

Down-Sampling (4B)

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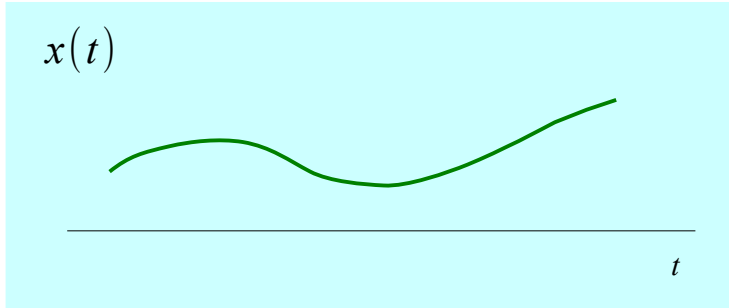
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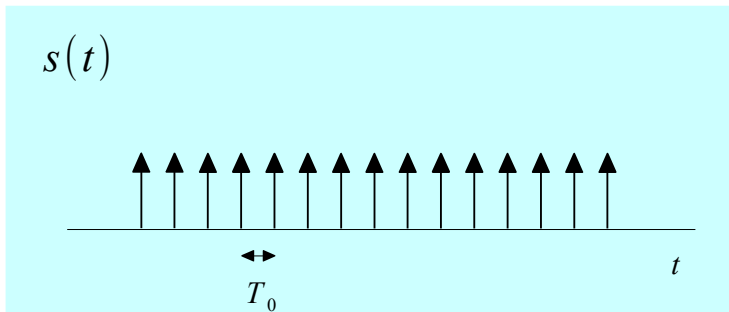
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Spectrum Replication (1)

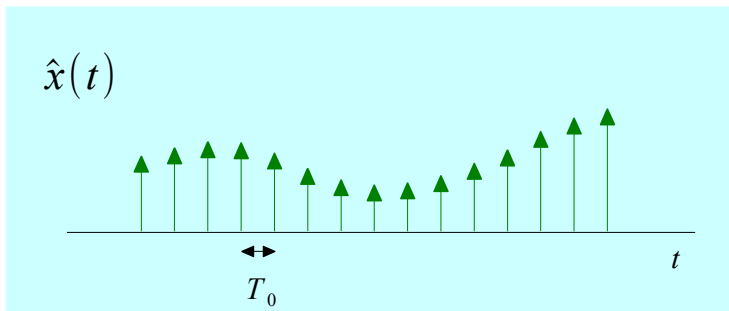
Ideal Sampling



X



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$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0) \delta(t-nT_0)$$

$$\begin{aligned} s(t) &= \sum_{n=-\infty}^{+\infty} \delta(t-nT_0) \\ &= \frac{1}{T_0} \sum_{m=-\infty}^{+\infty} e^{+j2\pi m f_s t} \end{aligned}$$

$$\hat{x}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} x(t) e^{+j2\pi m f_s t}$$

Shift Property



$$\hat{X}(f) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} X(f-m f_s)$$

Spectrum Replication (2)

$$S(f) = \frac{1}{T} \sum_{m=-\infty}^{+\infty} \delta(f - m f_s)$$

Convolution in Frequency

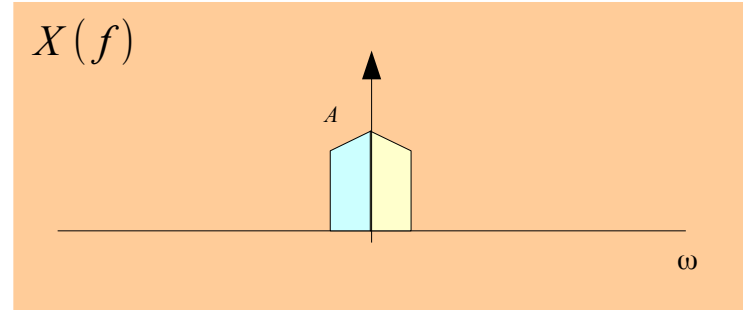
$$\hat{X}(f) = X(f) * S(f)$$

$$= \int_{-\infty}^{+\infty} X(f - f') S(f') d f'$$

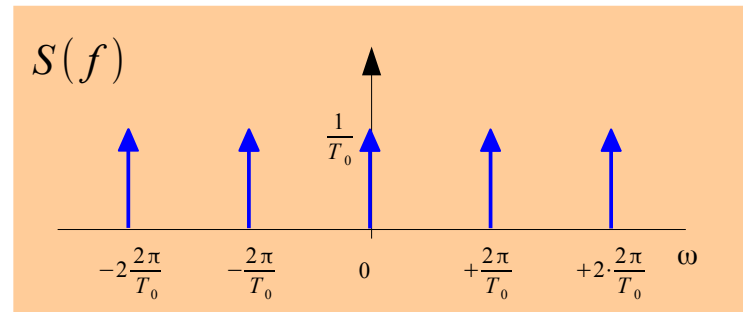
$$= \frac{1}{T_0} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(f - f') \delta(f' - m f_s) d f'$$

$$\hat{X}(f) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} X(f - m f_s)$$

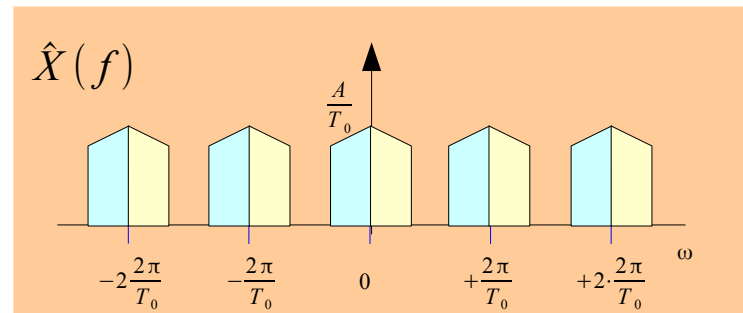
Frequency Domain



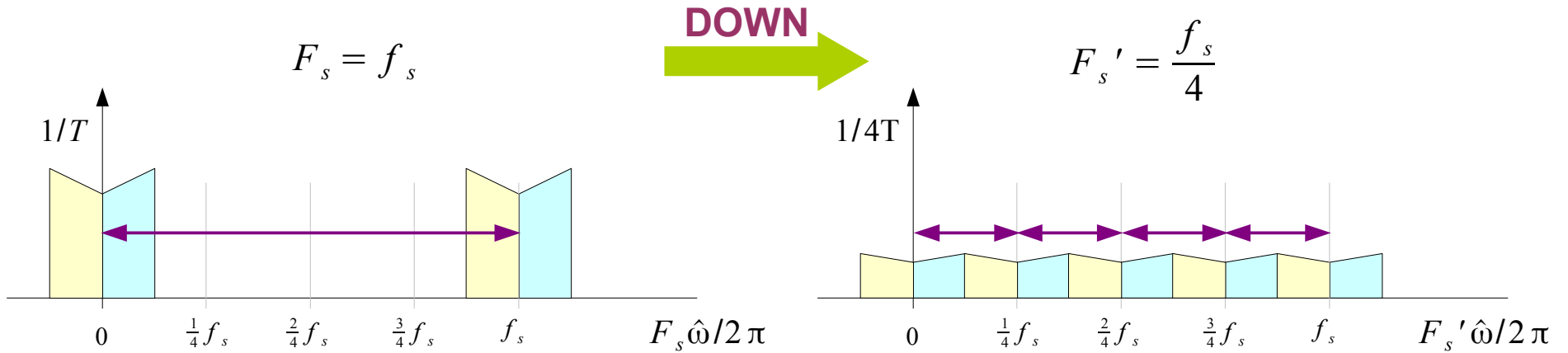
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Decreasing Sampling Frequency



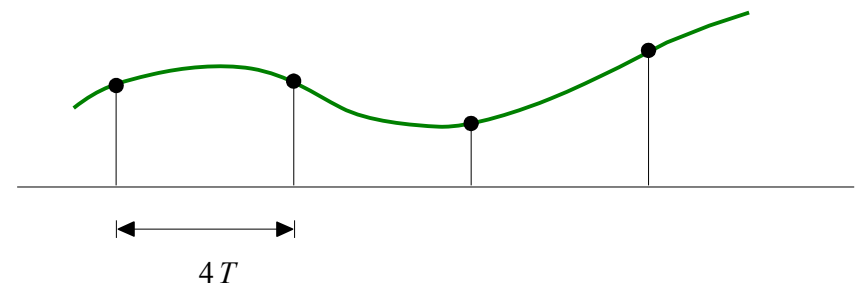
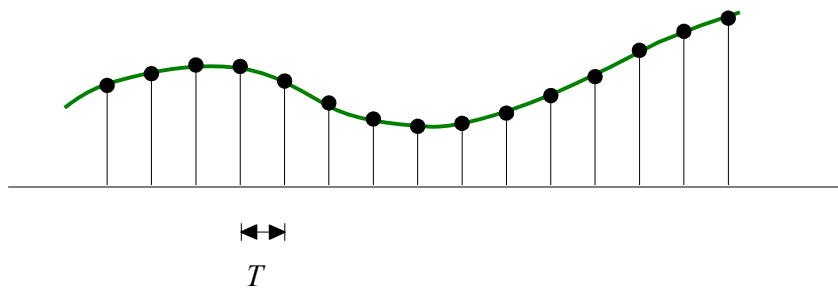
Sampling Frequency $F_s = f_s$

Sampling Time $T = \frac{1}{f_s}$

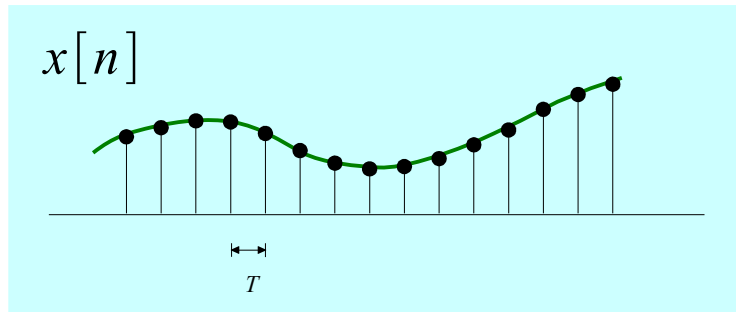


Sampling Frequency $F_s' = \frac{f_s}{4}$

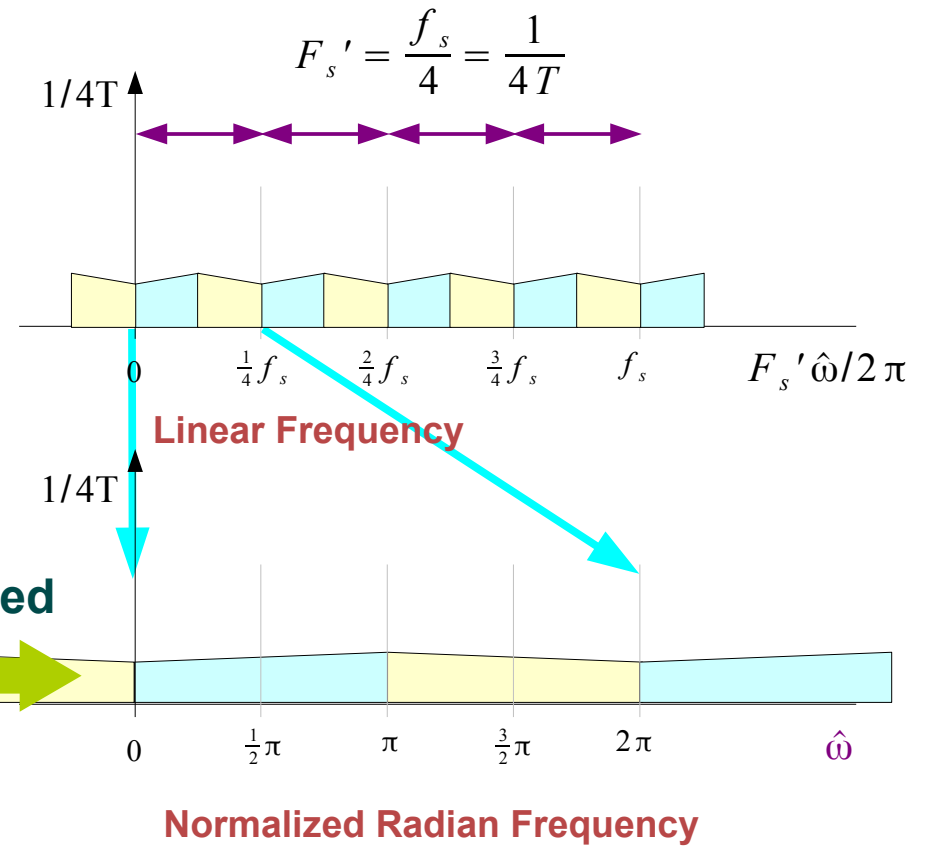
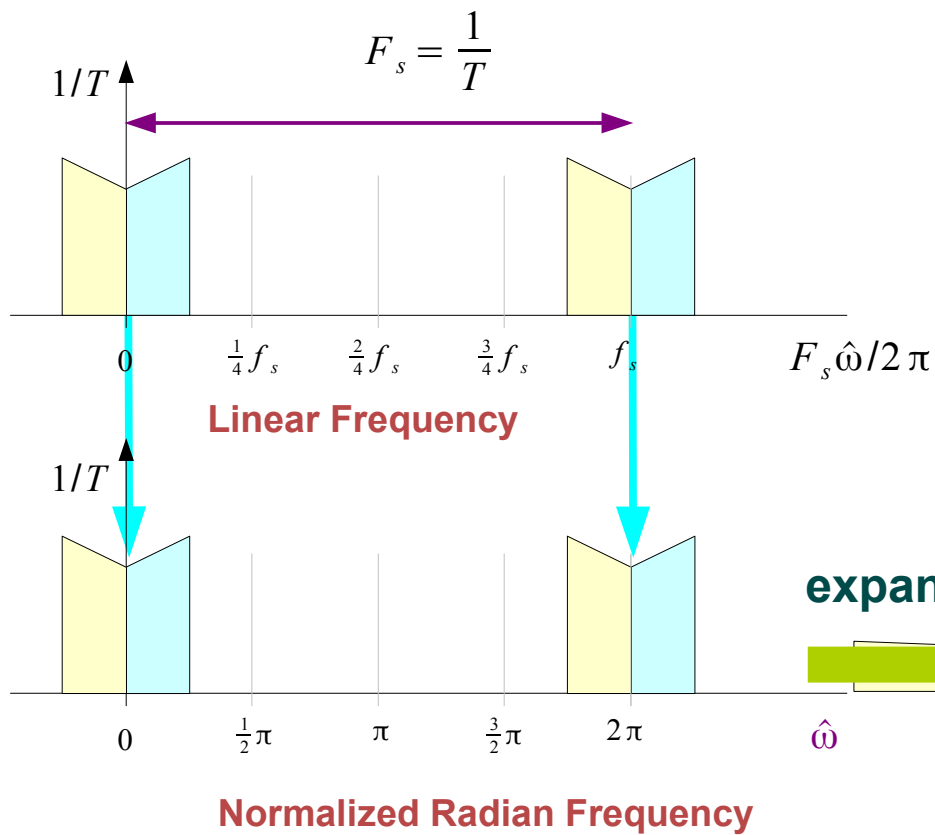
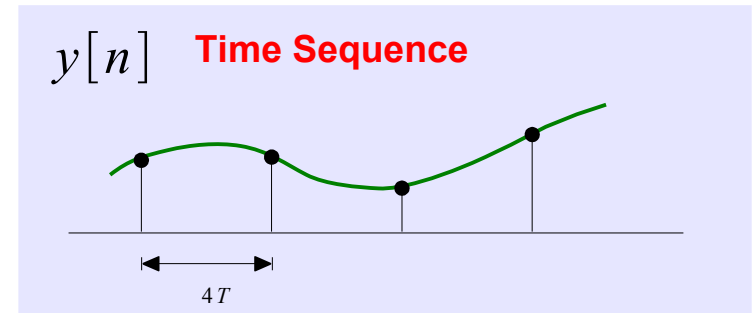
Sampling Time $T' = 4T$



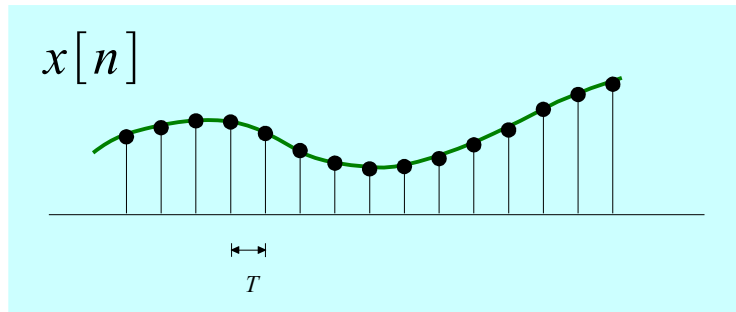
Coarse Sequence & Spectrum



DOWN



Normalized Radian Frequency

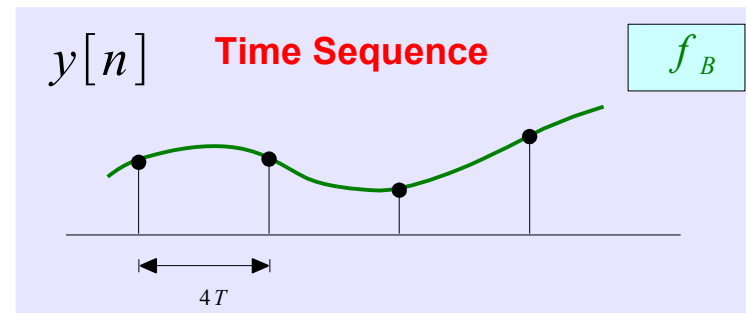


$$\hat{\omega} = \omega \cdot T_s = \frac{\omega}{1/T_s}$$

$$\hat{\omega} = \frac{\omega}{f_s} = 2\pi \frac{f}{f_s}$$

↑ ↑
Normalized to f_s

Normalized Radian Frequency

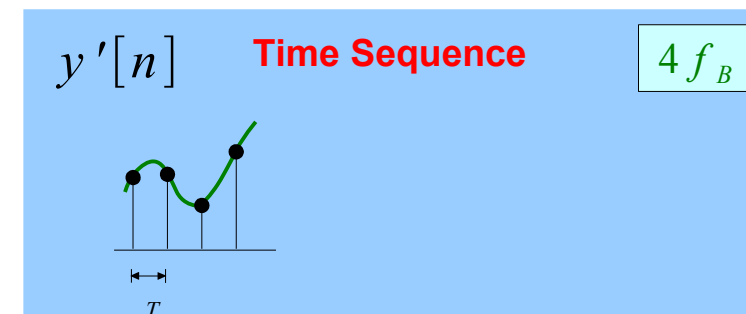


$$\frac{f}{f_s} = \frac{f_B}{1/4T} = f_B \cdot 4T$$

The Same

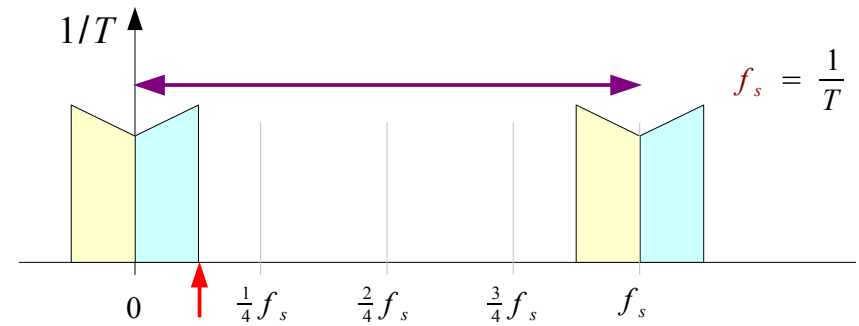
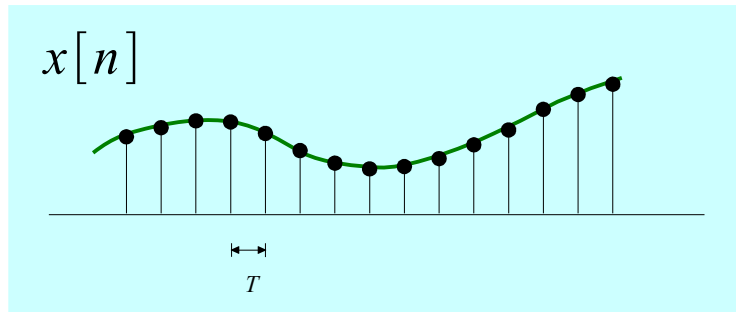
- Time Sequence
- Normalized Radian Frequency

$$\frac{f}{f_s} = \frac{4f_B}{1/T} = f_B \cdot 4T$$

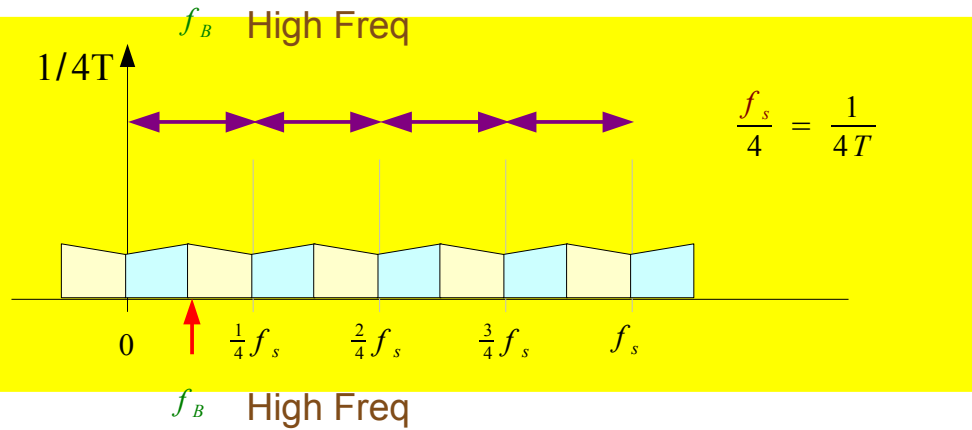
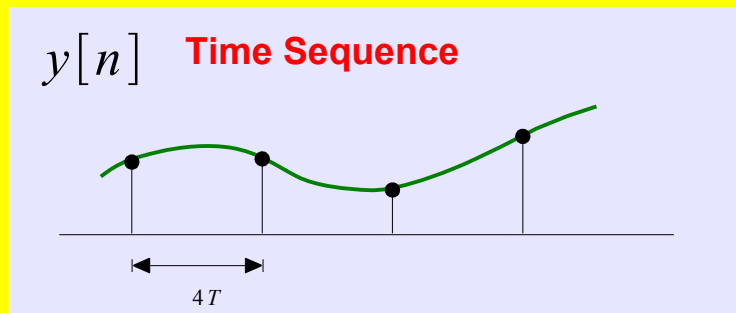


The Highest Frequency:

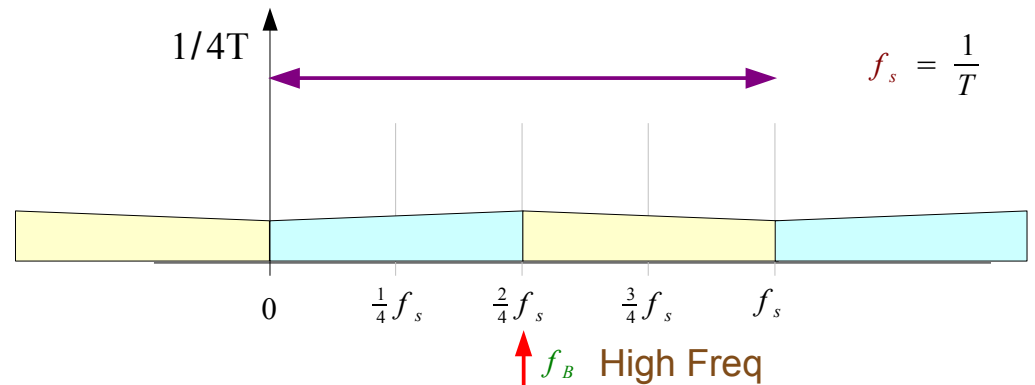
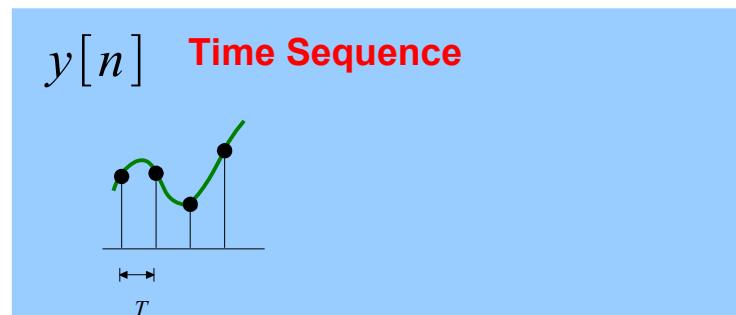
Coarse Sequence Spectrum – Linear Frequency



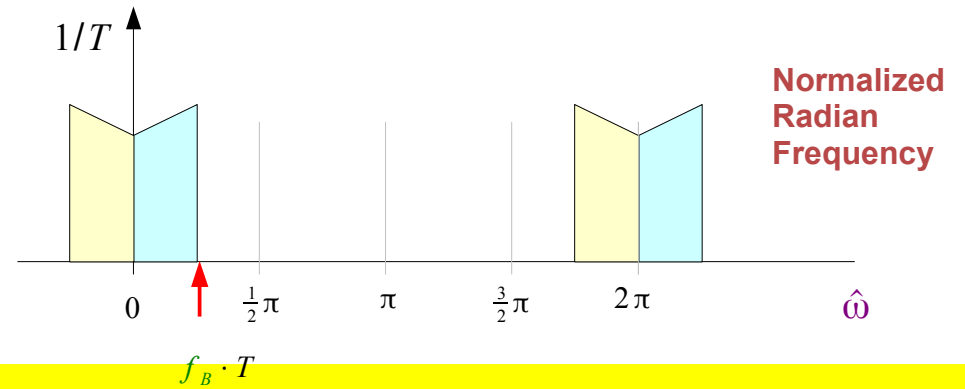
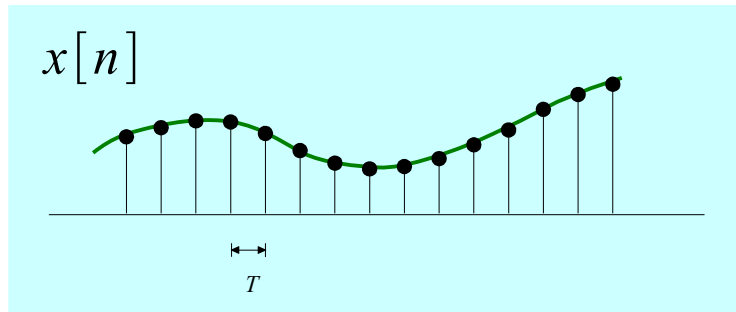
DOWN



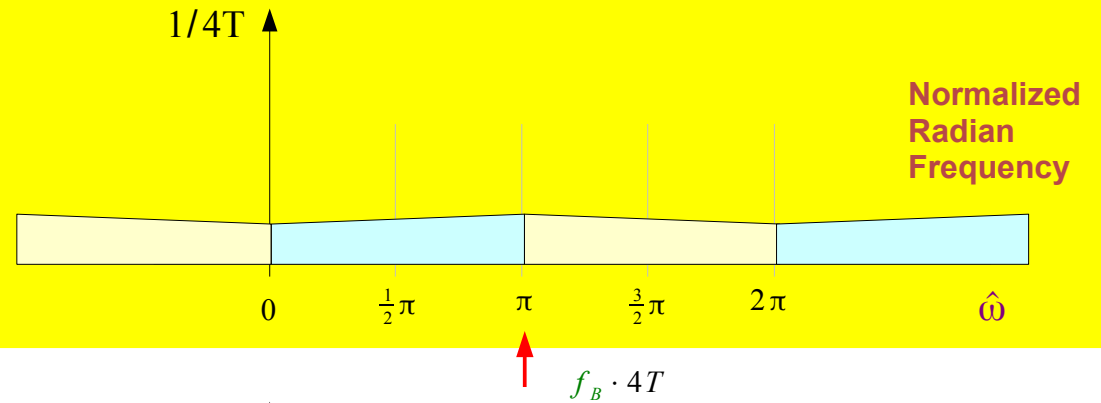
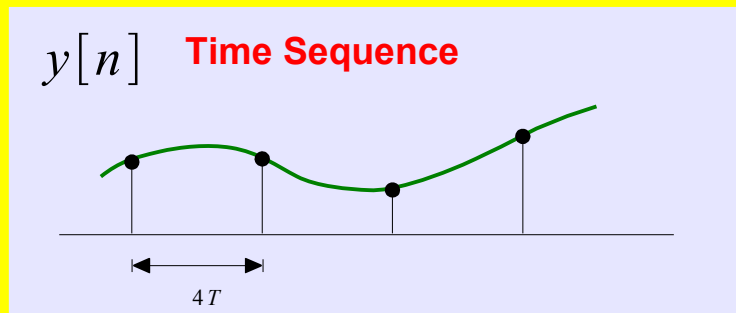
|| The Same Time Sequence



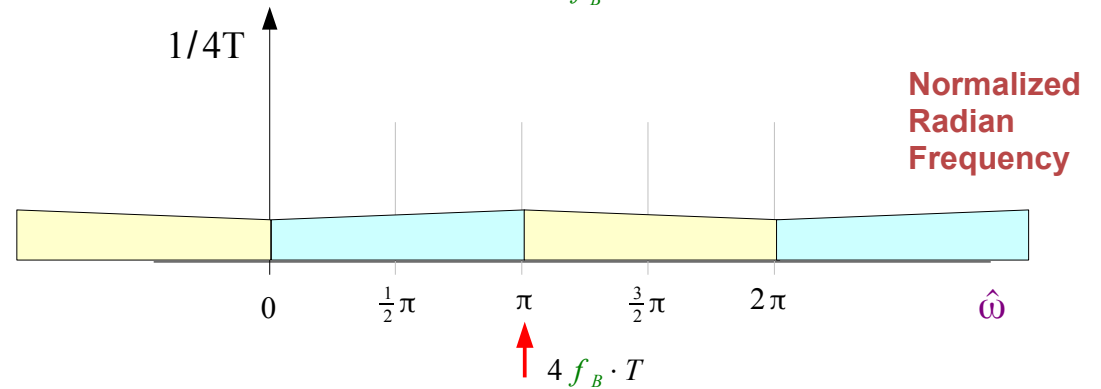
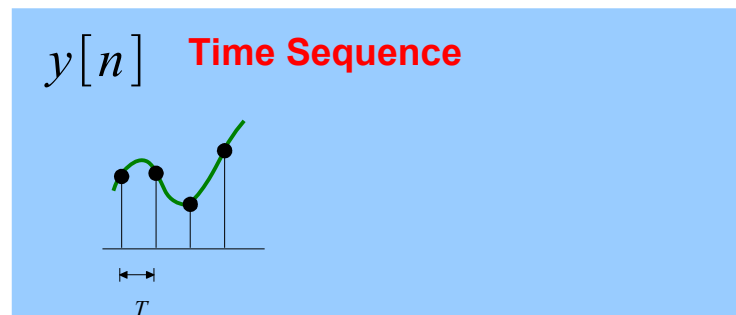
Coarse Sequence Spectrum – Normalized Frequency



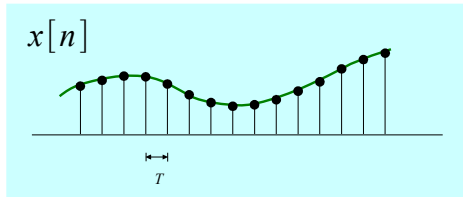
DOWN



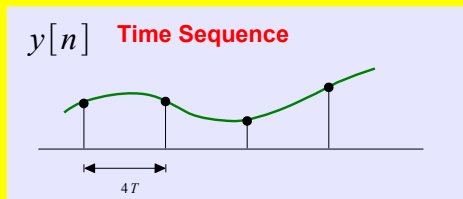
|| The Same Time Sequence



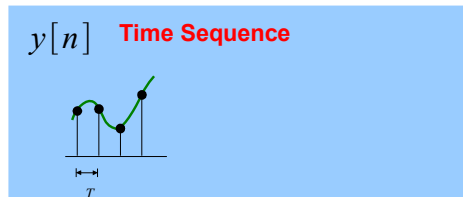
Coarse Sequence Spectrum – Linear Frequency



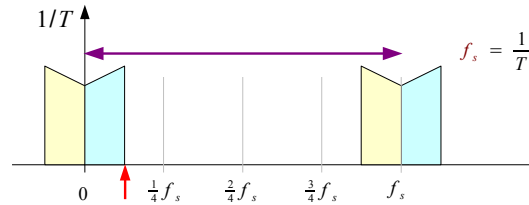
DOWN



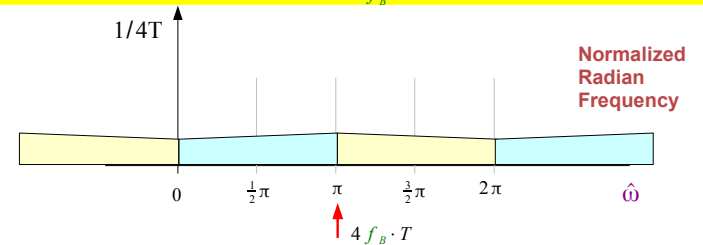
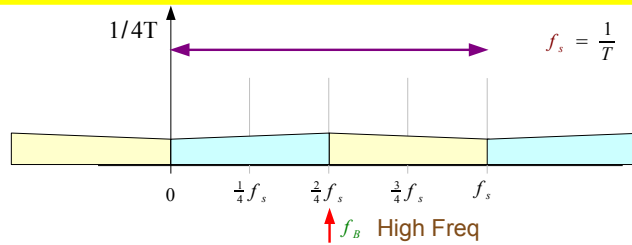
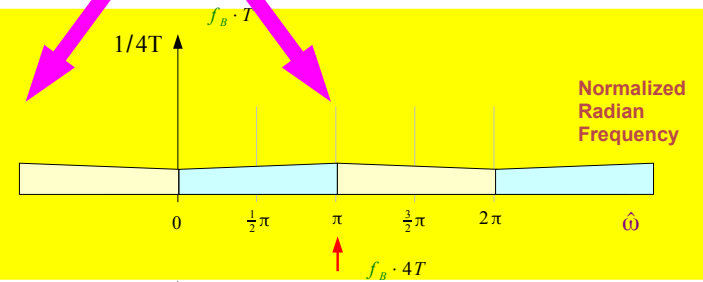
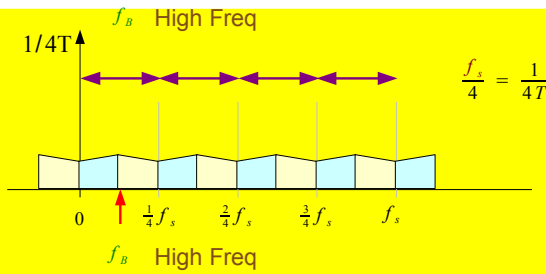
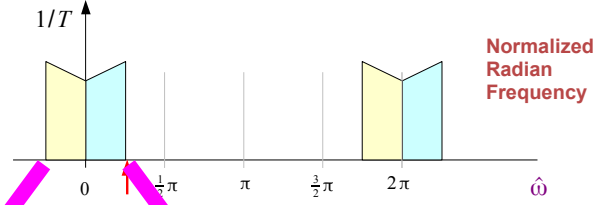
|| The Same Time Sequence



Linear Frequency



Normalized Radian Frequency

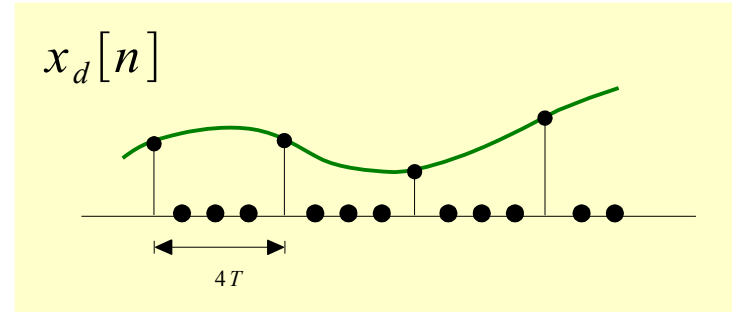
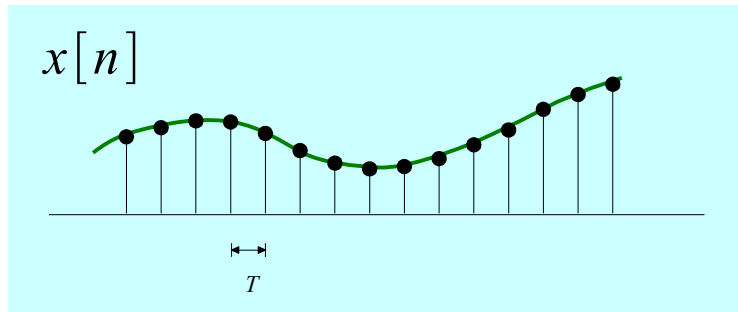


Coarse Sequence Spectrum

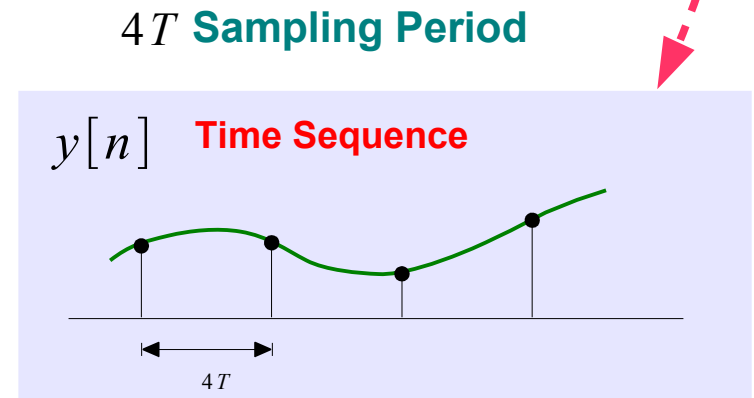
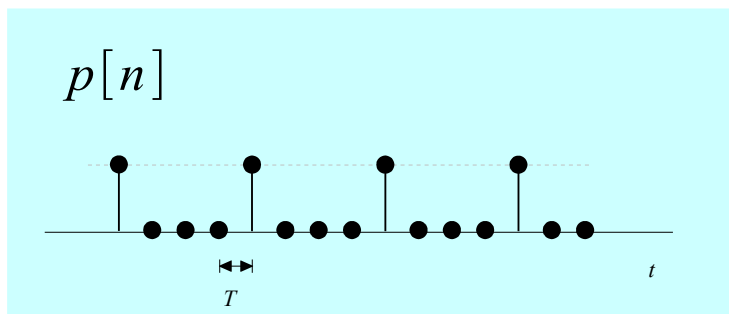
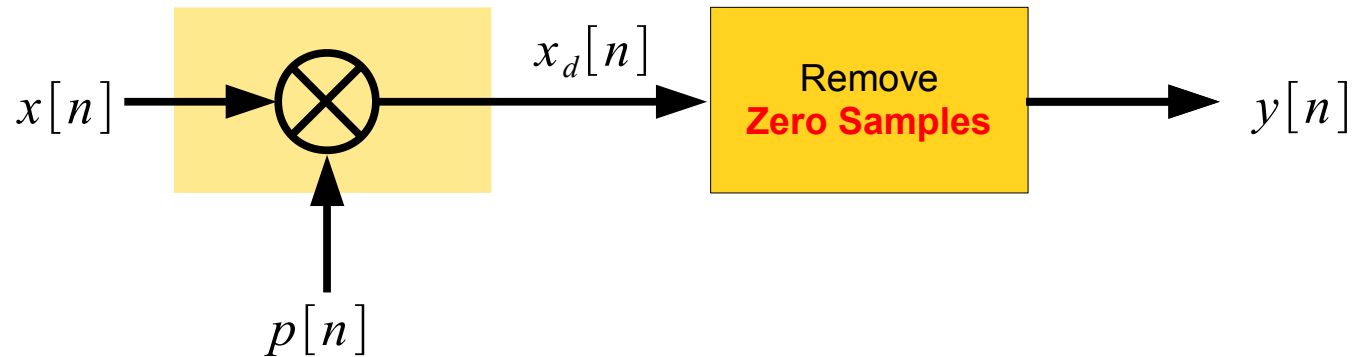
➔ Expanded

Normalized Radian Frequency

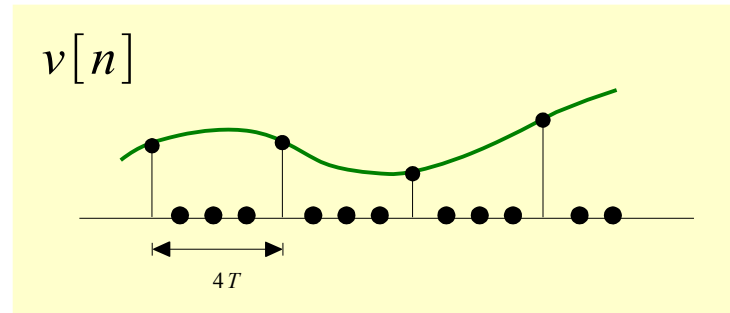
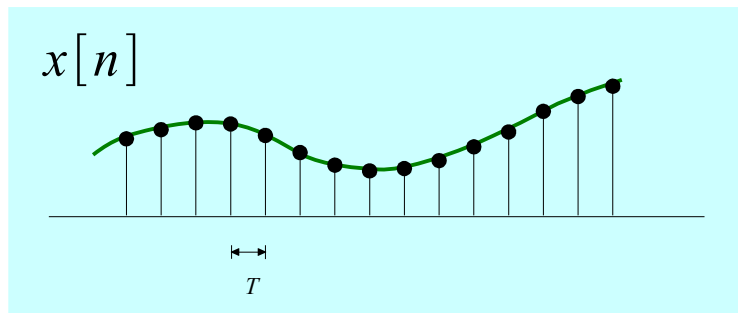
Coarse Sequence Generation



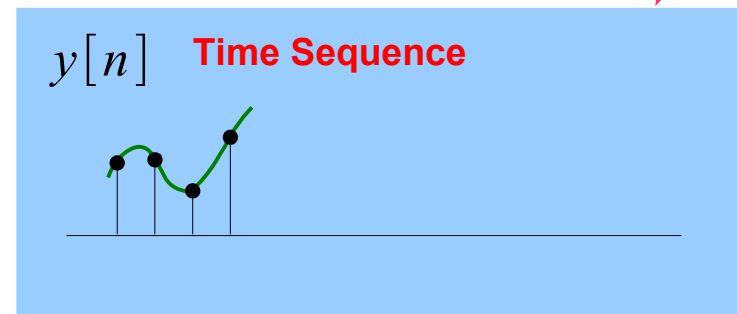
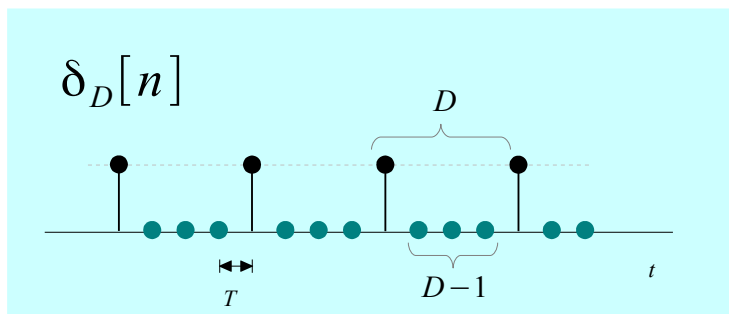
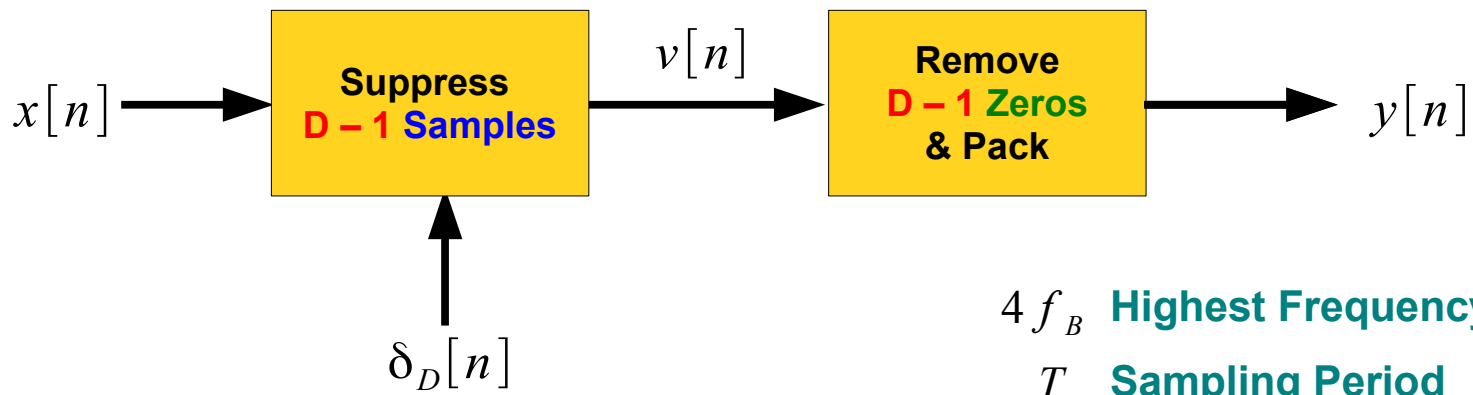
T Sampling Period



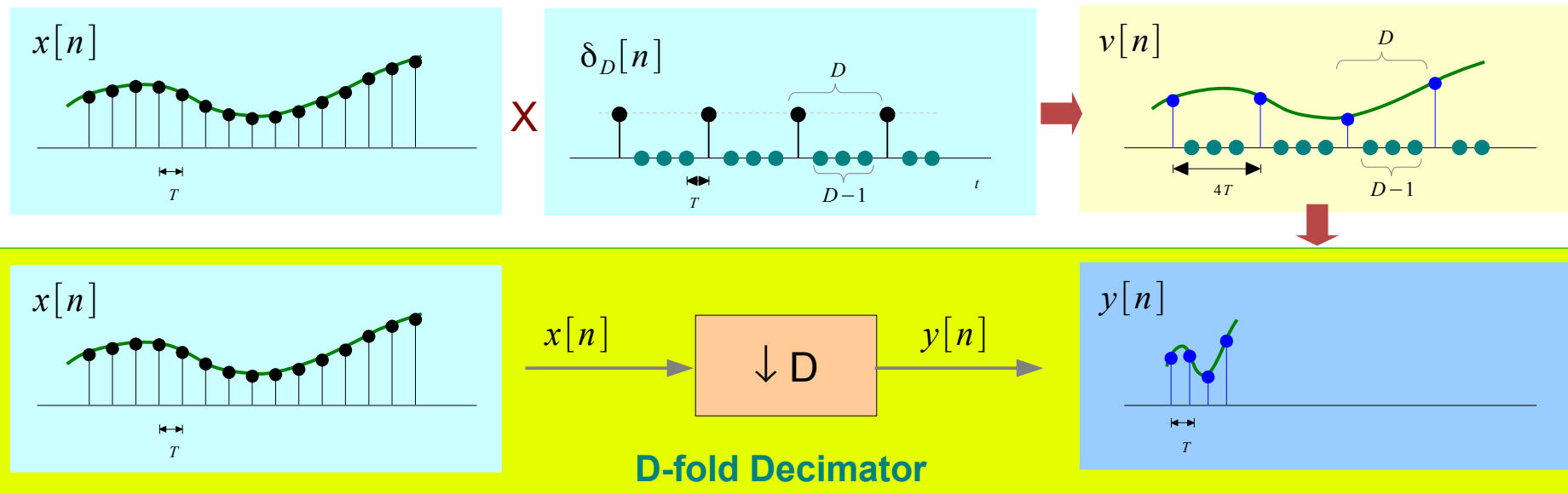
Down Sampling in Two Steps



T Sampling Period



Down-Sampling Operator



$$y[n] = S_{1/D} x[n] = x[nD]$$

Decrease
sampling
frequency
by a factor of $1/D$

Increase
sampling
period
by a factor of D

$$D = 2$$

$$y[0] = x[0 \cdot 2]$$

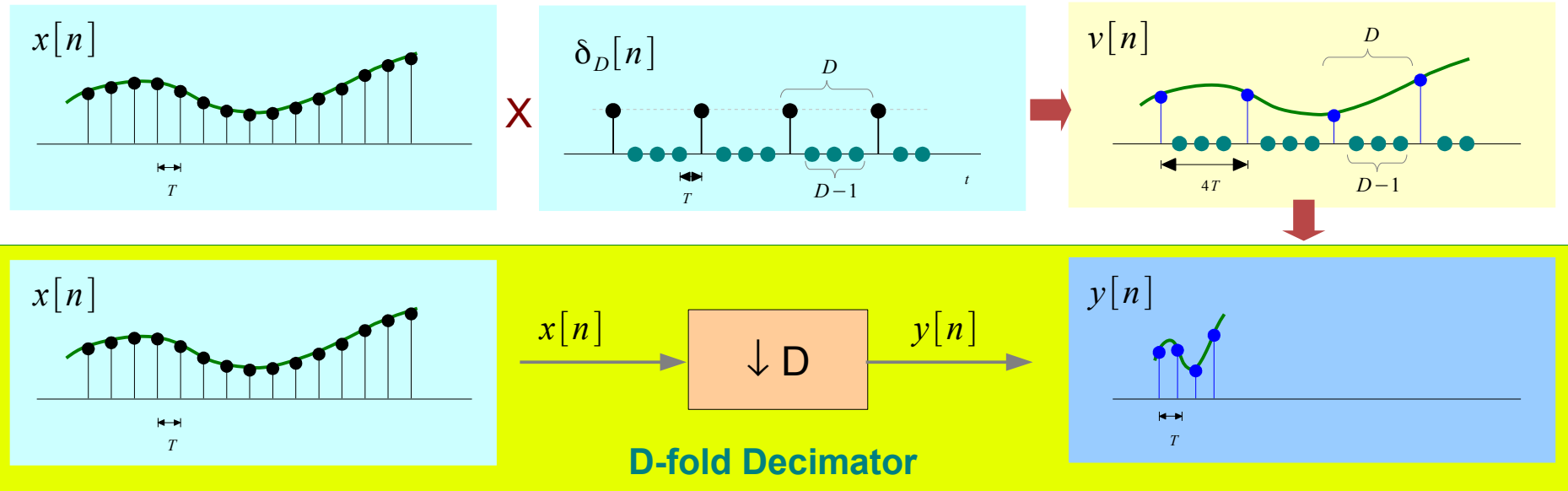
$$y[1] = x[1 \cdot 2]$$

$$y[2] = x[2 \cdot 2]$$

$$y[3] = x[3 \cdot 2]$$

...

Down-Sampling Operator



$$y[n] = S_{1/D} x[n] = x[nD]$$

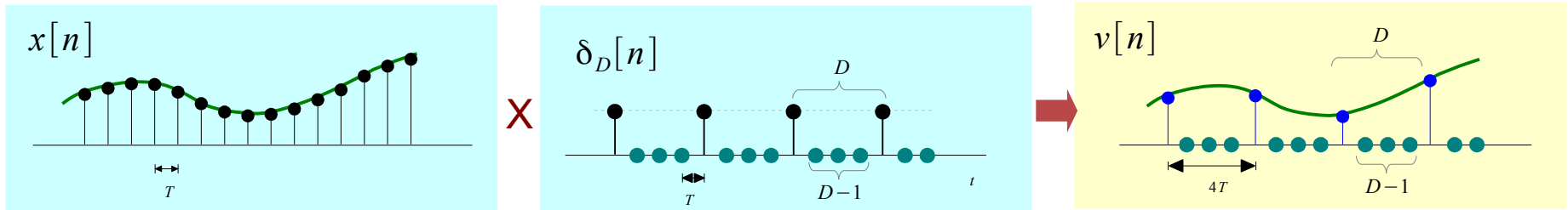
$$x[n] = e^{j\hat{\omega}n} \quad \rightarrow \quad y[n] = e^{j\hat{\omega}Dn}$$

$$-\pi \leq \hat{\omega} \leq +\pi \quad \quad -\pi \leq \hat{\omega}D \leq +\pi \quad \text{stretched}$$

$$-\frac{\pi}{D} \leq \hat{\omega}_1 \leq +\frac{\pi}{D} \quad \quad -\pi \leq \hat{\omega}_1 D \leq +\pi \quad \text{w/o aliasing}$$

$$\hat{\omega}_2 > +\frac{\pi}{D} \quad \quad \hat{\omega}_2 D > +\pi \quad \text{with aliasing}$$

Suppressing D-1 Samples (1)



$$\delta_D[n] = \frac{1}{D} \sum_{k=0}^{D-1} e^{-j2\pi kn/D} = \begin{cases} \frac{1}{D} D = 1 & \text{if } \text{mod}(n, D) = 0 \\ \frac{1}{D} \frac{1 - e^{-j2\pi n}}{1 - e^{-j2\pi n/D}} = 0 & \text{otherwise} \end{cases} \quad (e^{-j2\pi k(n/D)} = e^{-j2\pi km} = 1)$$

$$v[n] = \delta_D[n] x[n] = \frac{1}{D} \sum_{k=0}^{D-1} x[n] e^{-j2\pi kn/D}$$

$$\hat{\omega}_k = 2\pi k/D$$

$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

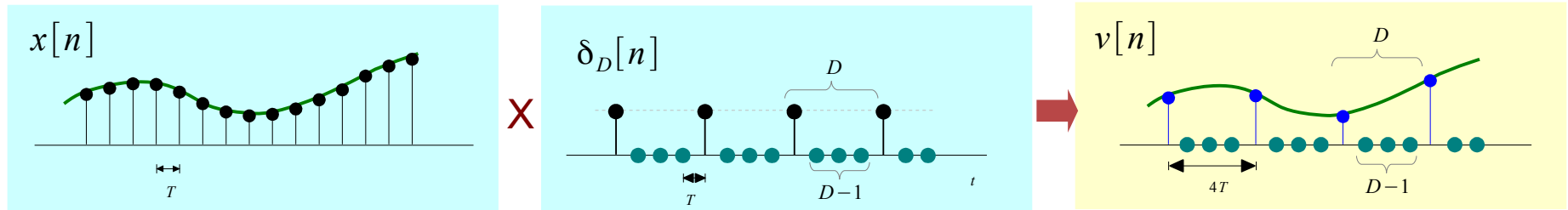
$$Z\{x[n] e^{-j\hat{\omega}_k n}\} = \sum_{n=-\infty}^{+\infty} x[n] (e^{j\hat{\omega}_k} \cdot z)^{-n} = X(e^{j\hat{\omega}_k} \cdot z)$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

$$X(e^{-j2\pi k/D} \cdot z) = X(e^{-j\omega_k} e^{j\hat{\omega}}) = X(e^{j(\hat{\omega} - \omega_k)})$$

$$z = e^{j\hat{\omega}}$$

Removing D-1 Samples and Packing (2)



$$V(z) = \dots + v[0]z^0 + 0 + \dots + 0 + v[D]z^{-D} + 0 + \dots + 0 + v[2D]z^{-2D} + \dots$$

$$V(z) = \sum_{n=-\infty}^{+\infty} v[n]z^{-n} = \sum_{m=-\infty}^{+\infty} v[mD]z^{-mD} = F(z^D)$$

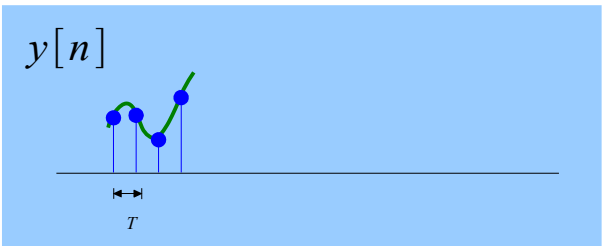


$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

$$\begin{aligned} Y(z) &= \dots + v[0]z^0 + v[D]z^{-1} + v[2D]z^{-2} + v[3D]z^{-3} + \dots \\ &= \dots + y[0]z^0 + y[1]z^{-1} + y[2]z^{-2} + y[3]z^{-3} + \dots \end{aligned}$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

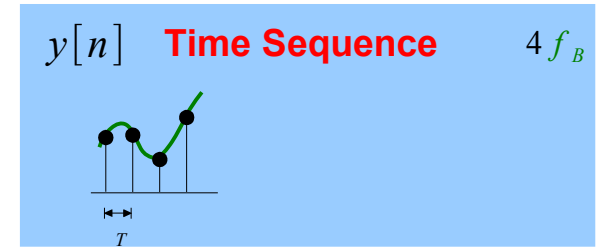
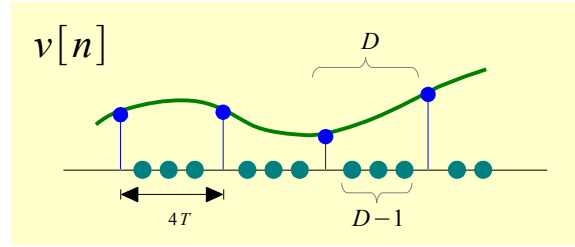
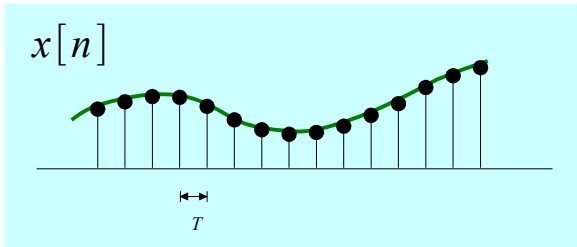
$$V(z^{1/D}) = \sum_{n=-\infty}^{+\infty} v[n]z^{-n/D} = \sum_{m=-\infty}^{+\infty} v[mD]z^{-m}$$



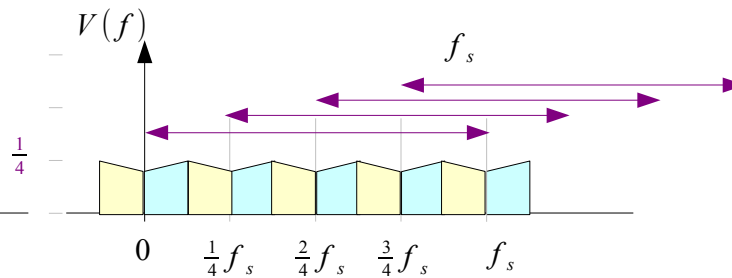
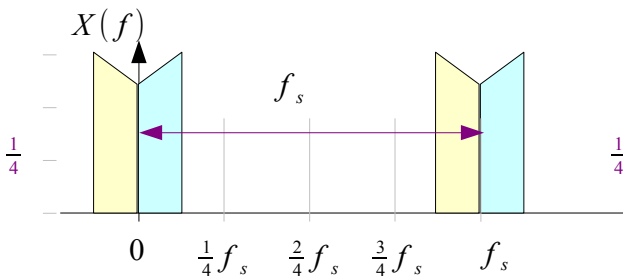
$$Y(z) = V(z^{1/D}) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D})$$

$$Y(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(\frac{\hat{\omega}}{D} - k \frac{2\pi}{D}\right)$$

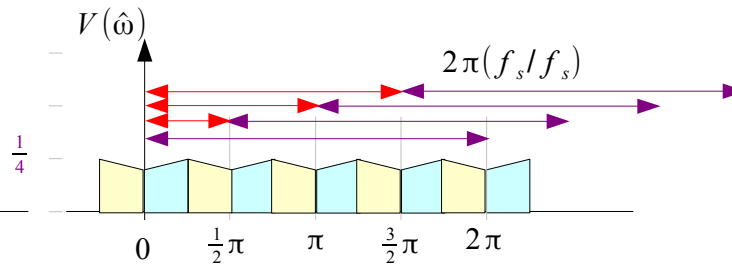
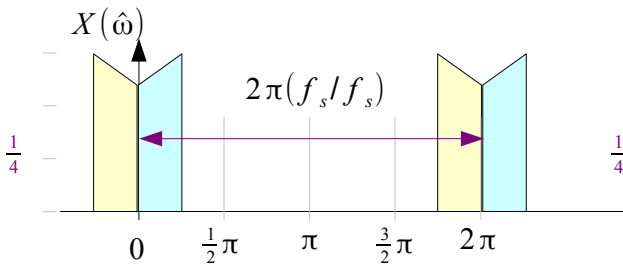
Down-Sampling Spectra (1)



$$\begin{aligned}
 X(z) & & V(z) &= \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z) & Y(z) &= \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D}) \\
 X(\hat{\omega}) & & V(\hat{\omega}) &= \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D) & Y(\hat{\omega}) &= \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)
 \end{aligned}$$

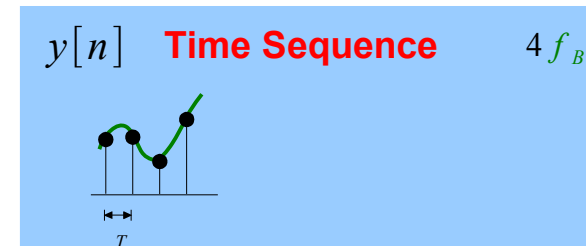
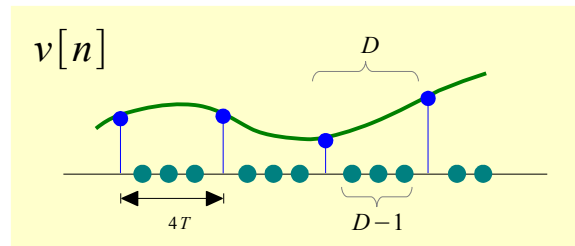
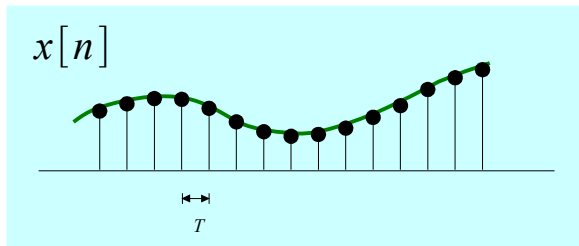


$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$



$$\frac{2\pi \cdot 1}{4} \quad \frac{2\pi \cdot 2}{4} \quad \frac{2\pi \cdot 3}{4} \quad 2\pi k/D$$

Down-Sampling Spectra (2)



$$X(z)$$

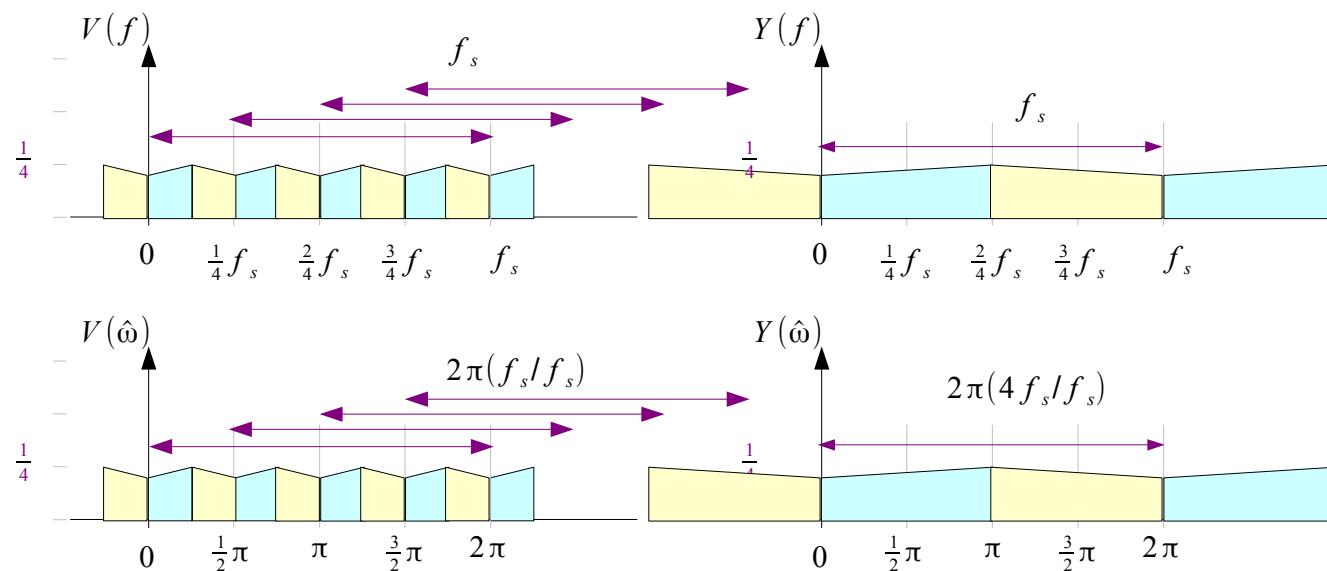
$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

$$Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D})$$

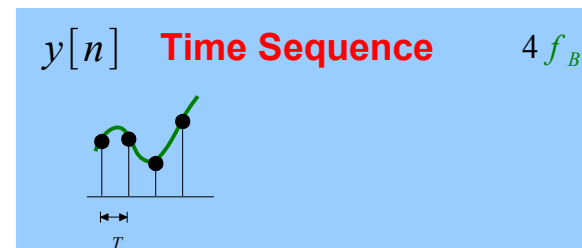
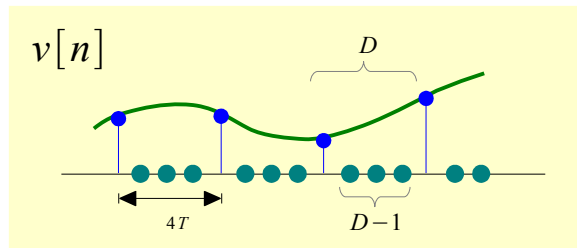
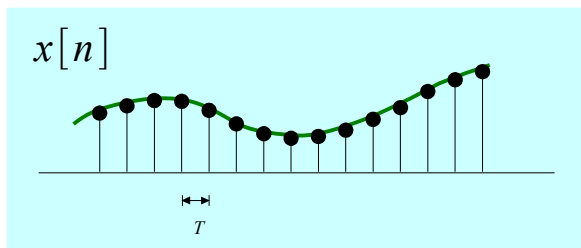
$$X(\hat{\omega})$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

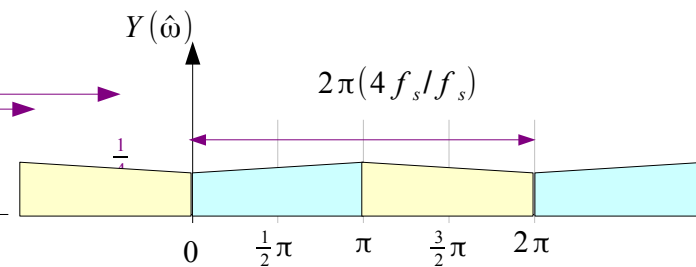
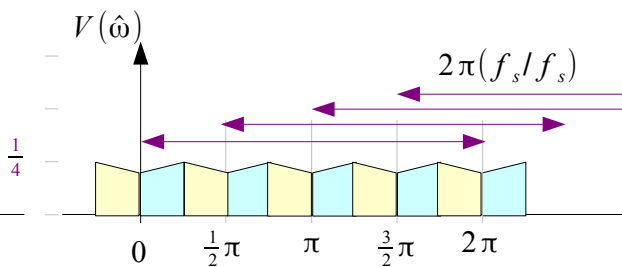
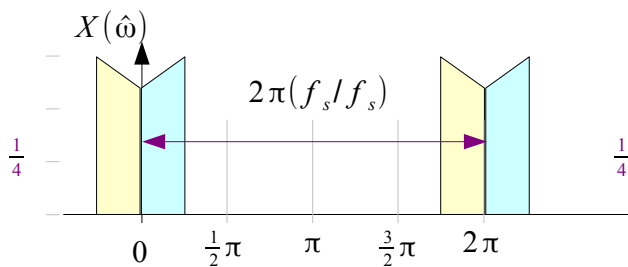
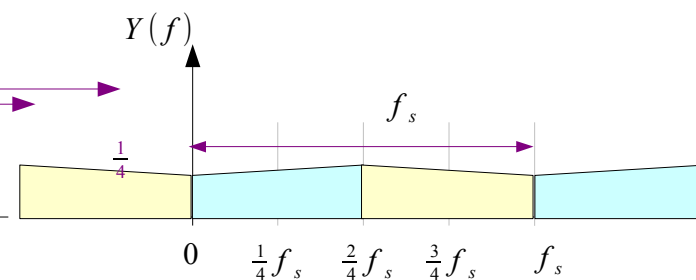
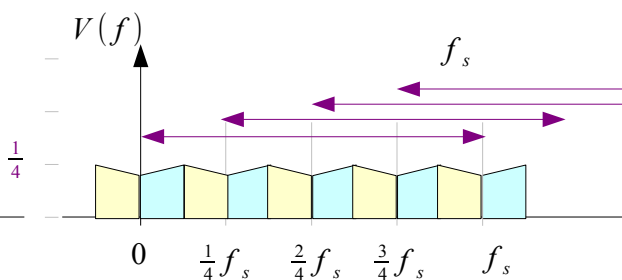
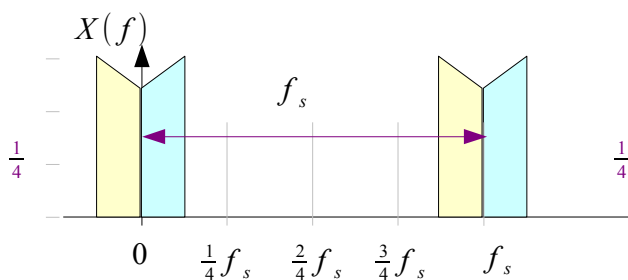
$$Y(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)$$



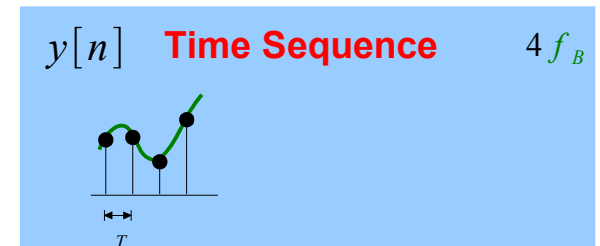
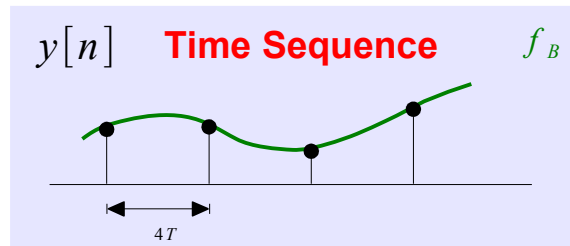
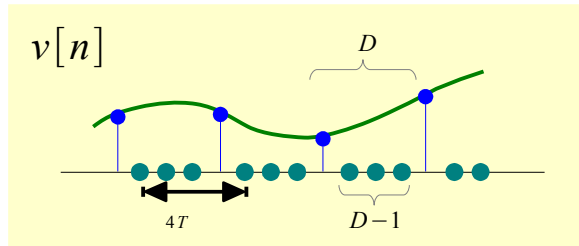
Down-Sampling Spectra (2)



$$\begin{aligned}
 X(z) & & V(z) &= \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z) & Y(z) &= \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D}) \\
 X(\hat{\omega}) & & V(\hat{\omega}) &= \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D) & Y(\hat{\omega}) &= \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)
 \end{aligned}$$



Z-Transform & Removing D-1 Zeros (3)

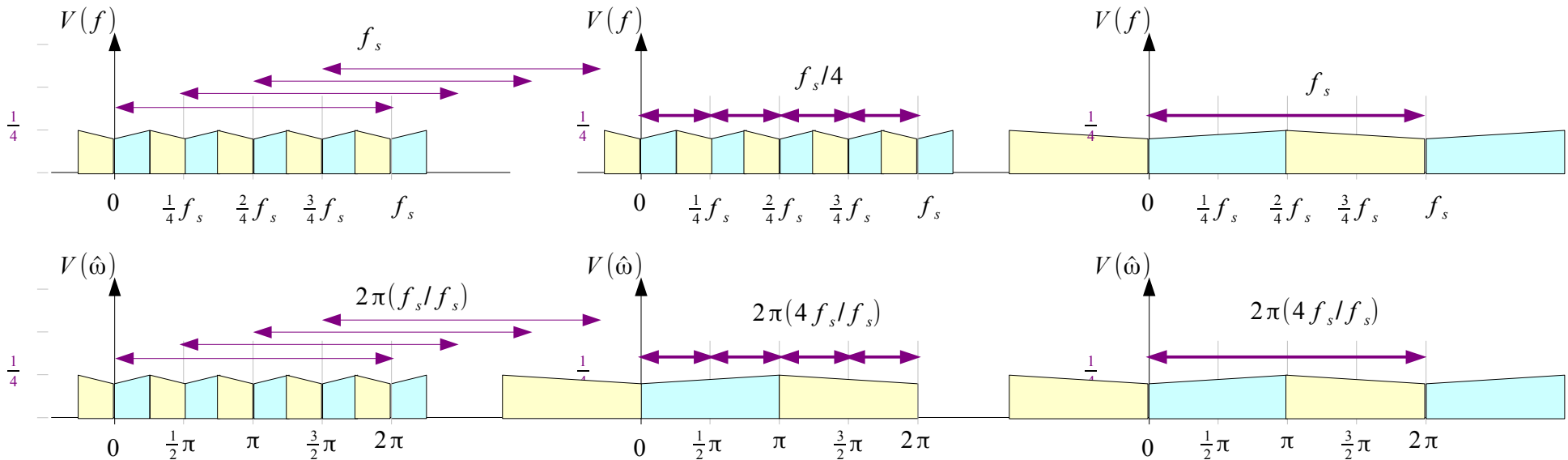


$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

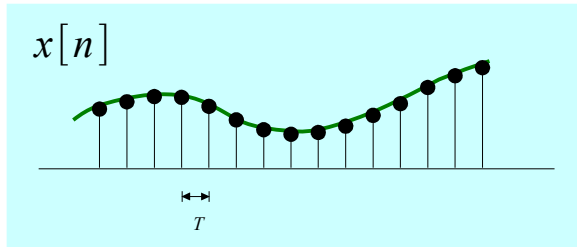
$$Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D})$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

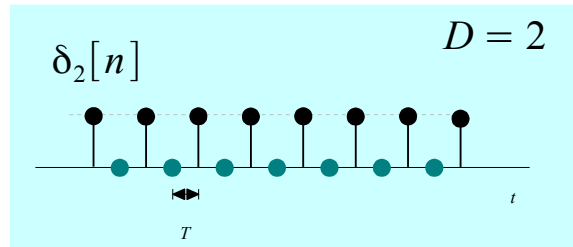
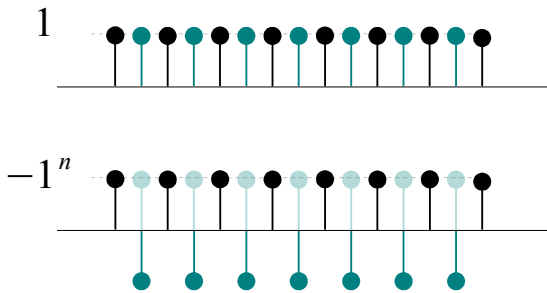
$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)$$



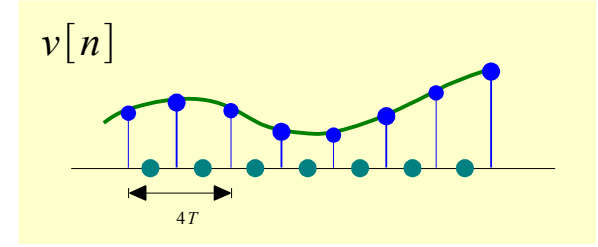
Example When $D=2$ (1)



$$x[n] = e^{j\omega n}$$



$$\begin{aligned} \delta_2[n] &= \frac{1}{2}(1 + (-1)^n) \\ &= \frac{1}{2}(1 + e^{-j\pi n}) \\ &\quad (e^{-j\pi} = -1) \end{aligned}$$



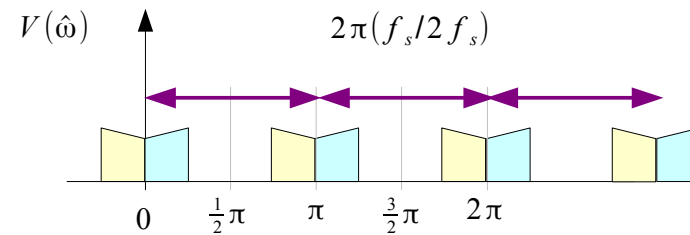
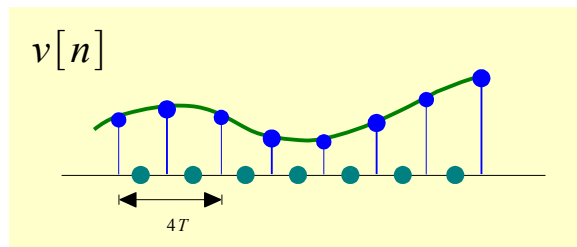
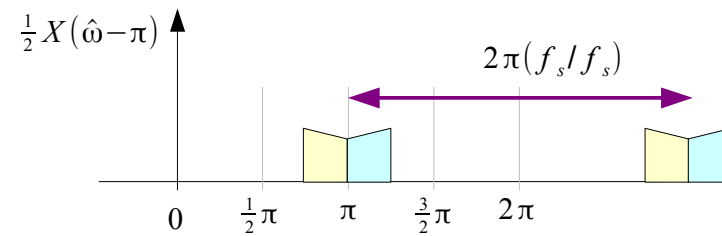
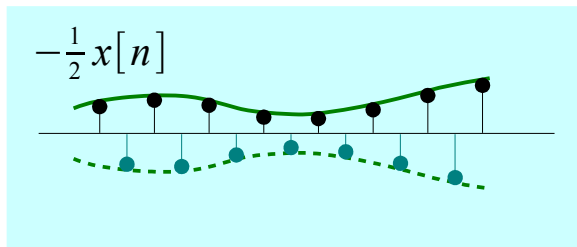
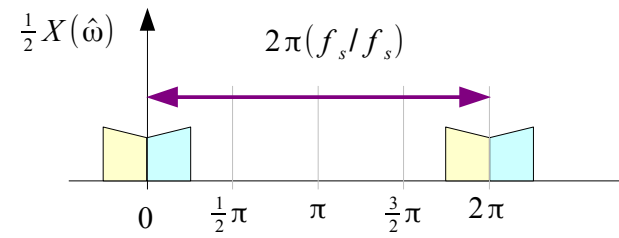
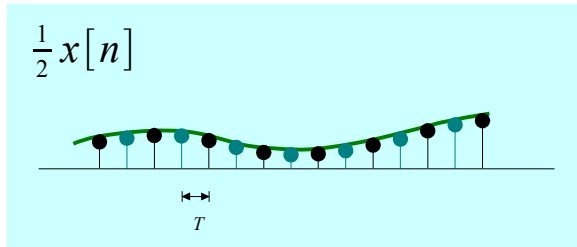
$$\begin{aligned} v[n] &= \frac{1}{2}x[n] + \frac{1}{2}e^{-j\pi n}x[n] \\ &= \frac{1}{2}e^{j\omega n} + \frac{1}{2}e^{-j\pi n}e^{j\omega n} \\ &= \frac{1}{2}e^{j\omega n} + \frac{1}{2}e^{+j(\omega-\pi)n} \end{aligned}$$

$$\begin{aligned} V(z) &= \frac{1}{2} \sum_{n=-\infty}^{+\infty} (x[n]z^{-n} + x[n](-z)^{-n}) \\ &= \frac{1}{2}X(z) + \frac{1}{2}X(-z) \end{aligned}$$

$$V(e^{j\hat{\omega}}) = \frac{1}{2}X(e^{j\hat{\omega}}) + \frac{1}{2}X(e^{-j\pi}e^{j\hat{\omega}})$$

$$V(\hat{\omega}) = \frac{1}{2}X(\hat{\omega}) + \frac{1}{2}X(\hat{\omega} - \pi)$$

Example When D=2 (2)



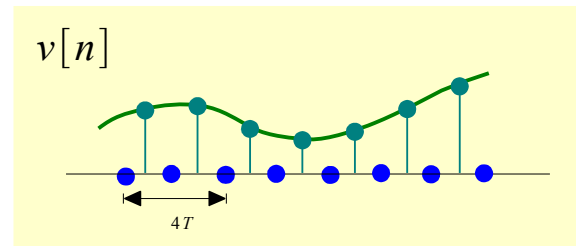
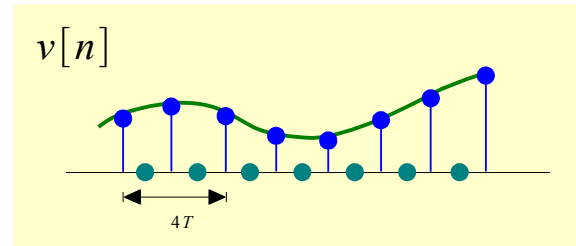
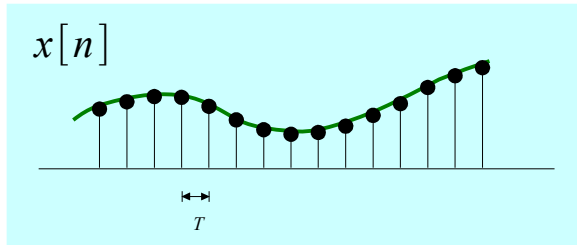
$$v[n] = \frac{1}{2}x[n] + \frac{1}{2}e^{-j\pi n}x[n]$$

$$V(z) = \frac{1}{2}X(z) + \frac{1}{2}X(-z)$$

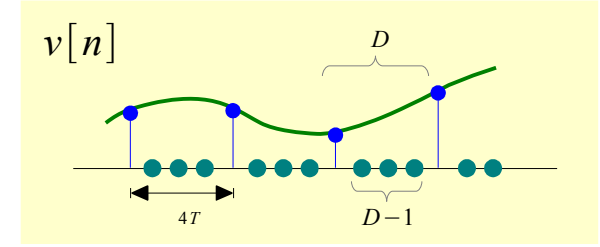
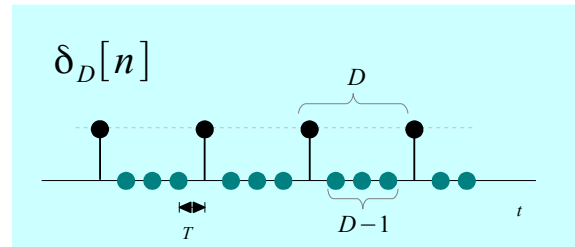
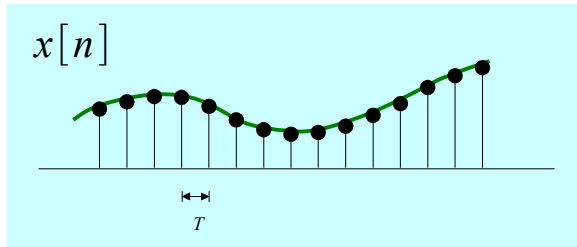
$$V(e^{j\hat{\omega}}) = \frac{1}{2}X(e^{j\hat{\omega}}) + \frac{1}{2}X(e^{-j\pi}e^{j\hat{\omega}})$$

$$V(\hat{\omega}) = \frac{1}{2}X(\hat{\omega}) + \frac{1}{2}X(\hat{\omega} - \pi)$$

Example When $D=2$ (3)

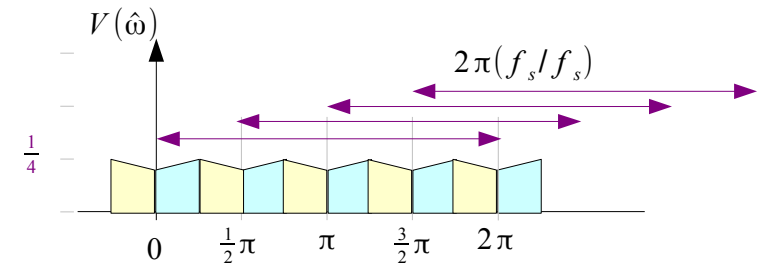
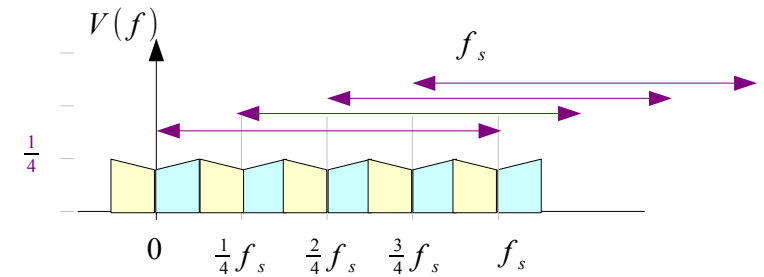
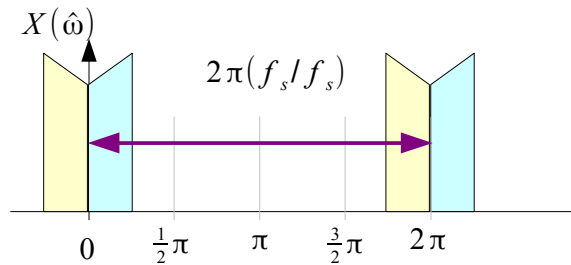
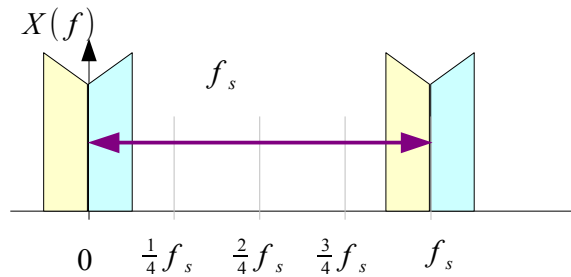


Z-Transform & Suppressing D-1 Samples (2)

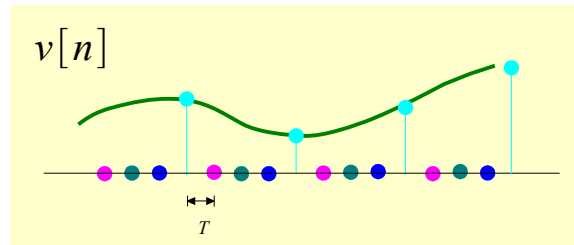
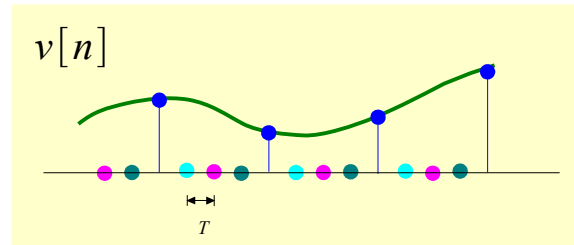
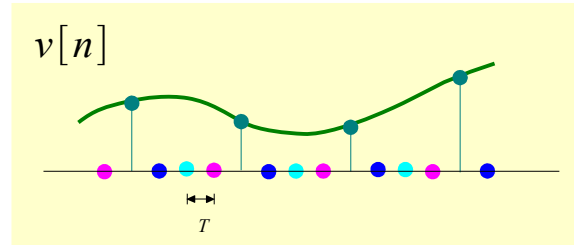
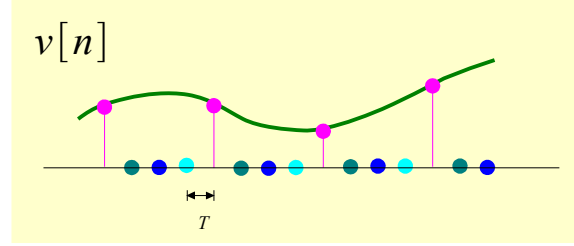
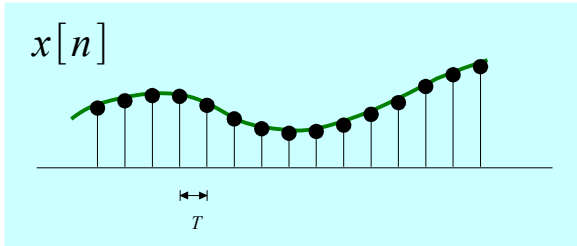


$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

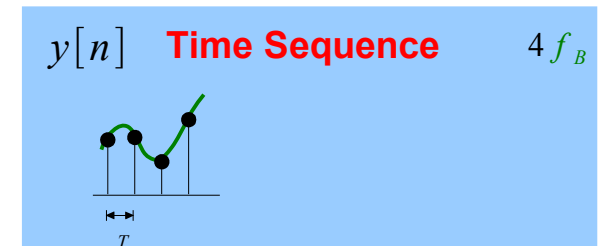
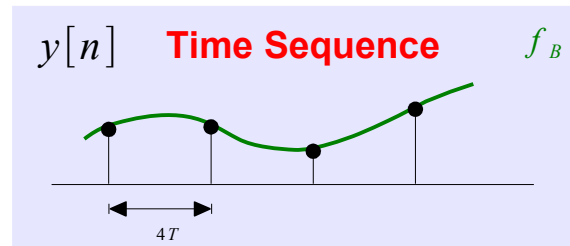
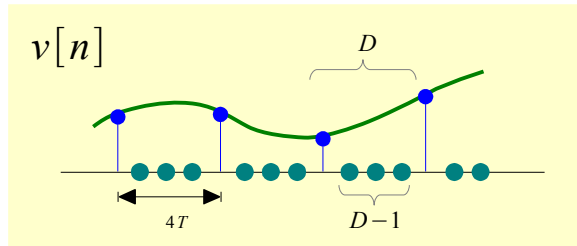
$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$



Example When $D=2$ (3)



Z-Transform & Removing D-1 Zeros (1)



$$\begin{aligned}
 Y(z) &= \cdots + \underline{y[0]} + \underline{y[1]}z^{-1} + \underline{y[2]}z^{-2} + \cdots + \underline{y[n]}z^{-n} + \cdots \\
 &= \cdots + \underline{v[0]} + \underline{v[D]}z^{-1} + \underline{v[2D]}z^{-2} + \cdots + \underline{v[nD]}z^{-n} + \cdots
 \end{aligned}$$

$$Y(z) = V(z^{1/D})$$

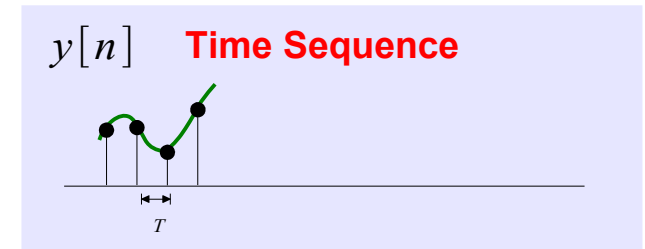
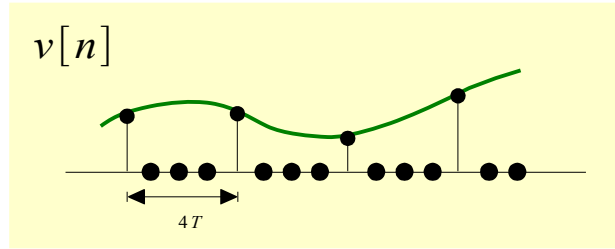
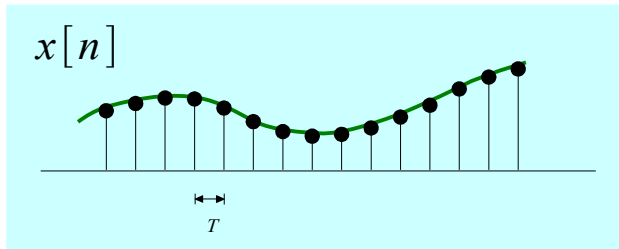
$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

$$Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D})$$

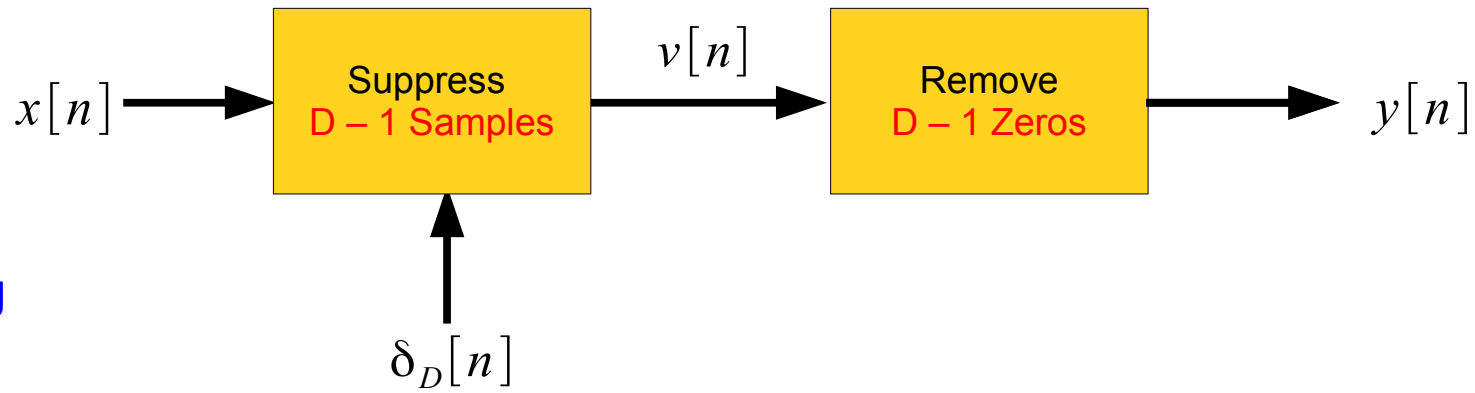
$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)$$

Down Sampling



T Sampling Period



Ideal Sampling

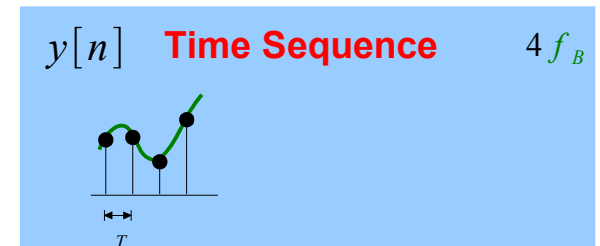
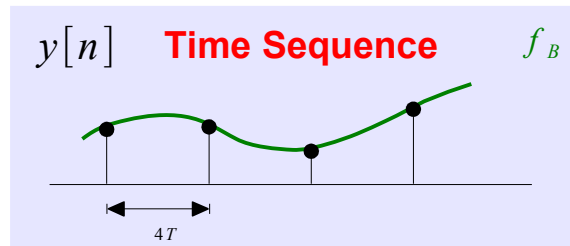
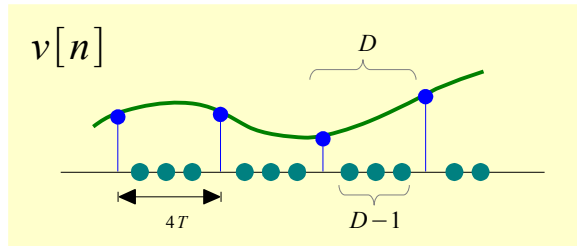
Amplitude Axis Scaling
 $1 / D$

Time Axis Scaling
 $1 / D$

Spectrum Replication
 f_s / D

Stretching Spectrum
 ω / D

Z-Transform & Removing D-1 Zeros (1)



$$\begin{aligned}
 Y(z) &= \cdots + \underline{y[0]} + \underline{y[1]}z^{-1} + \underline{y[2]}z^{-2} + \cdots + \underline{y[n]}z^{-n} + \cdots \\
 &= \cdots + \underline{v[0]} + \underline{v[D]}z^{-1} + \underline{v[2D]}z^{-2} + \cdots + \underline{v[nD]}z^{-n} + \cdots
 \end{aligned}$$

$$Y(z) = V(z^{1/D})$$

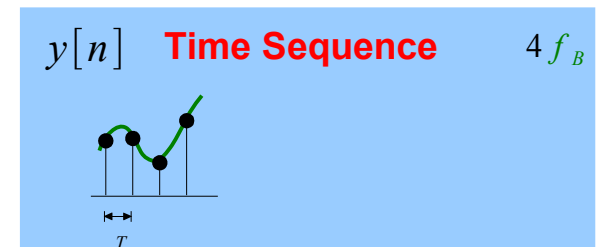
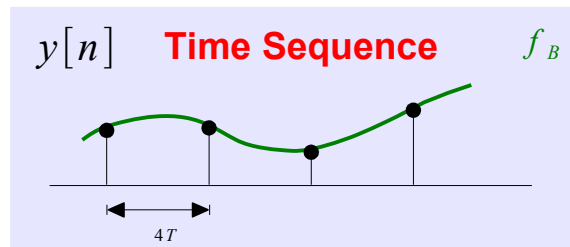
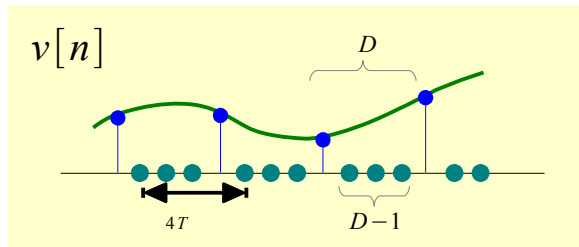
$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

$$Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D})$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)$$

Z-Transform & Removing D-1 Zeros (2)

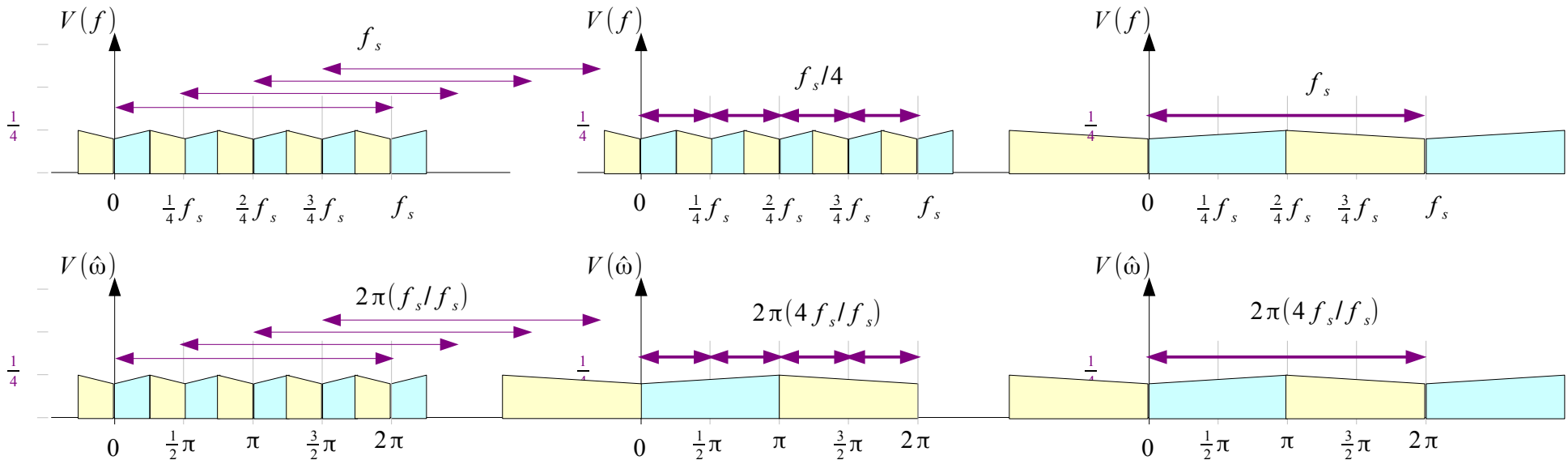


$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

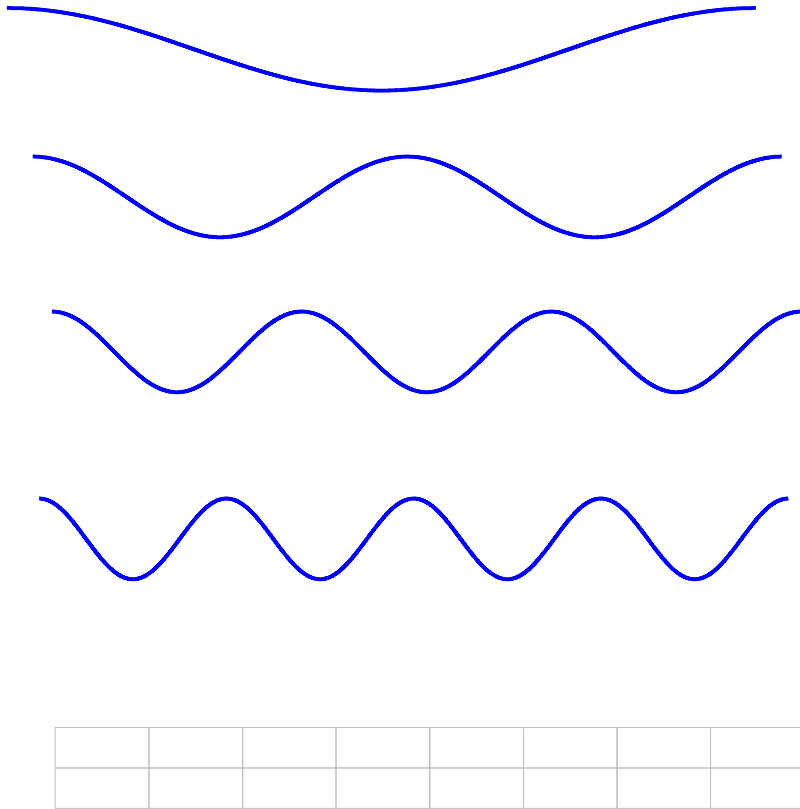
$$Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D})$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

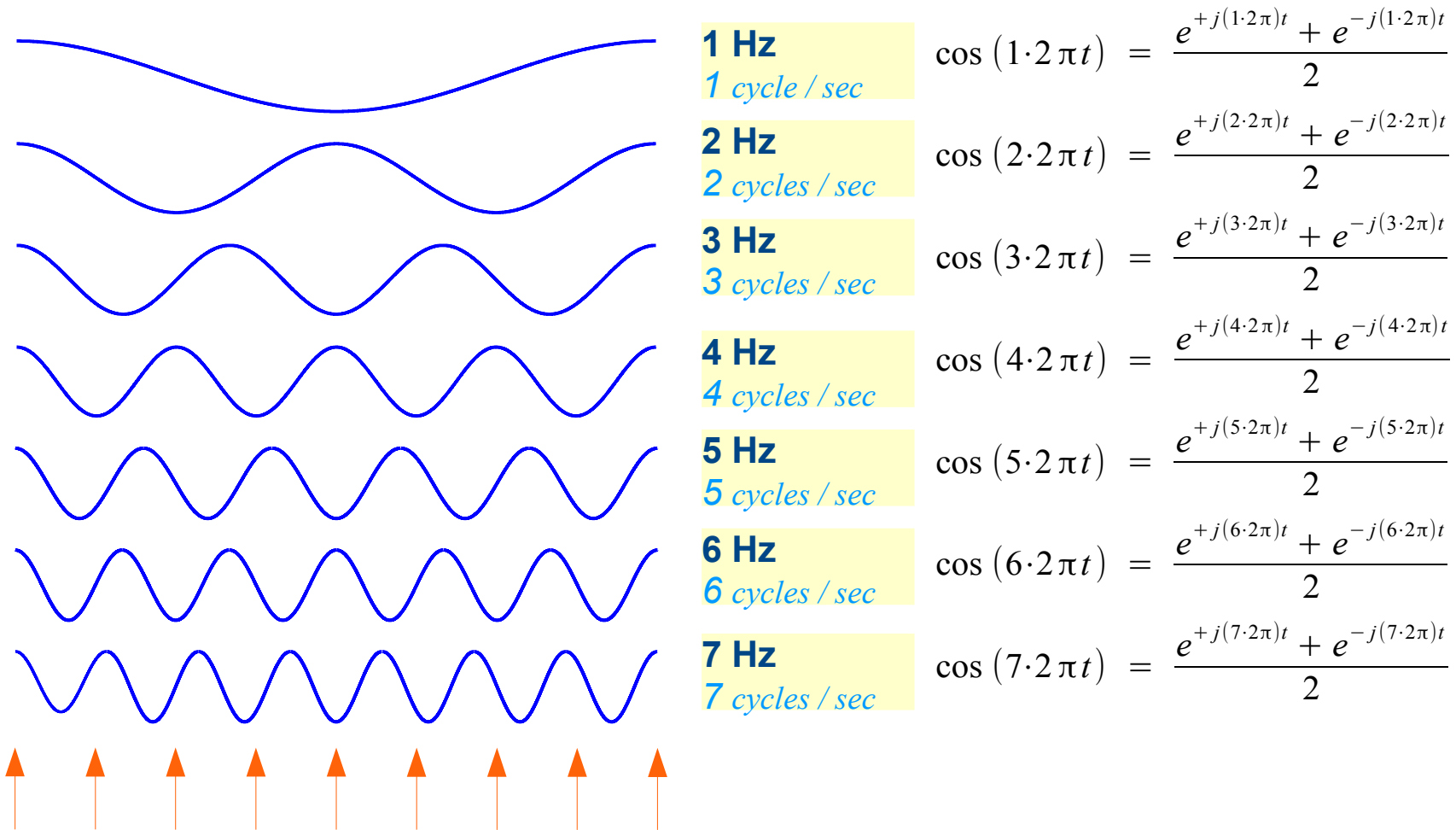
$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)$$



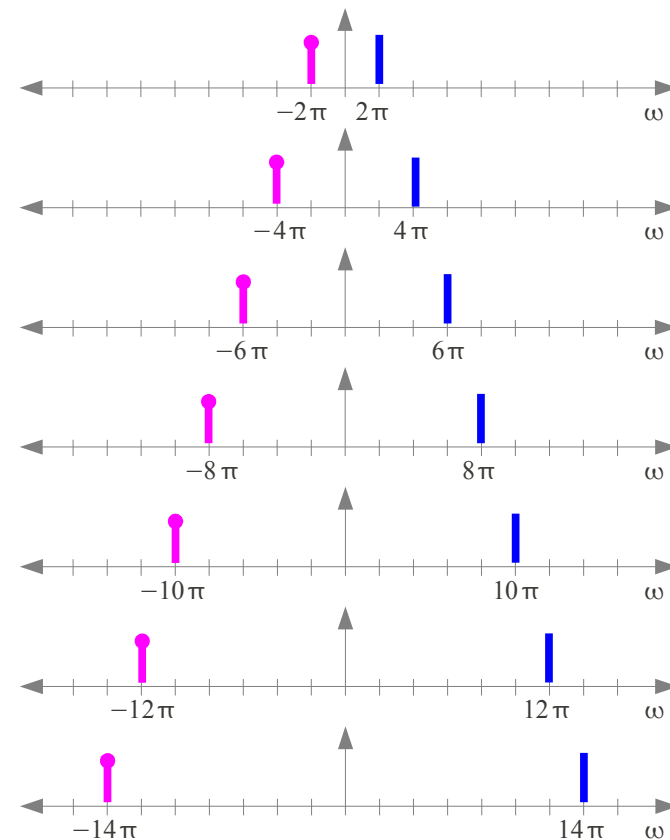
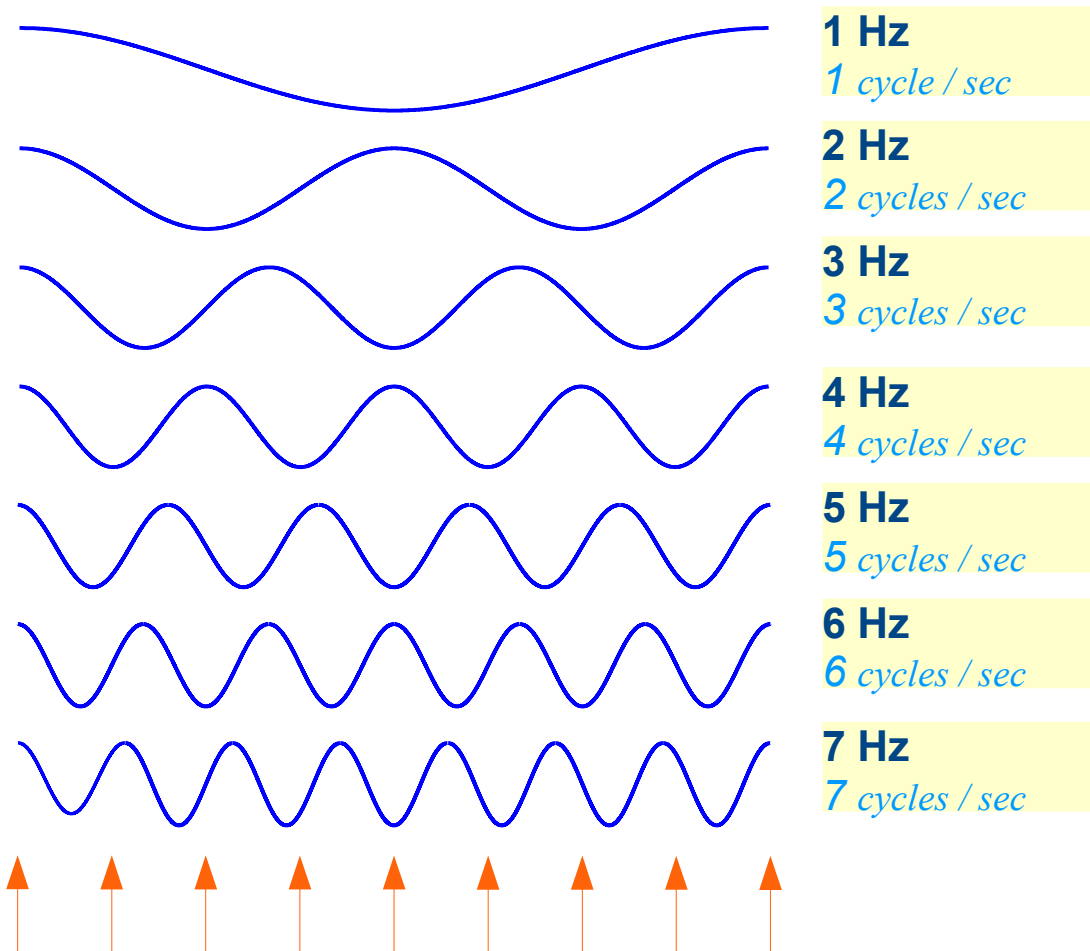
Measuring Rotation Rate



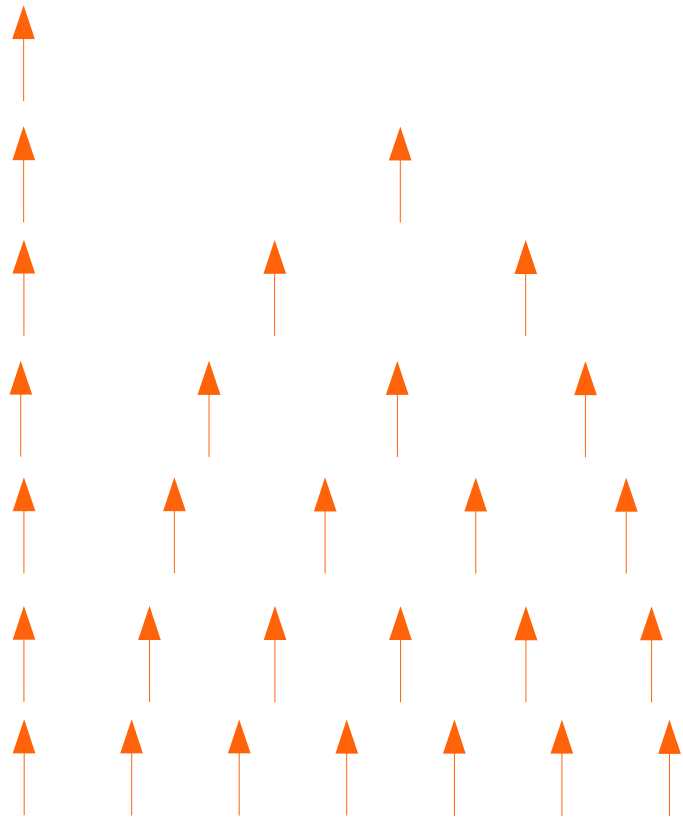
Signals with Harmonic Frequencies (1)



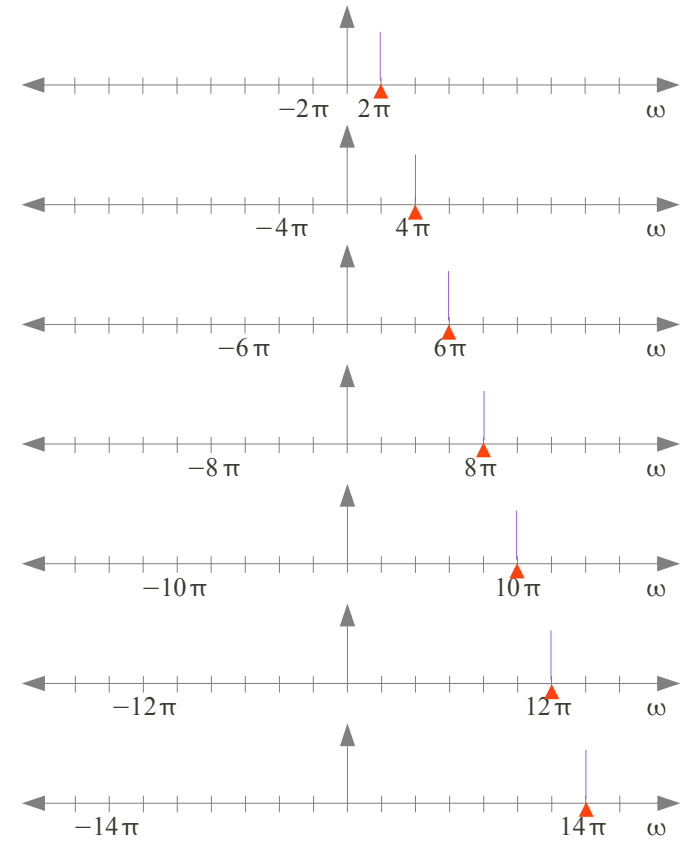
Signals with Harmonic Frequencies (2)



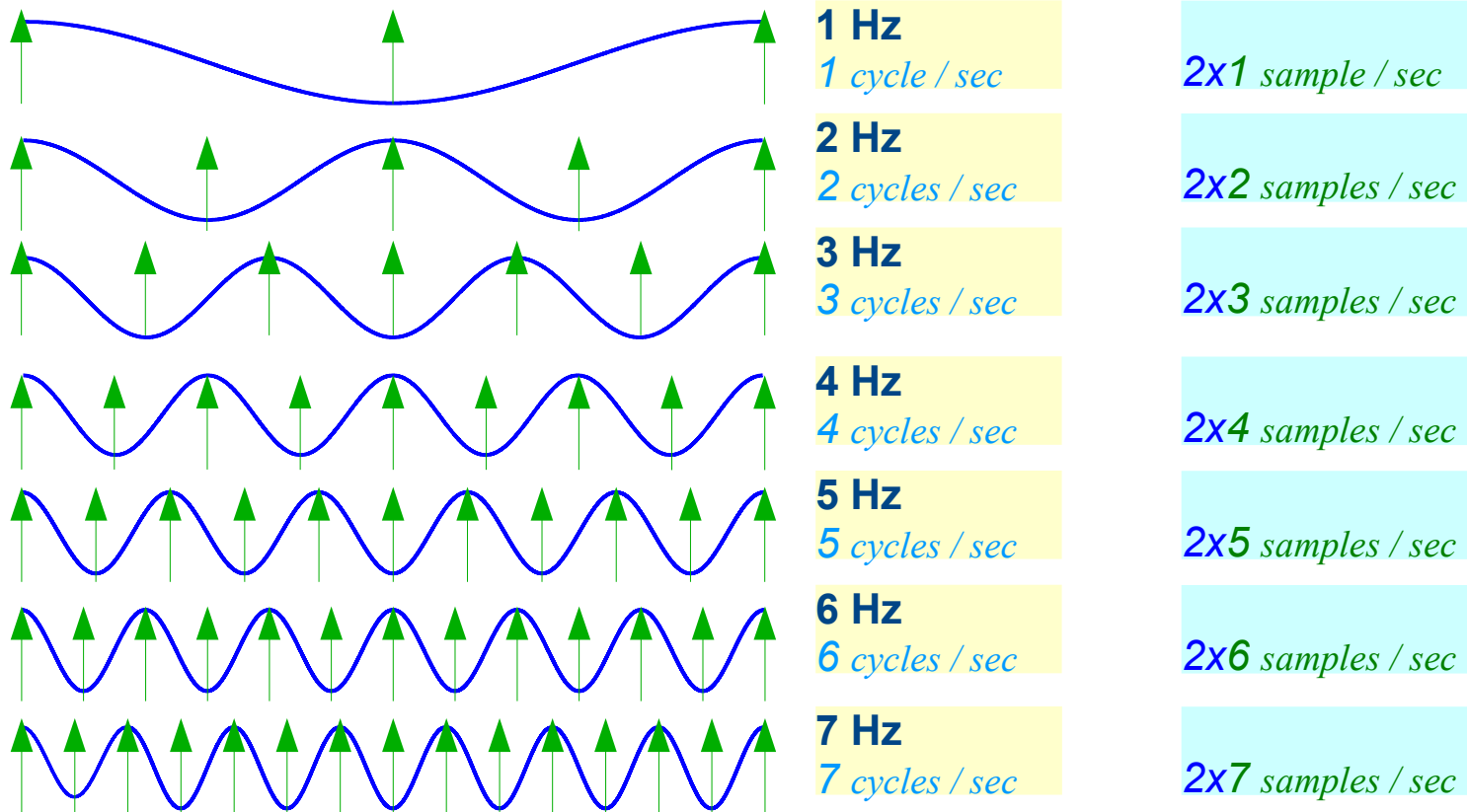
Sampling Frequency



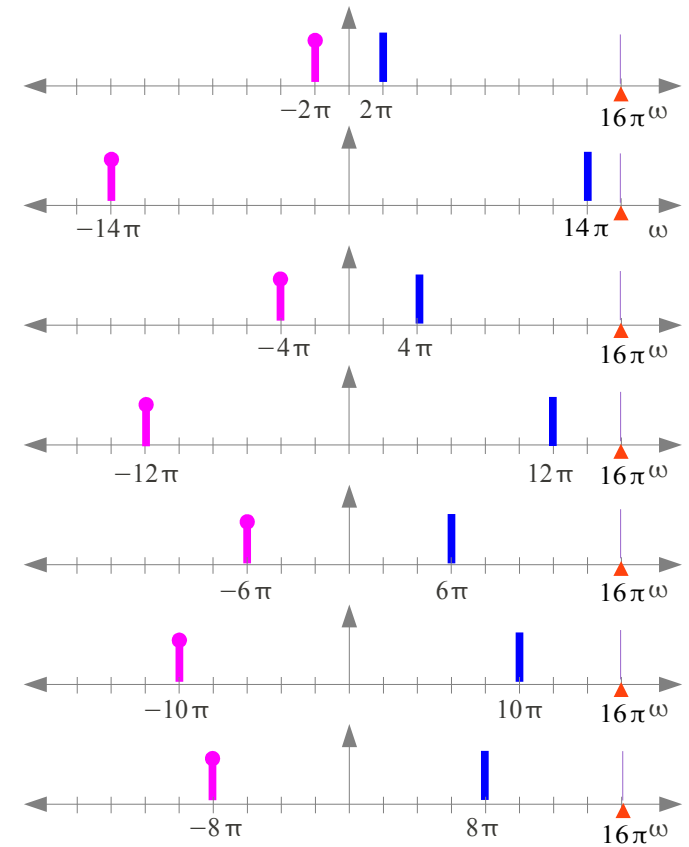
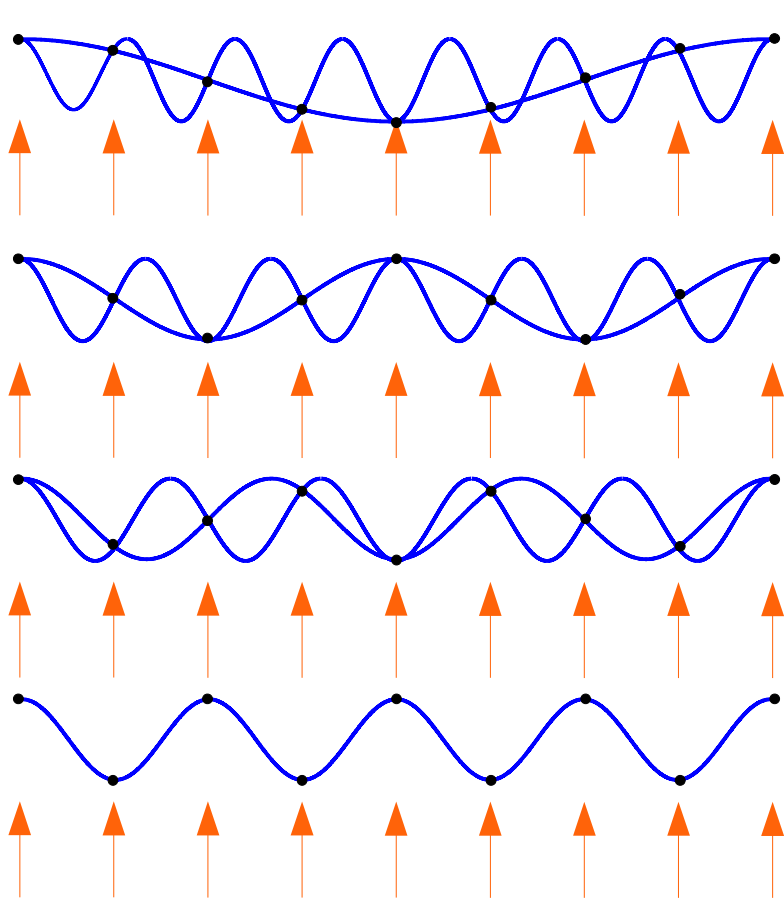
- 1 Hz
1 sample / sec
- 2 Hz
2 samples / sec
- 3 Hz
3 samples / sec
- 4 Hz
4 samples / sec
- 5 Hz
5 samples / sec
- 6 Hz
6 samples / sec
- 7 Hz
7 samples / sec



Nyquist Frequency

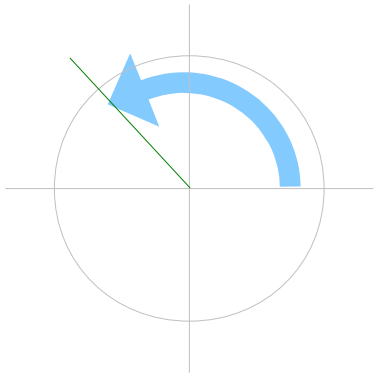


Aliasing



Sampling

$$\omega_s = 2\pi f_s \text{ (rad/sec)}$$

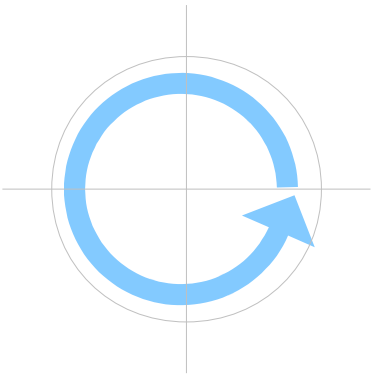


$$\omega_1 = 2\pi f_1$$

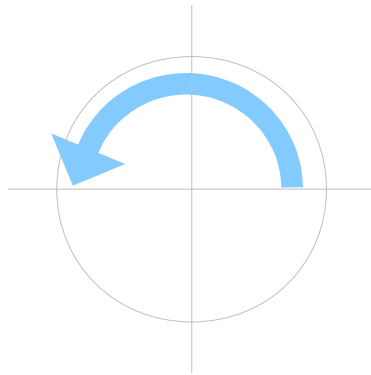
$$\omega_1 = \frac{\omega_s}{2} \text{ (rad/sec)}$$

$$f_1 = \frac{f_s}{2} \text{ (rad/sec)}$$

$$2\pi \text{ (rad)} / T_s \text{ (sec)}$$



$$\pi \text{ (rad)} / T_s \text{ (sec)}$$

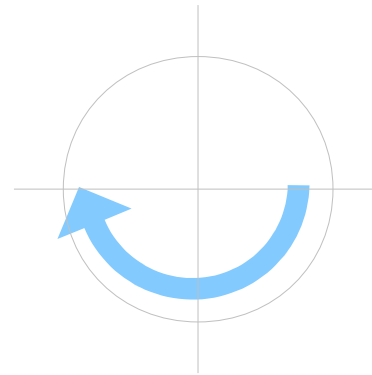


$$\omega_2 = 2\pi f_2$$

$$\omega_2 = -\frac{\omega_s}{2} \text{ (rad/sec)}$$

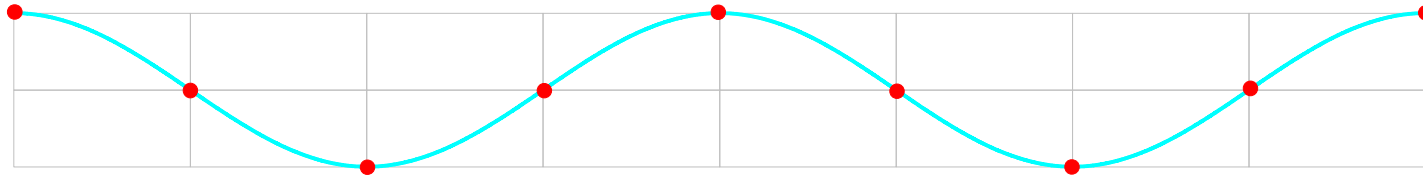
$$f_2 = -\frac{f_s}{2} \text{ (rad/sec)}$$

$$-\pi \text{ (rad)} / T_s \text{ (sec)}$$

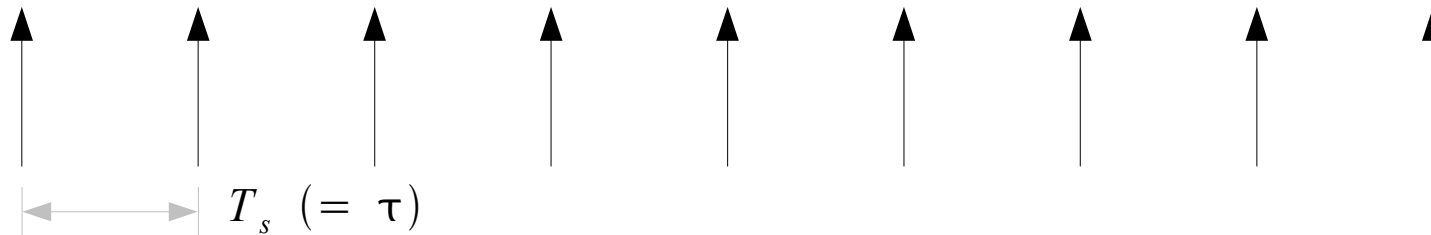


Sampling

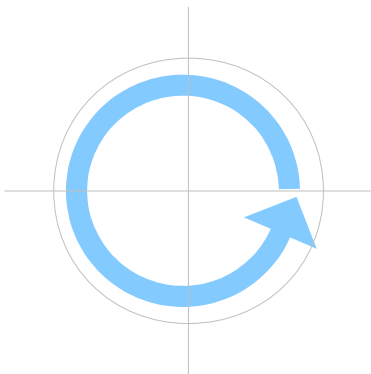
$$\omega_1 = 2\pi f_1 \text{ (rad/sec)}$$



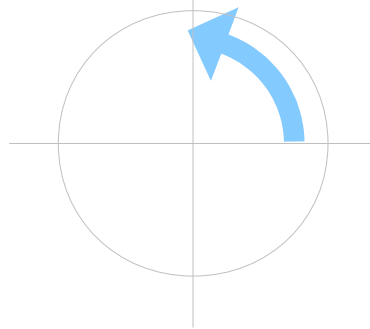
$$\omega_s = 2\pi f_s \text{ (rad/sec)}$$



$$2\pi \text{ (rad)} / T_s \text{ (sec)}$$



$$\frac{\pi}{2} \text{ (rad)} / T_s \text{ (sec)}$$



For the period of T_s
Angular displacement $\frac{\pi}{2}$ (rad)

$$\begin{aligned} \hat{\omega} &= \omega \cdot T_s \text{ (rad)} \\ &= 2\pi f_1 \cdot T_s \text{ (rad)} \\ &= 2\pi \frac{f_s}{4} \cdot T_s \text{ (rad)} \\ &= \frac{\pi}{2} \text{ (rad)} \end{aligned}$$

Angular Frequencies in Sampling

continuous-time signals

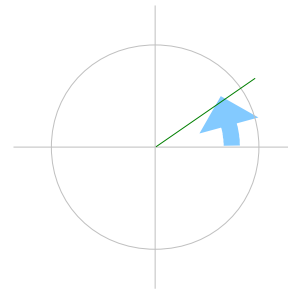
Signal Frequency

$$f_0 = \frac{1}{T_0}$$

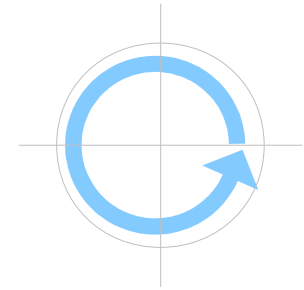
Signal Angular Frequency

$$\omega_0 = 2\pi f_0 \text{ (rad/sec)}$$

For 1
second
 $2\pi f_0 \text{ (rad/sec)}$



For 1
revolution
 $2\pi \text{ (rad)}$
 $T_0 \text{ (sec)}$



sampling sequence

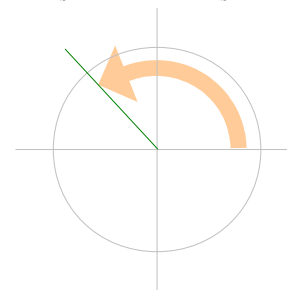
Sampling Frequency

$$f_s = \frac{1}{T_s}$$

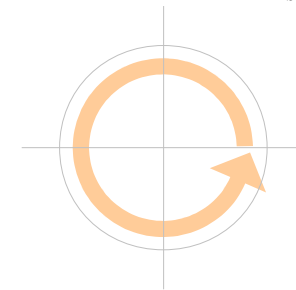
Sampling Angular Frequency

$$\omega_s = 2\pi f_s \text{ (rad/sec)}$$

For 1
second
 $2\pi f_s \text{ (rad/sec)}$



For 1
revolution
 $2\pi \text{ (rad)}$
 $T_s \text{ (sec)}$



References

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- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
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- [4] R. Cristi, "Modern Digital Signal Processing"