Complex Functions (1A)

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Derivatives

the complex function f is defined in a neighborhood of a point z_0

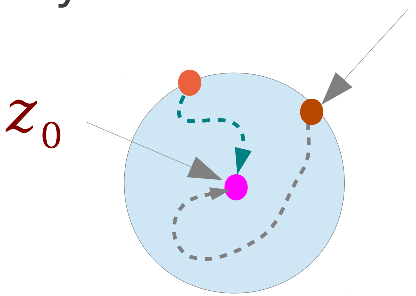
Derivative of f at z_0

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided that this limit exists

f is said to be **differentiable** at z_0

Δz can approach zero from any convenient direction



Analyticity

$\left\{ \begin{array}{l} f \end{array} \right.$ **differentiable** at z_0

f **differentiable** at <u>every point</u> in some **neighborhood** of z_0

⇒ the complex function f is said to be **analytic** at a point z_0

A complex function can be <u>differentiable</u> *at a point* z_0
but <u>differentiable</u> *nowhere else*

A function that is **analytic** at **every** point z **:analytic function**

Analytic Functions

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

$$\Delta f = f(z + \Delta z) - f(z)$$

$$\Delta z = \Delta x + i \Delta y$$

$f(z)$: **analytic** in a region ⟺ $f(z)$ has a (unique) derivative at every point of the region

$f(z)$: **analytic** at a point $z = a$ ⟺ $f(z)$ has a (unique) derivative at every point of some small circle about $z = a$

Singular Point

Regular point of $f(z)$ ⟺

a point at which $f(z)$ is analytic

Singular point of $f(z)$ ⟺

a point at which $f(z)$ is not analytic

Isolated Singular point of $f(z)$ ⟺

a point at which $f(z)$ is analytic everywhere else inside some small circle about the singular point

Cauchy-Riemann Condition (1)

Necessary Condition for Analyticity

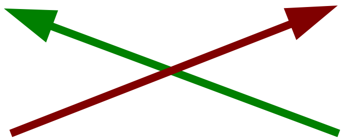
$f(z) = u(x,y) + i\,v(x,y)$: **differentiable** at a point $z = x + i\,y$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$f(z) = u(x,y) + i\,v(x,y)$

$\frac{\partial}{\partial x}$ $\qquad \frac{\partial}{\partial y}$

$f(z) = u(x,y) + i\,v(x,y)$



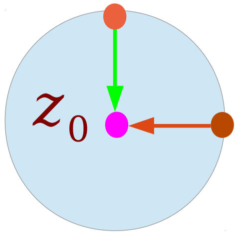
$\frac{\partial}{\partial x}$ $\qquad \frac{\partial}{\partial y}$

Cauchy-Riemann Condition (2)

$f(z) = u(x,y) + iv(x,y)$: **differentiable** at a point $z = x + iy$

$f'(z)$ exists $\quad f'(z) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \qquad \Delta z = \Delta x + i\Delta y$

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta z}$$



horizontal approach $\quad \Delta z \to 0 \Rightarrow \Delta x \to 0, \quad \Delta y = 0$

vertical approach $\quad \Delta z \to 0 \Rightarrow \Delta y \to 0, \quad \Delta x = 0$

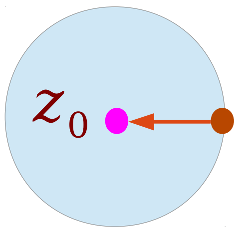
the same $f'(z)$

Cauchy-Riemann Condition (3)

horizontal approach $\Delta z \to 0 \Rightarrow \Delta x \to 0 \quad \Delta y = 0$

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta z}$$

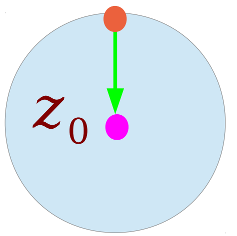
$$= \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$



vertical approach $\Delta z \to 0 \Rightarrow \Delta y \to 0 \quad \Delta x = 0$

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$



$$\Rightarrow \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

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To Be Analytic (1)

$f(z) = u(x,y) + iv(x,y)$: **analytic** in a domain D

$\Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$f(z) = u(x,y) + iv(x,y)$: **analytic** in a domain D

$\Leftarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

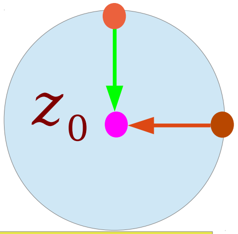
$u(x,y),\ v(x,y)$: **continuous** on in a domain D

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$: **continuous** on in a domain D

To Be Analytic (2)

if the real functions u(x,y) and v(x,y) are continuous and have continuous first order partial derivatives in a neighborhood of z, and if u and v satisfy the Cauchy-Riemann equations at the point z, then the complex function f(z) = u(x,y) + iv(x,y) is **differentiable** at z and f'(z) is as belows.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$



$f(z) = u(x,y) + iv(x,y)$: **analytic** in a domain D

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$u(x,y),\ v(x,y)$: **continuous** on in a domain D

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$: **continuous** on in a domain D

Derivatives

$f(z) = u(x,y) + iv(x,y)$: **analytic** in a region R

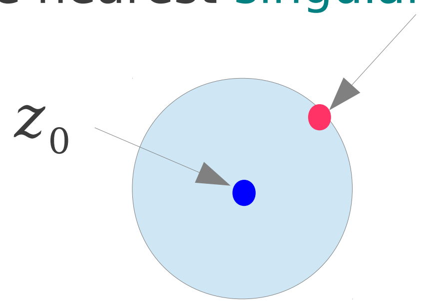
➡ derivatives of all orders at points inside region

$f'(z_0),\ f''(z_0),\ f^{(3)}(z_0),\ f^{(4)}(z_0),\ f^{(5)}(z_0),\ \cdots$

➡ Taylor series expansion about any point z_0 inside the region

The power series converges inside the circle about z_0

This circle extends to the nearest singular point



Laplace Equation

$f(z) = u(x,y) + i v(x,y)$: **analytic** in a region R

⇒ $u(x,y)$, $v(x,y)$ satisfy Laplace's equation in the region

harmonic functions

$u(x,y)$, $v(x,y)$ satisfy Laplace's equation in simply connected region

⇒ Real / imaginary part of an analytic function $f(z)$

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”