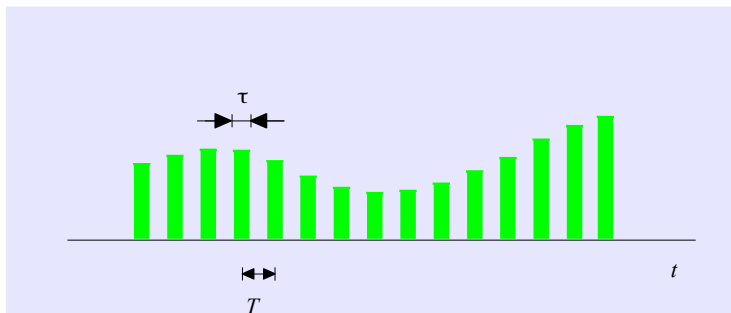
## Practical Sampling



X

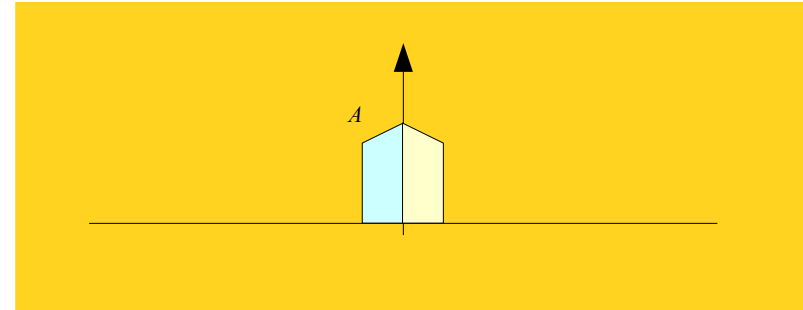


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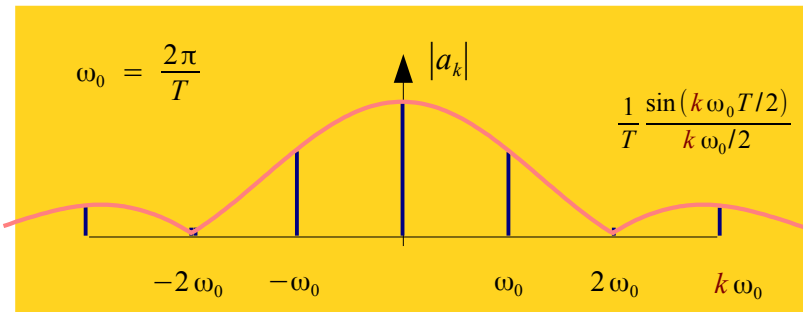
## Frequency Domain

CTFT



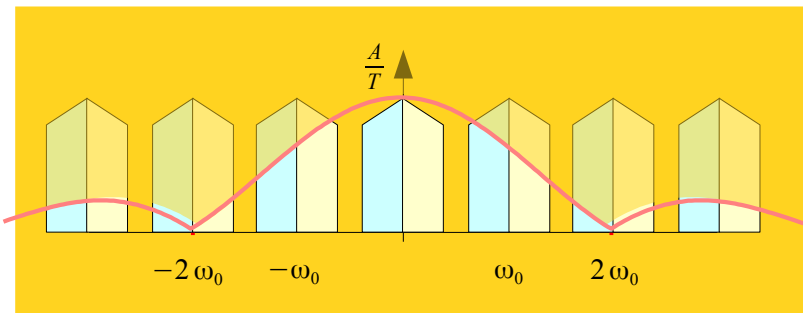
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CTFT



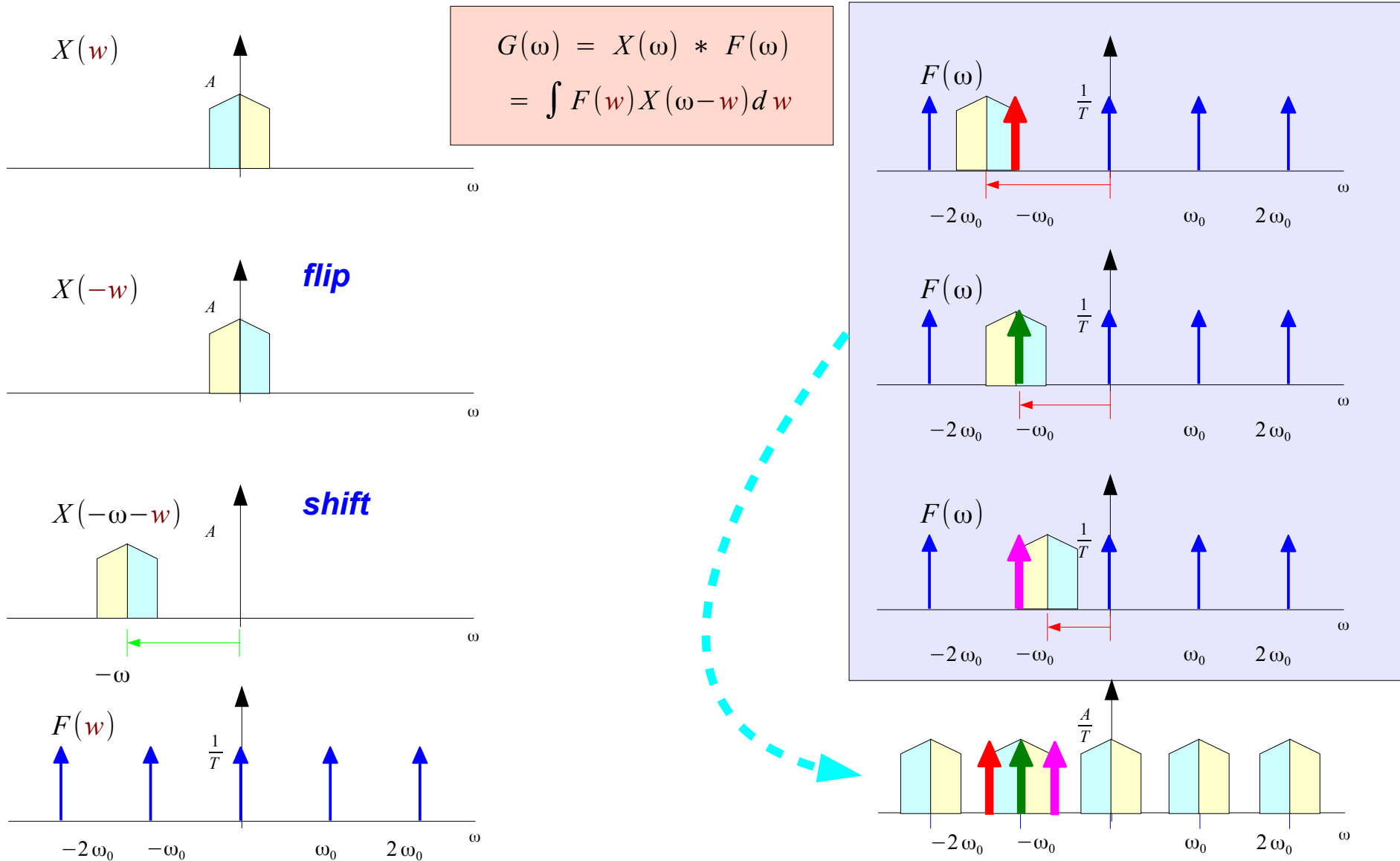
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CTFT



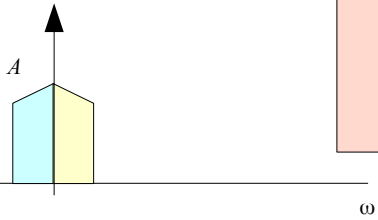


# Convolution with Impulse Train



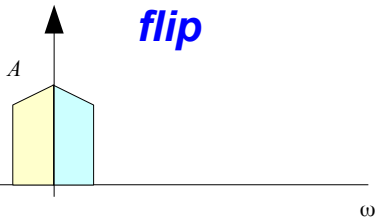
# Convolution with Sinc Impulse Train

$X(\omega)$



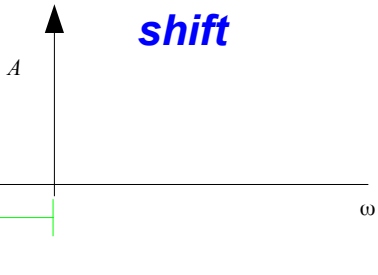
$$G(\omega) = X(\omega) * F(\omega) = \int F(\omega') X(\omega - \omega') d\omega'$$

$X(-\omega)$



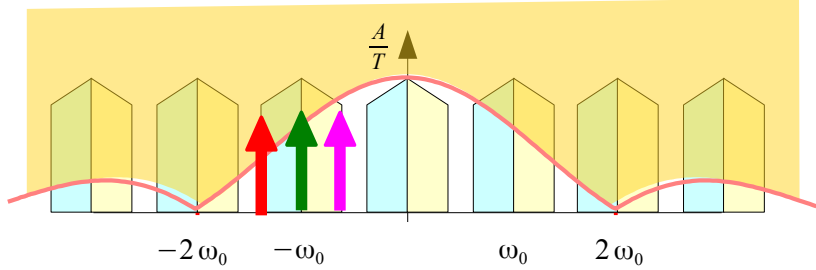
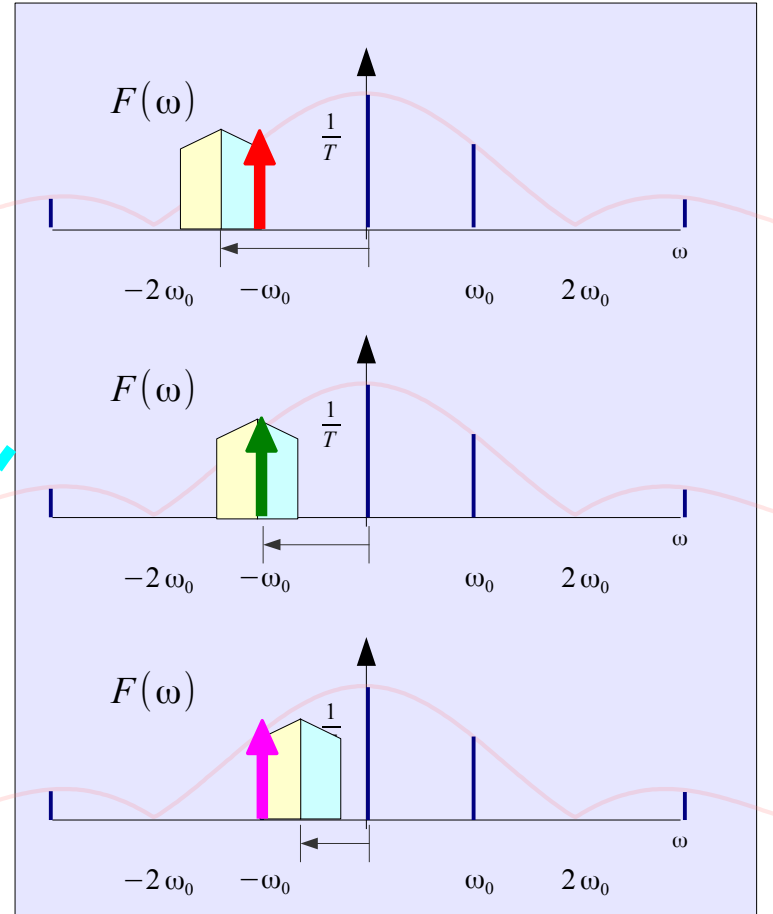
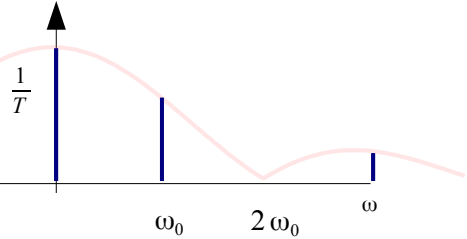
*flip*

$X(-\omega - \omega')$

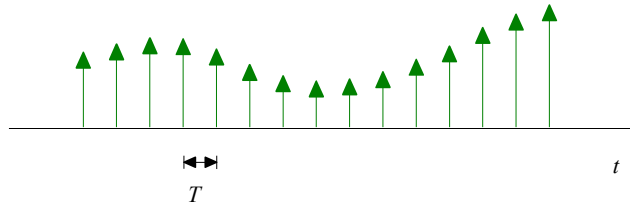


*shift*

$F(\omega)$

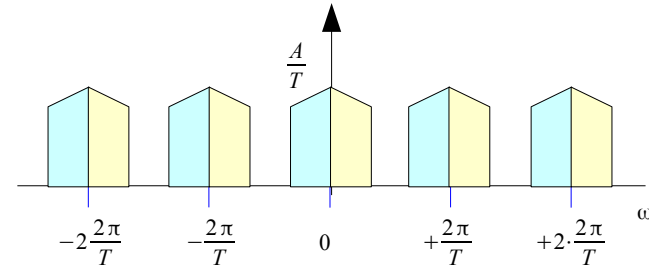


# CTFT of Sampled Signal



$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

CTFT



$$\begin{aligned} \hat{X}(f) &= \int_{-\infty}^{+\infty} \hat{x}(t) e^{-j2\pi f t} dt \\ &= \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT) e^{-j2\pi f t} dt \\ &= \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi f T n} \end{aligned}$$

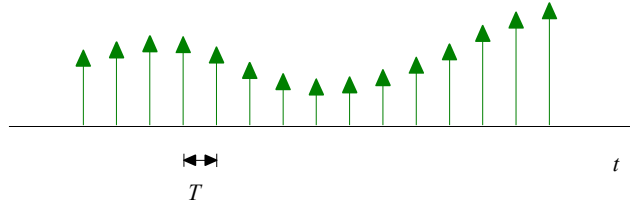
$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

CTFT



$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi f T n}$$

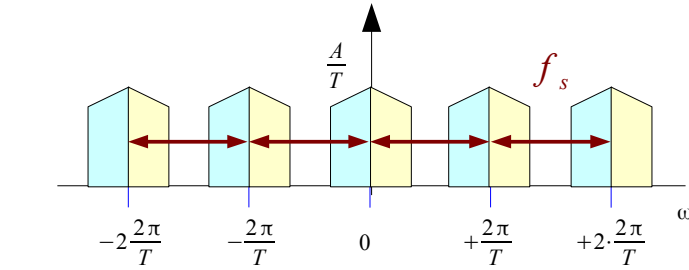
# Periodicity in Frequency



$$f_s = \frac{1}{T} \quad 2\pi f_s = \frac{2\pi}{T} = \omega_0$$

$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

CTFT



$$2\pi f = \omega$$

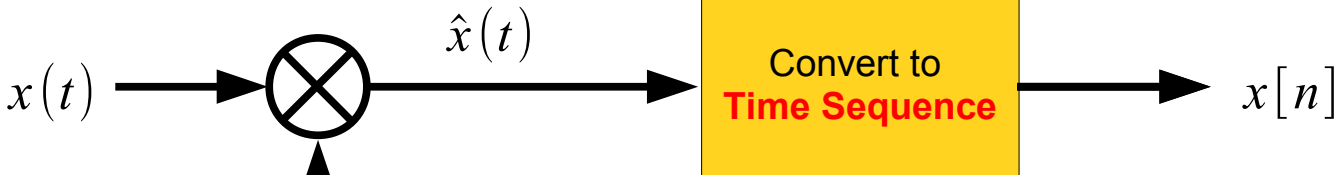
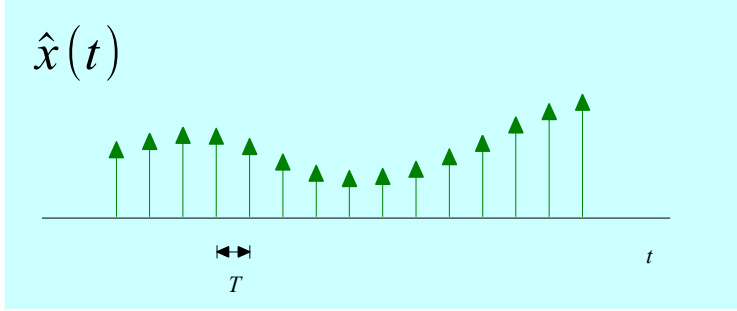
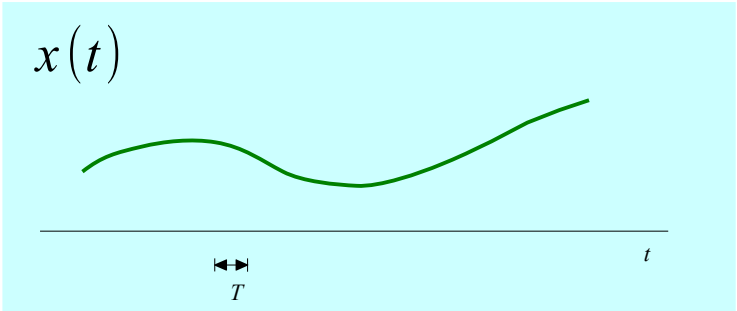
$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi f T n}$$

$$e^{-j2\pi(f+f_s)Tn} = e^{-j2\pi(f)Tn} \quad \leftarrow f_s T = 1$$

Period = Sampling Frequency  $f_s$

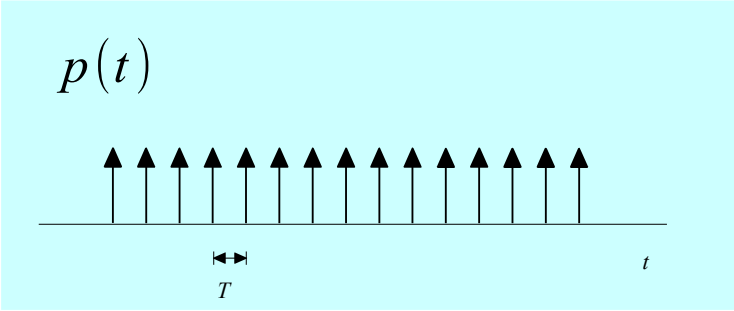
$$\hat{X}(f) = \hat{X}(f + f_s)$$

# Time Sequence

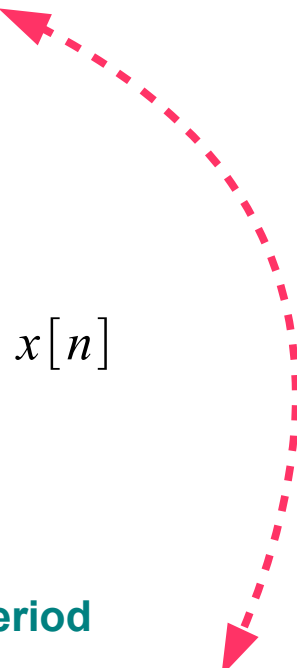
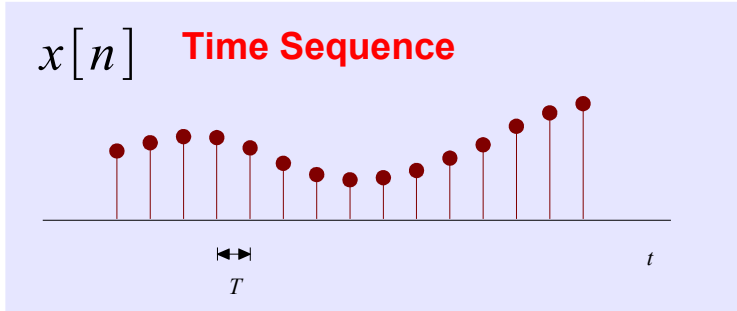


**Ideal Sampling**

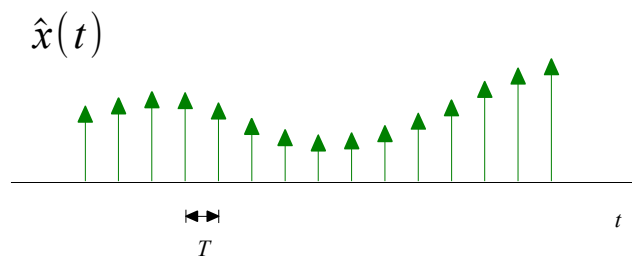
$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$



$T$  Sampling Period

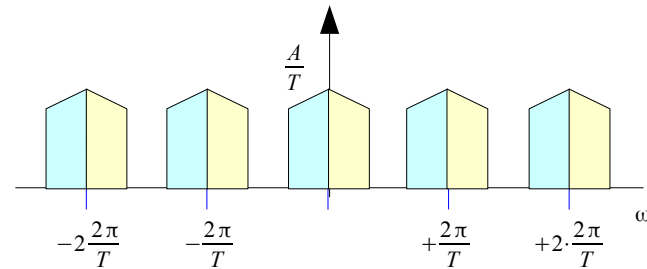


# DTFT of a Time Sequence

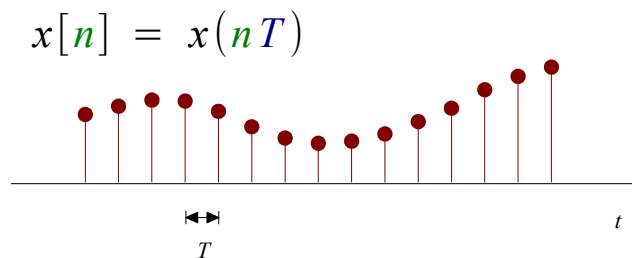


$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

CTFT

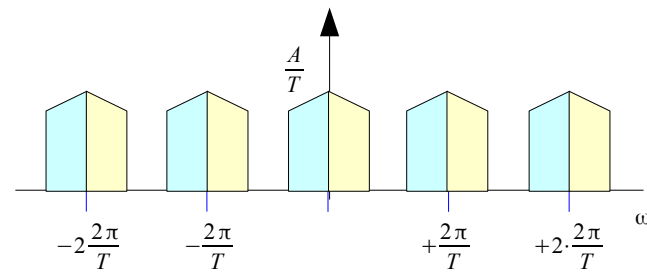


$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi fTn}$$



$$x[n], \text{ Sampling Period } T$$

DTFT

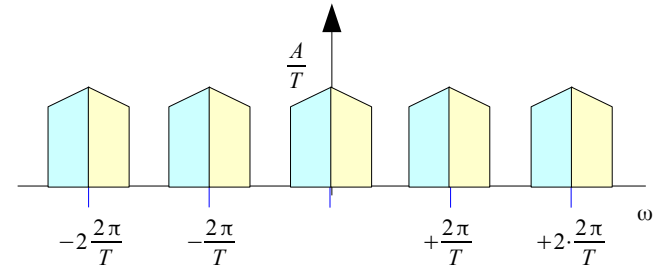
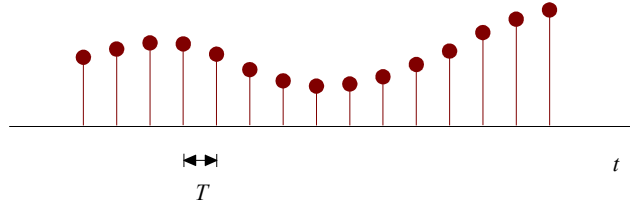


$$X(f) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi fTn}$$

Here,  $X(f)$  does not denote the CTFT of  $x(t)$

# Discrete Time Fourier Transform (1)

$$x[n] = x(nT)$$



$x[n]$  , Sampling Period  $T$

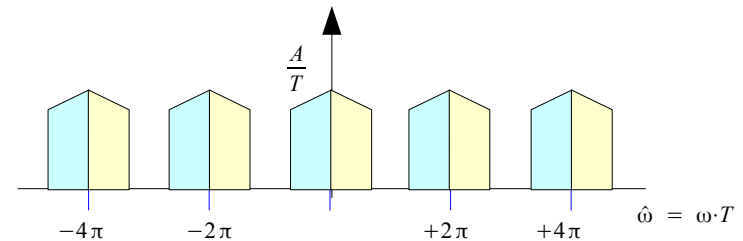
**DTFT**



$$X(f) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi f T n}$$

**Normalized Angular Frequency**

$$2\pi f T = \frac{2\pi f}{1/T} = 2\pi \frac{f}{f_s} = \hat{\omega}$$



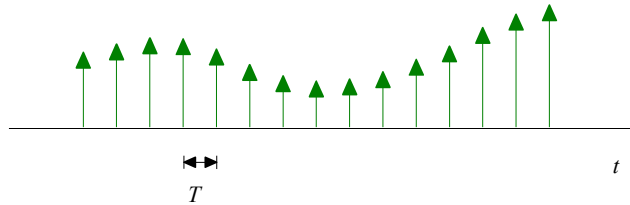
$x[n]$  , Sampling Period  $T$

**DTFT**



$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega} n}$$

# Discrete Time Fourier Transform (2)



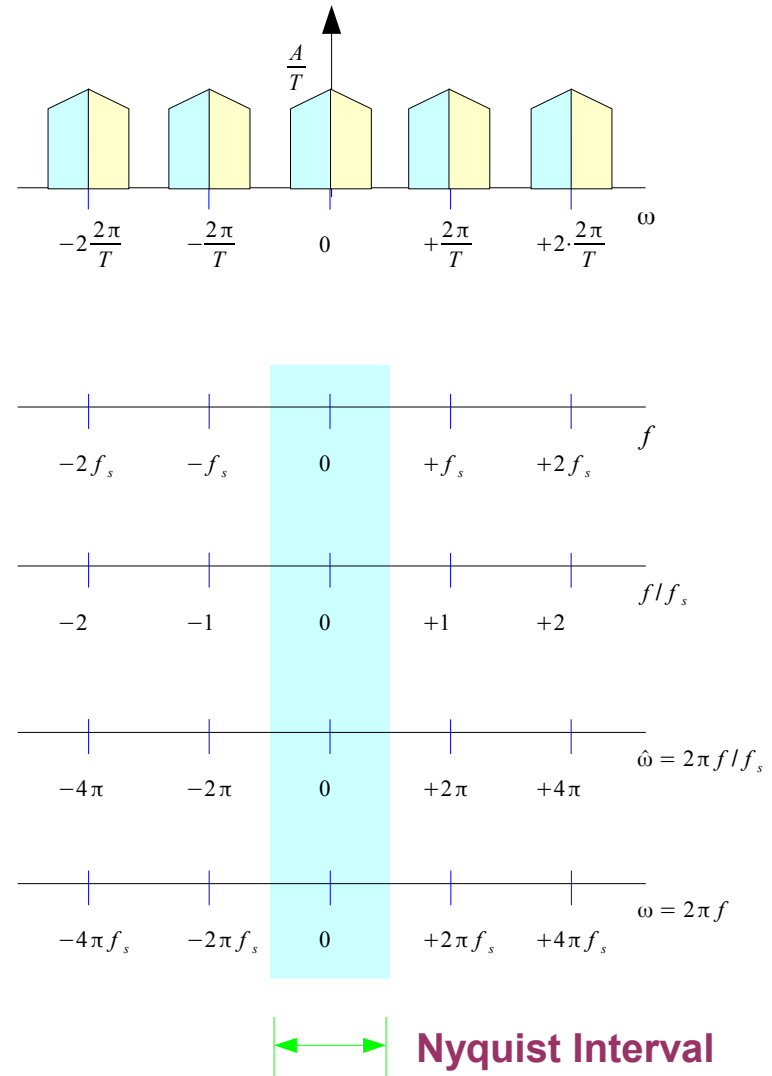
$$f_s = \frac{1}{T} \quad 2\pi f_s = \frac{2\pi}{T} = \omega_0$$

$$X(f) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi f T n}$$

Normalized Angular Frequency

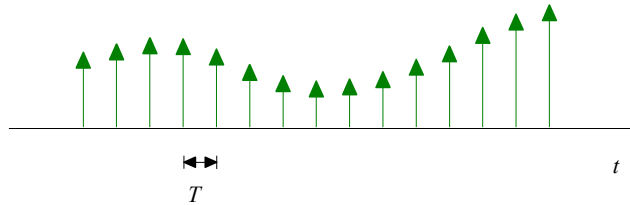
$$2\pi f T = \frac{2\pi f}{1/T} = 2\pi \frac{f}{f_s} = \hat{\omega}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega} n}$$





# Discrete Time Fourier Transform (3)



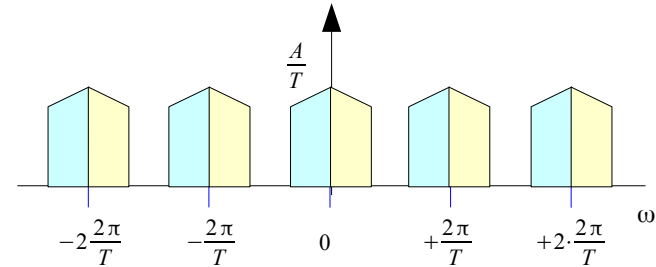
$$f_s = \frac{1}{T} \quad 2\pi f_s = \frac{2\pi}{T} = \omega_0$$

$$X(f) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi f T n}$$

Normalized Angular Frequency

$$2\pi f T = \frac{2\pi f}{1/T} = 2\pi \frac{f}{f_s} = \hat{\omega}$$

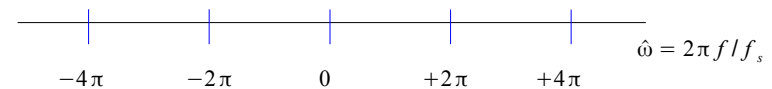
$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega} n}$$



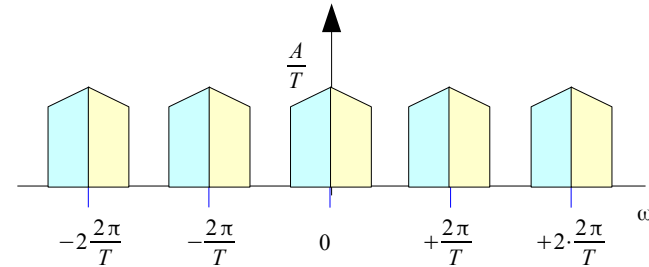
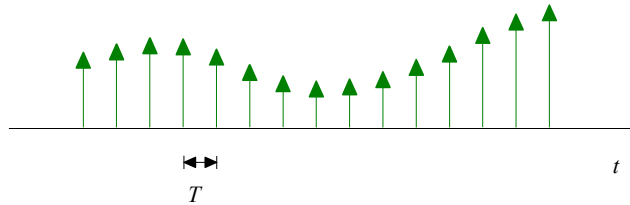
$\hat{X}(f)$  Absolute Frequency



$\hat{X}(e^{j\hat{\omega}})$  Normalized Angular Frequency  
unit circle →  
emphasize the periodic nature



# Fourier Series Interpretation



$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

**CTFT**



$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi fTn}$$

$$x(nT) = \frac{1}{f_s} \int_{+f_s/2}^{-f_s/2} \hat{X}(f) e^{+j2\pi fTn} df$$

**CTFS**



$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi fTn}$$

$$= \int_{-\pi}^{+\pi} \hat{X}(\omega) e^{+j\omega n} \frac{d\omega}{2\pi}$$

$$\omega = 2\pi f / f_s \quad \frac{df}{f_s} = \frac{d\omega}{2\pi}$$

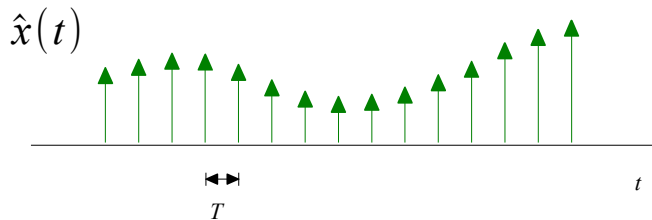
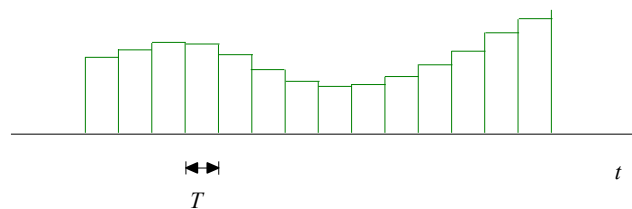
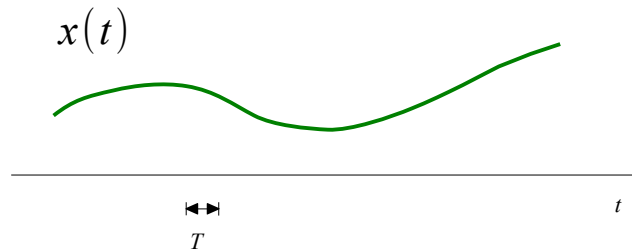
**Fourier Series Coefficients**  $x(nT)$

$\hat{X}(f)$  **Continuous Periodic Function**

**View as a Fourier Series Expansion**

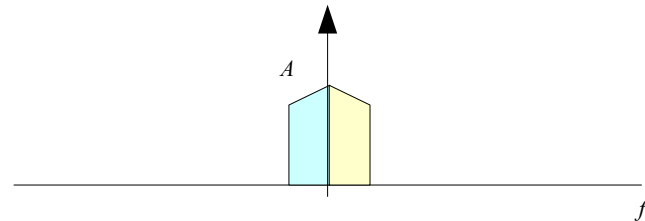
# Numerical Approximation

$$X(f) = \lim_{T \rightarrow 0} T \hat{X}(f)$$



$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

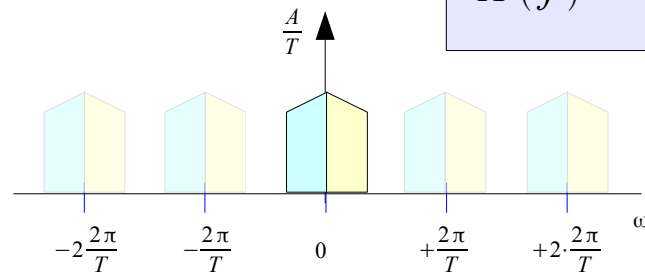
CTFT



$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{+j2\pi f t} dt$$

$$\approx \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi f T n} \cdot T$$

$$X(f) \approx T \hat{X}(f)$$



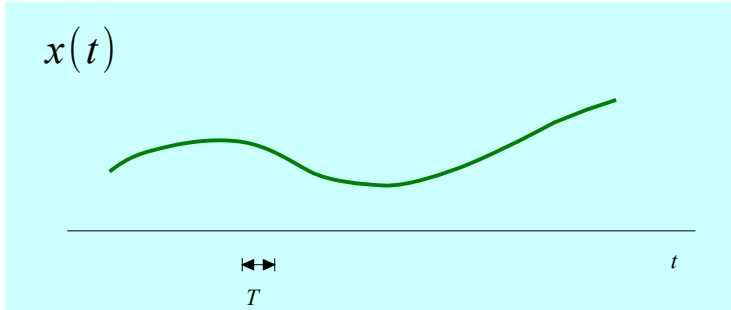
CTFT



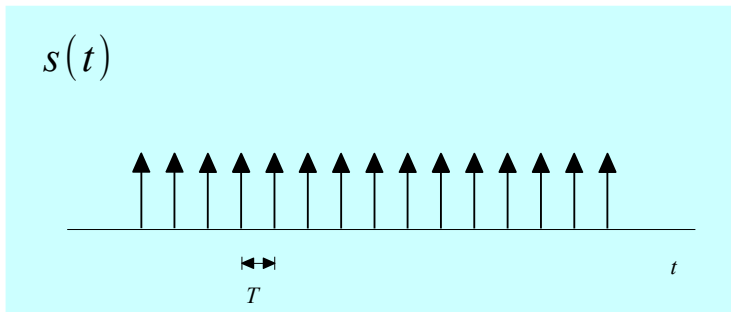
$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi f T n}$$

# Spectrum Replication (1)

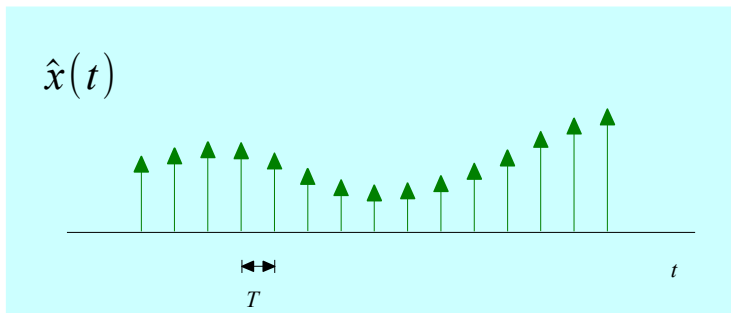
## Ideal Sampling



**X**



**||**



$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

$$s(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT)$$

$$= \frac{1}{T} \sum_{m=-\infty}^{+\infty} e^{+j2\pi m f_s t}$$

$$\hat{x}(t) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} x(t) e^{+j2\pi m f_s t}$$

Shift Property



$$\hat{X}(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X(f - m f_s)$$

# Spectrum Replication (2)

$$S(f) = \frac{1}{T} \sum_{m=-\infty}^{+\infty} \delta(f - m f_s)$$

## Convolution in Frequency

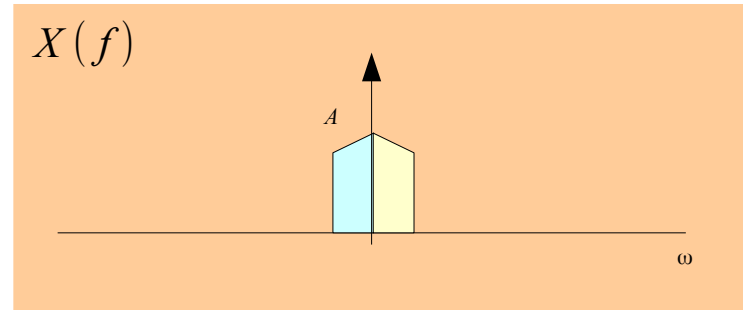
$$\hat{X}(f) = X(f) * S(f)$$

$$= \int_{-\infty}^{+\infty} X(f - f') S(f') d f'$$

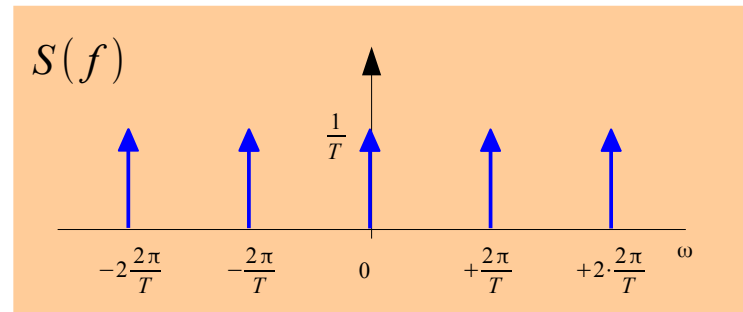
$$= \frac{1}{T} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(f - f') \delta(f' - m f_s) d f'$$

$$\hat{X}(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X(f - m f_s)$$

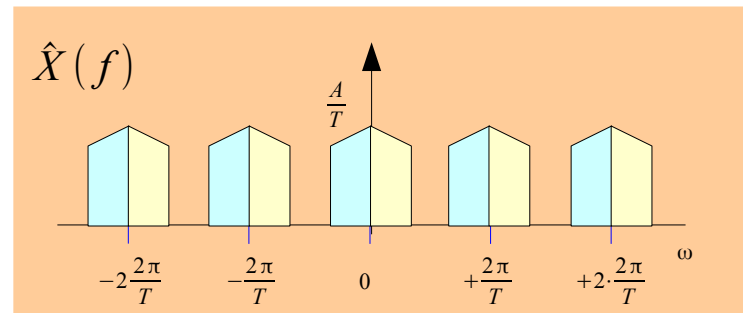
## Frequency Domain



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## References

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- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] A “graphical interpretation” of the DFT and FFT, by Steve Mann
- [4] R. G. Lyons, Understanding Digital Signal Processing, 1997
- [5] AVR121: Enhancing ADC resolution by oversampling
- [6] S.J. Orfanidis, Introduction to Signal Processing  
[www.ece.rutgers.edu/~orfanidi/intro2sp](http://www.ece.rutgers.edu/~orfanidi/intro2sp)