Matched Filter (3B)

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12/4/12

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Gaussian Random Process

Thermal Noise — zero-mean white Gaussian random process

$n(t)$ random function
the value at time t is characterized by
Gaussian probability density function

➡ $z(t) = a + n(t)$

$$p(n) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{n}{\sigma}\right)^2\right]$$

➡ $$p(z) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{z-a}{\sigma}\right)^2\right]$$

σ^2 variance of n

$\sigma = 1$ normalized (standardized)
Gaussian function

Central Limit Theorem

sum of statistically independent random variables
approaches Gaussian distribution
regardless of individual distribution functions

White Gaussian Noise (1)

Thermal Noise

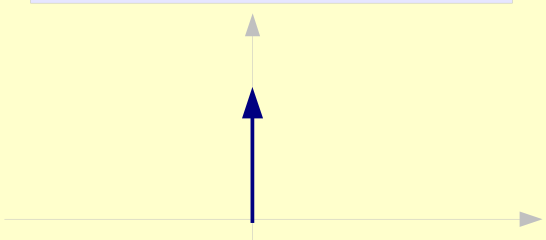
power spectral density is the same for all frequencies

$$G_n(f) = \frac{N_0}{2}$$ watts / hertz

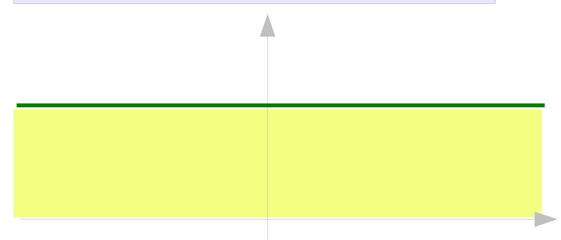
equal amount of noise power per unit bandwidth

uniform spectral density ➡ White Noise

$$R_n(t) = \frac{N_0}{2}\,\delta(t)$$



$$G_n(f) = \frac{N_0}{2}$$



$\delta(t)$ totally uncorrelated, noise samples are independent

memoryless channel

White Gaussian Noise (2)

$n($

Thermal Noise

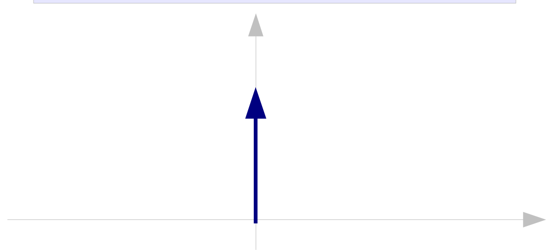
power spectral density is the same for all frequencies

$$G_n(f) = \frac{N_0}{2}$$ watts / hertz

equal amount of noise power per unit bandwidth

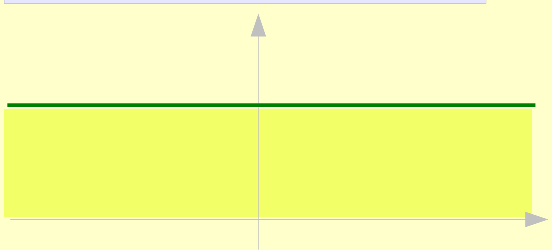
uniform spectral density ➡ White Noise

$$R_n(t) = \frac{N_0}{2}\delta(t)$$



⬌

$$G_n(f) = \frac{N_0}{2}$$



average power

$$P_x^T = \frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)\,dt = \int_{-\infty}^{+\infty} G_x(f)\,df$$

$$P_n = \int_{-\infty}^{+\infty} \frac{N_0}{2}\,df = \infty$$

White Gaussian Noise (3)

$n($

Additive White Gaussian Noise (AWGN)

additive and no multiplicative mechanism

average power

$P_n = \infty$

$R_n(t) = \frac{N_0}{2}\delta(t)$

$\Updownarrow$

$G_n(f) = \frac{N_0}{2}$

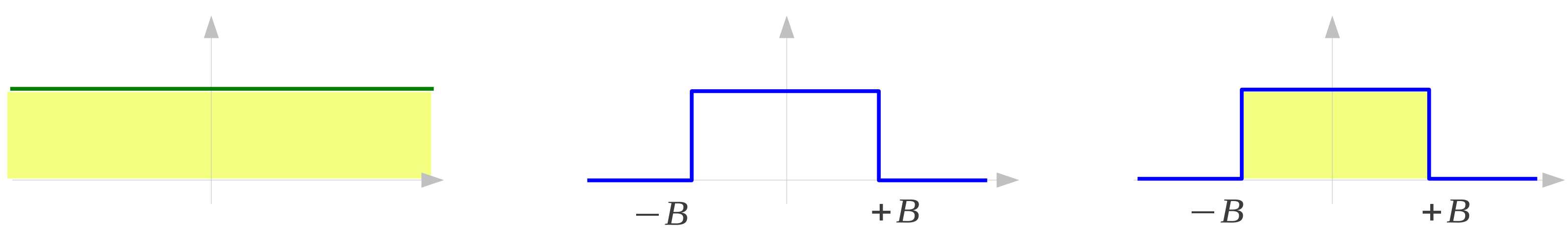
R.V		R.V
$X(t)$ →	$h(t)$	→ $Y(t)$
$R_{xx}(\tau)$ →	$h(t) * h^*(-t)$	→ $R_{yy}(\tau)$
$\Updownarrow$		$\Updownarrow$
$S_{xx}(\omega)$ →	$H(\omega)H^*(\omega)$	→ $S_{yy}(\omega)$

average power

$$P_{n0} = \int_{-B}^{+B} \frac{N_0}{2}\, df$$

$$= \frac{N_0}{2} \cdot 2B$$

$$= N_0 B$$



White Gaussian Noise (4)

$$n_0(t) = n(t) * h(t)$$

$$G_n(f) = \frac{N_0}{2}$$

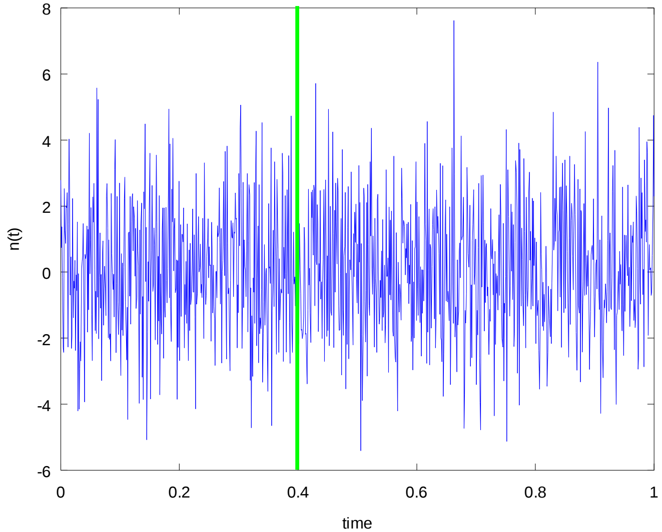
$$G_{n0}(f) = G_n(f)\,|H(f)|^2 = \begin{cases} \dfrac{N_0}{2}\,|H(f)|^2 & \text{for } |f| < f_u \\ 0 & \text{otherwise} \end{cases}$$

Average output noise power $$\sigma_0^2 = \overline{n_0^2(t)} = \frac{N_0}{2}\int_{-\infty}^{+\infty} |H(f)|^2\, df$$

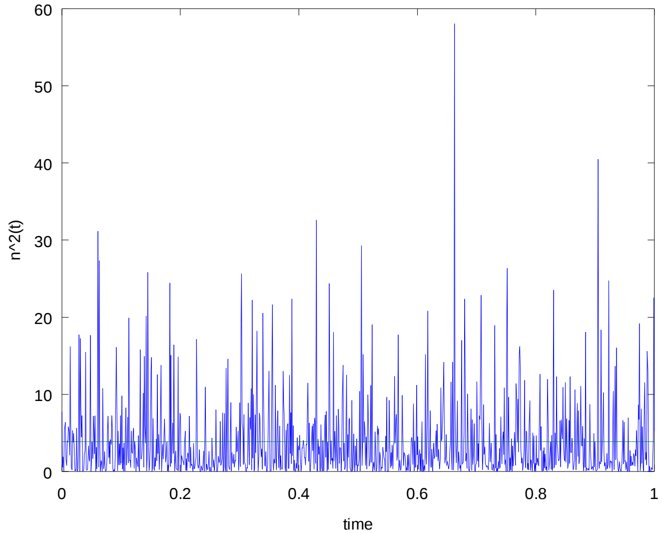
RMS $$\sigma_0 = \sqrt{\overline{n_0^2(t)}} = \sqrt{\frac{1}{T}\int_{-T/2}^{+T/2} n_0^2(t)\, dt}$$

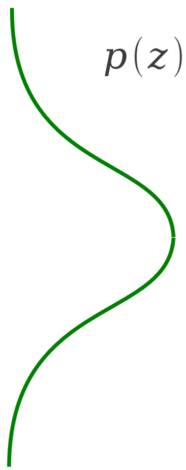
Gaussian Random Process

$n(t)$



$n^2(t)$

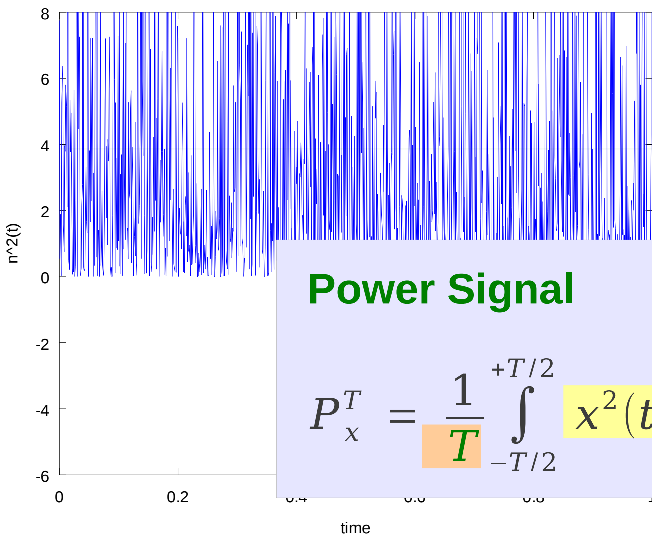




$$p(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2\right]$$

$m = 0$

$\sigma^2 \neq 0$



$n^2(t)$

Power Signal

$$P_x^T = \frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)\,dt$$

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Matched Filter (1)

to find a filter h(t) that gives max signal-to-noise ratio

$r(t) = s_i(t) + n(t)$

$z(t) = a_i(t) + n_0(t)$

variance of $n_0(t)$ ➡ σ_0^2 avg noise power

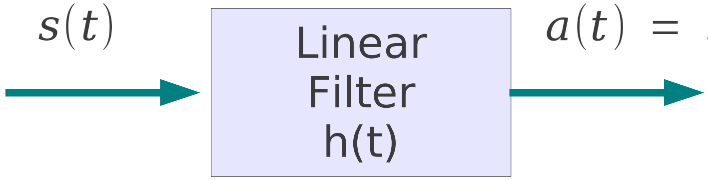
$$\frac{a_i^2(T)}{\overline{n_0^2(t)}}$$

$$\frac{\text{instantaneous signal power}}{\text{average noise power}} \Rightarrow \left(\frac{S}{N}\right)_T = \frac{a_i^{\,2}}{\sigma_0^{\,2}}$$

assume $H_0(f)$ a filter transfer function that maximizes $\left(\frac{S}{N}\right)_T$

Matched Filter (2)

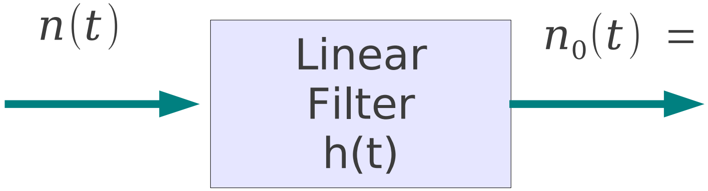
$s(t)$



$$a(t) = s(t) * h(t)$$

$$S(f) \qquad A(f) = S(f)H(f) \quad \Longleftrightarrow \quad a(t) = \int_{-\infty}^{+\infty} S(f)H(f)e^{j2\pi f t}\, df$$

$n(t)$



$$n_0(t) = n(t) * h(t)$$

$$G_n(f) = \frac{N_0}{2}$$

$$G_{n0}(f) = G_n(f)\,|H(f)|^2 = \begin{cases} \frac{N_0}{2}|H(f)|^2 & \text{for } |f| < f_u \\ 0 & \text{otherwise} \end{cases}$$

Average output noise power $\quad \sigma_0 = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H(f)|^2\, df$

Matched Filter (3)

instantaneous signal power a_i^2 ← $a(t) = \int_{-\infty}^{+\infty} S(f)H(f)e^{j2\pi f t}\,df$

average output noise power $\sigma_0 = \frac{N_0}{2}\int_{-\infty}^{+\infty} |H(f)|^2\,df$

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{\left|\int_{-\infty}^{+\infty} H(f)S(f)e^{+j2\pi f T}\,df\right|^2}{N_0/2\int_{-\infty}^{+\infty}|H(f)|^2\,df}$$

Does not depend on the particular shape of the waveform

Cauchy Schwarz's Inequality

$$\left|\int_{-\infty}^{+\infty} f_1(x) f_2(x)\,dx\right|^2 \le \int_{-\infty}^{+\infty}|f_1(x)|^2\,dx\int_{-\infty}^{+\infty}|f_2(x)|^2\,dx \qquad \text{'=' holds when } f_1(x) = k f_2^*(x)$$

$$\left|\int_{-\infty}^{+\infty} H(f)S(f)e^{+j2\pi ft}\,dx\right|^2 df \le \int_{-\infty}^{+\infty}|H(f)|^2\,df\int_{-\infty}^{+\infty}|S(f)e^{+j2\pi f T}|^2\,df \qquad |e^{+j2\pi f T}| = 1$$

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{\left|\int_{-\infty}^{+\infty} H(f)S(f)e^{+j2\pi f T}\,df\right|^2}{N_0/2\int_{-\infty}^{+\infty}|H(f)|^2\,df} \le \frac{\int_{-\infty}^{+\infty}|H(f)|^2\,df\int_{-\infty}^{+\infty}|S(f)e^{+j2\pi f T}|^2\,df}{N_0/2\int_{-\infty}^{+\infty}|H(f)|^2\,df} = \frac{2}{N_0}\int_{-\infty}^{+\infty}|S(f)|^2\,df$$

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Matched Filter (4)

Two-sided power spectral density of input noise ➡ $\dfrac{N_0}{2}$

Average noise power $\sigma_0 = \dfrac{N_0}{2} \int_{-\infty}^{+\infty} |H(f)|^2 \, df$

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{\left| \int_{-\infty}^{+\infty} H(f) S(f) e^{+j2\pi f T} \, df \right|^2}{N_0/2 \int_{-\infty}^{+\infty} |H(f)|^2 \, df}$$

Cauchy Schwarz's Inequality

$$\left(\frac{S}{N}\right)_T \le \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 \, df$$

$$max \left(\frac{S}{N}\right)_T = \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 \, df = \frac{2E}{N_0}$$

input signal energy (E)

power spectral density of input noise (N_0)

does not depend on the particular shape of the waveform

Matched Filter (5)

$$\left|\int_{-\infty}^{+\infty} H(f)S(f)e^{+j2\pi ft}\,dx\right|^2 df \le \int_{-\infty}^{+\infty} |H(f)|^2\,df \int_{-\infty}^{+\infty} |S(f)e^{+j2\pi fT}|^2\,df$$

$$\left(\frac{S}{N}\right)_T \le \frac{2}{N_0}\int_{-\infty}^{+\infty} |S(f)|^2\,df$$

$$max\left(\frac{S}{N}\right)_T = \frac{2}{N_0}\int_{-\infty}^{+\infty} |S(f)|^2\,df = \frac{2E}{N_0}$$

when complex conjugate relationship exists

$$H(f) = H_0(f) = k\,S^*(f)e^{-j2\pi fT}$$

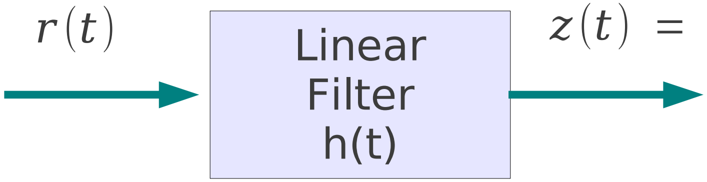
$$\Leftrightarrow\quad h(t) = h_0(t) = \begin{cases} k\,s(T-t) & 0 \le t \le T \\ 0 & elsewhere \end{cases}$$

$H_0(f)$ a filter transfer function that maximizes $\left(\frac{S}{N}\right)_T$

impulse response : delayed version of
the mirror image of the signal waveform

Convolution vs. Correlation Realization

$r(t)$



$z(t) = r(t) * h(t)$

$$z(t) = \int_0^t r(\tau) h(t - \tau) \, d\tau$$
$$= \int_0^t r(\tau) s(T - (t - \tau)) \, d\tau$$
$$= \int_0^t r(\tau) s(T - t + \tau) \, d\tau$$

$$z(T) = \int_0^T r(\tau) s(\tau) \, d\tau$$

shift position

convolution $z(t) = \int_0^t r(\tau) s(T - t + \tau) \, d\tau$ $\qquad$ $z(T) = \int_0^T r(\tau) s(\tau) \, d\tau$

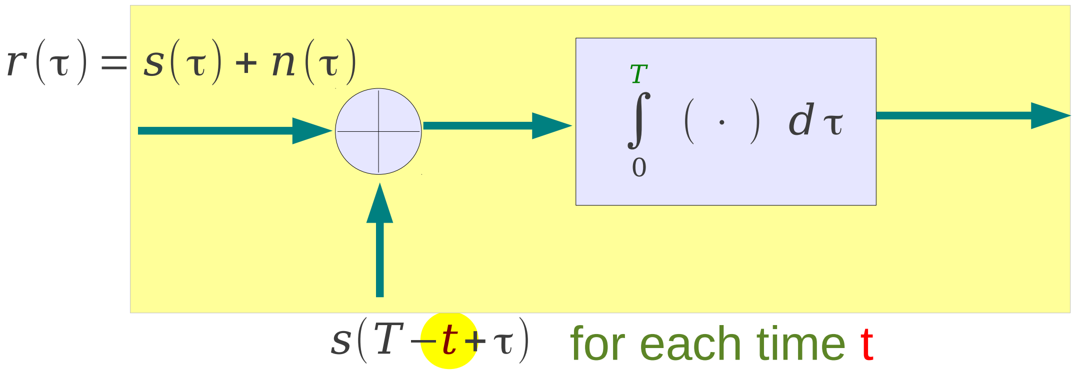
a sine-wave amplitude modulated
by a linear ramp

correlation $z(t) = \int_0^t r(\tau) s(\tau) \, d\tau$ $\qquad$ $z(T) = \int_0^T r(\tau) s(\tau) \, d\tau$

a linear ramp output

fixed position

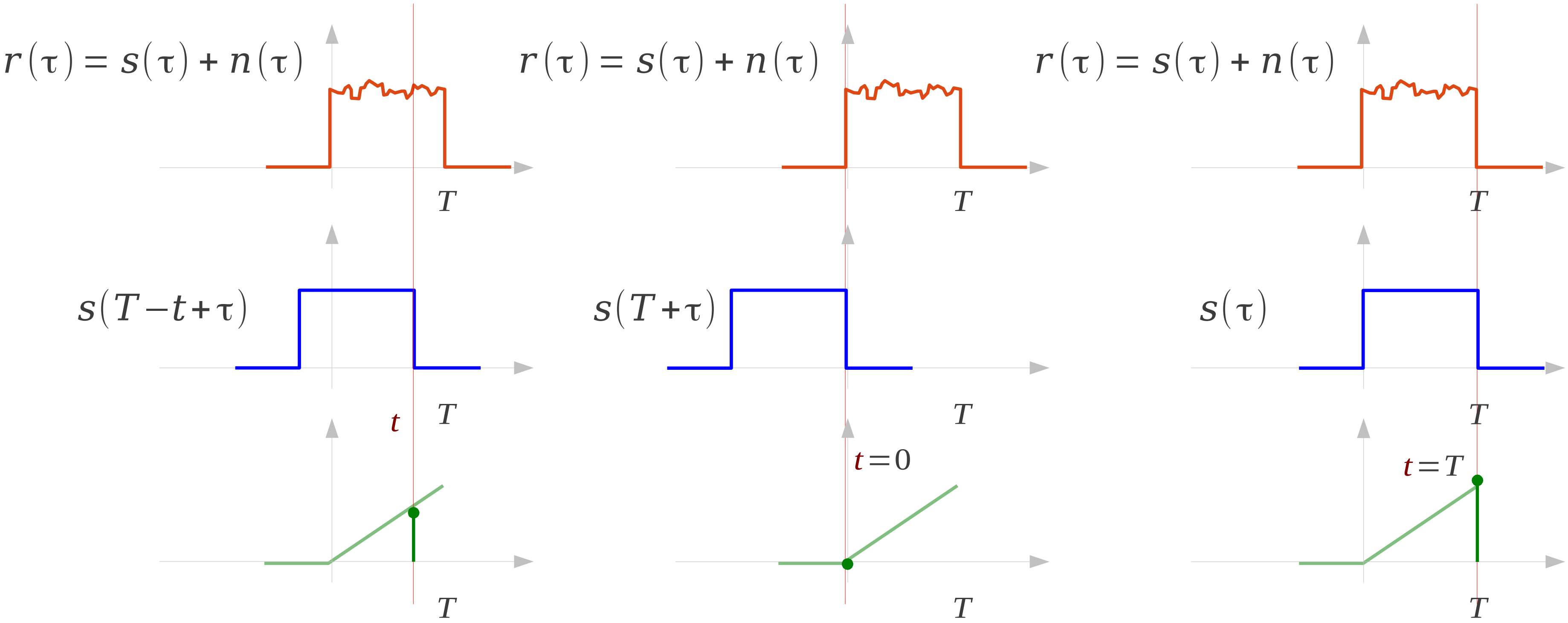
Convolution Realization



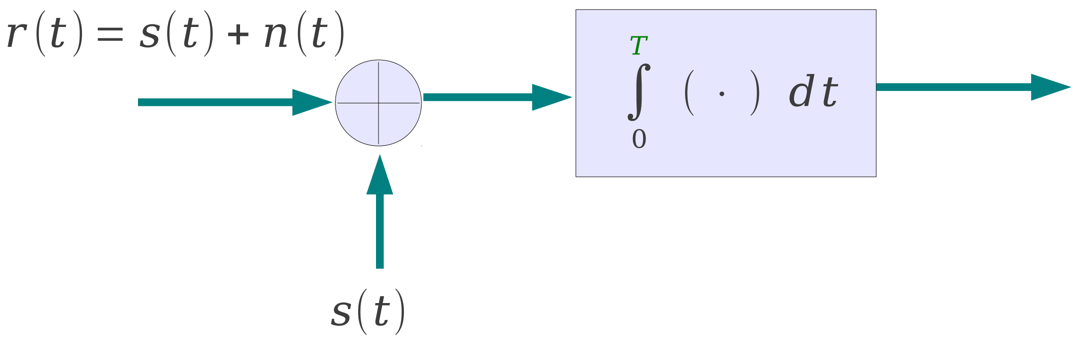
$$z(t) = \int_0^t r(\tau) h(t-\tau)\, d\tau$$

$$= \int_0^t r(\tau) s(T-(t-\tau))\, d\tau$$

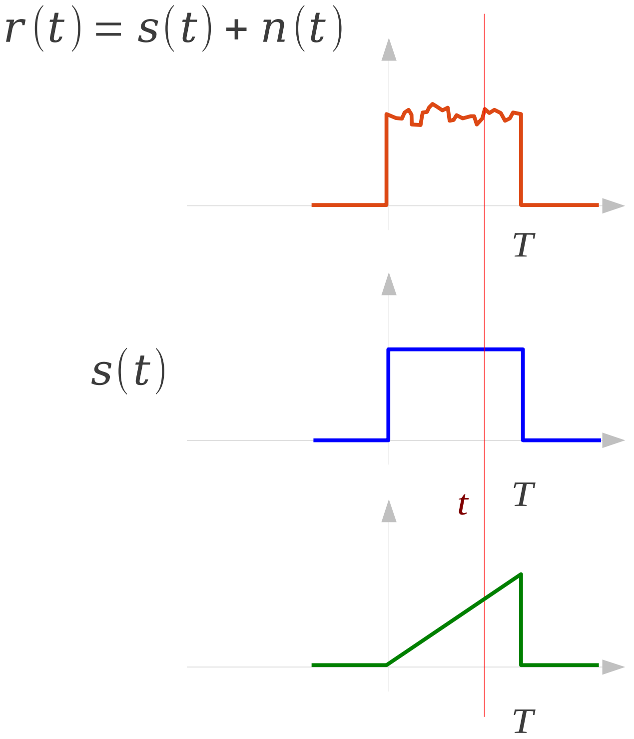
$$= \int_0^t r(\tau) s(T-t+\tau)\, d\tau$$



Correlation Realization (1)

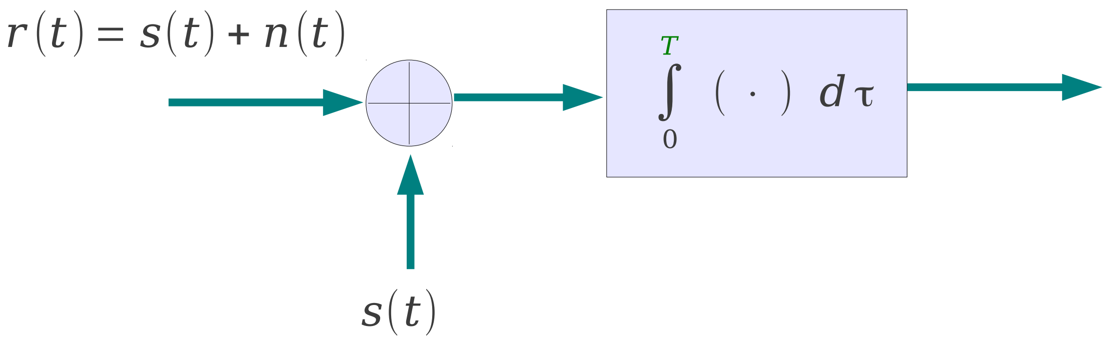


$$z(T) = \int_0^T r(t)s(t)\,dt$$



$$z(t) = \int_0^t r(\tau)s(\tau)\,d\tau$$

Correlation Realization (2)



$$z(T) = \int_0^T r(\tau)s(\tau)\,d\tau$$

$r(t) = s(t)$

$$z(T) = \int_0^T s^2(\tau)\,d\tau = E$$

$$\sigma_0^2 = \boldsymbol{E}[n_o(t)] = \boldsymbol{E}\left[\int_0^T n(t)s(t)\,dt \int_0^T n(\tau)s(\tau)\,d\tau\right]$$

$$= \boldsymbol{E}\left[\iint_0^T n(t)n(\tau)\, s(t)s(\tau)\,dt\,d\tau\right]$$

$$= \iint_0^T \boldsymbol{E}[n(t)n(\tau)]\, s(t)s(\tau)\,dt\,d\tau$$

$$= \iint_0^T \frac{N_0}{2}\delta(t-\tau)\, s(t)s(\tau)\,dt\,d\tau$$

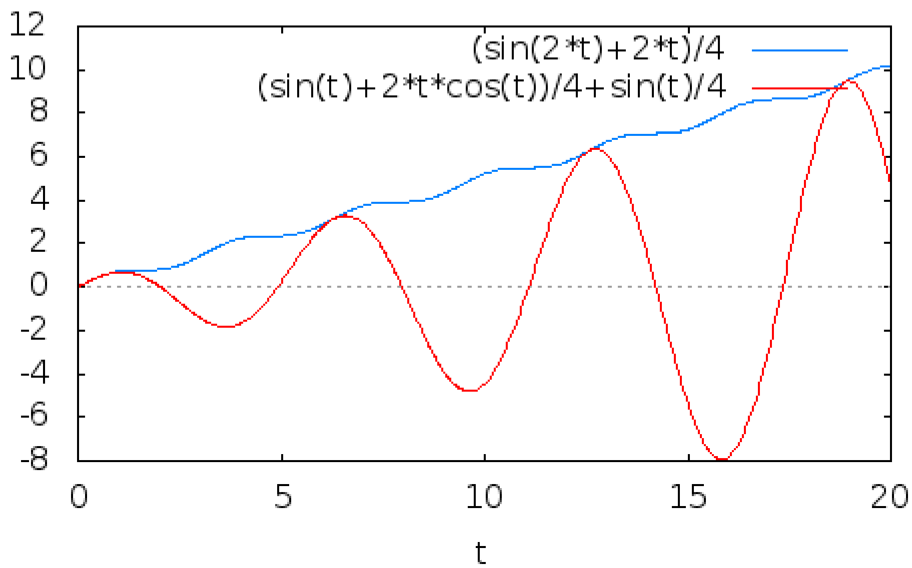
$$= \frac{N_0}{2}\int_0^T s^2(t)\,dt = \frac{N_0}{2}E$$

$$\frac{a_i^2(T)}{\overline{n_0^2(t)}} \qquad \left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2}$$

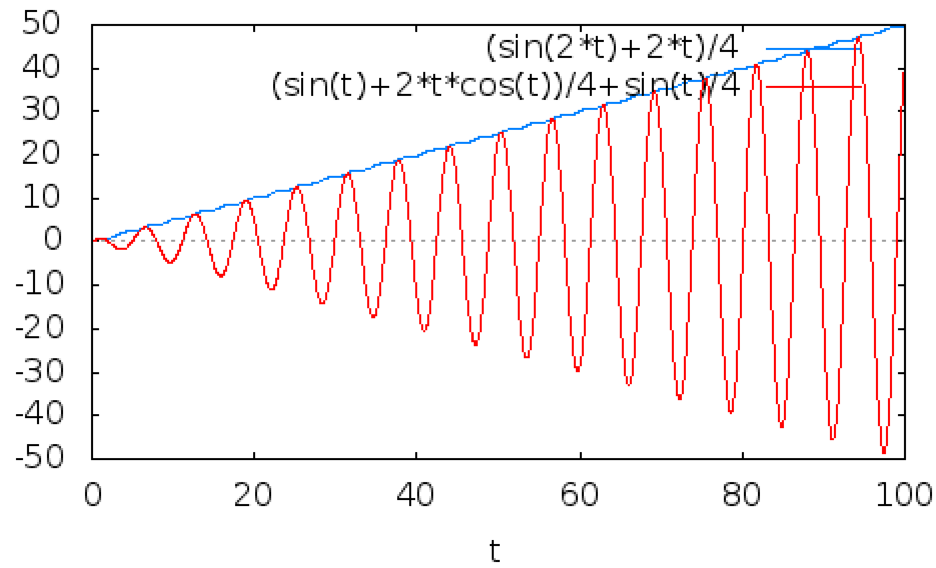
$$max\left(\frac{S}{N}\right)_T = \frac{2}{N_0}\int_{-\infty}^{+\infty}|S(f)|^2\,df = \frac{2E}{N_0}$$

Correlation and Convolution Examples (1)

correlation
convolution



correlation
convolution



z : integrate(cos(x)*cos(2*%pi - t +x), x, 0, t); convolution

(sin(t)+2*t*cos(t))/4+sin(t)/4

correlation

z : integrate(cos(x)*cos(x), x, 0, t);

(sin(2*t)+2*t)/4

Correlation and Convolution Examples (2)

$$s(t) \quad A\cos(\omega_0 t) \quad 0 \le t < T$$
$$\quad 0 \quad elsewhere$$

$$z(t) = \int_0^t r(\tau)h(t-\tau)\,d\tau$$
$$= \int_0^t r(\tau)s(T-(t-\tau))\,d\tau$$
$$= \int_0^t r(\tau)s(T-t+\tau)\,d\tau$$

$$z(t) = \int_0^t r(\tau)s(T-t+\tau)\,d\tau$$

$$when \;\; r(t) = s(t)$$

$$z(t) = \int_0^t s(\tau)s(T-t+\tau)\,d\tau$$
$$= A^2\int_0^t \cos(\omega_0\tau)\cos(\omega_0(T-t+\tau))\,d\tau$$
$$= \frac{A^2}{2}\int_0^t \cos(\omega_0(T-t)) + \cos(\omega_0(T-t+2\tau))\,d\tau$$
$$= \frac{A^2}{2}\left[\cos(\omega_0(T-t))\tau - \frac{1}{2\omega_0}\sin(\omega_0(T-t+2\tau))\right]_0^t$$
$$= \frac{A^2}{2}\left[\cos(\omega_0(T-t))t - \frac{1}{2\omega_0}\{\sin(\omega_0(T+t)) - \sin(\omega_0(T-t))\}\right]$$

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] B. Sklar, “Digital Communications: Fundamentals and Applications”