*where $\mu_k = \mathbb{E}[X^k]$ is the kth moment of X. The relationship can be seen as follows:*

$$
\begin{aligned}
m_X(t) &= \mathbb{E}\left[e^{tX}\right] = \int e^{tx} p_X(x)dx \\
&= \int \sum_{k=0}^{\infty} \frac{tx^k}{k!} p_X(x)dx = \sum_{k=0}^{\infty} \frac{1}{k!} \int t^k x^k p_X(x)dx \\
&= \sum_{k=0}^{\infty} \frac{t^k \mu_k}{k!} = \mu_0 + t\mu_1 + \frac{t^2}{2}\mu_2 + \cdots . \qquad (2.6)
\end{aligned}
$$

Wikipedia's article on Normal distr. refers to McPherson 1990 or 1980 for this

*If $X \sim \mathcal{N}(0, \sigma^2)$ is a Gaussian random variable, then $m_X(t) = e^{\sigma^2 t^2/2}$.*

First use of "~" ?

### 2.2.5 Random Number Generation

One of the basis tasks in stochastic simulations is to generate a sequence of random numbers satisfying a desired probability distribution. To this end, one first seeks to generate a random sequence with a common uniform distribution in (0, 1). There are many available algorithms that have been well studied. The algorithms in practical implementations are all deterministic (typically using recursion) and can therefore at best mimic properties of uniform random variables. For this reason, the sequence of outputs is called a sequence of *pseudorandom numbers*. Despite some defects in the early work, the algorithms for generating pseudorandom numbers have been much improved. The readers of this book will do well with existing software, which is fast and certainly faster than self-made high-level-language routines, and will seldom be able to improve it substantially. Therefore, we will not spend time on this subject, and we refer interested readers to references such as [38, 59, 65, 87].

$F_X(U) \leqslant F_X(\ldots$

$\mathbb{P}(U \leqslant u)$ = $F_U(u) = 1$

Since $\mathbb{P}(U \leqslant u) = 1$ ⇒ ∃U always for any u

For nonuniform random variables, the most straightforward technique is via the inversion of a distribution function. Let $F_X(x) = P(X \le x)$ be the distribution function of $X$. For the simple case where $F_X$ is strictly increasing and continuous, then $x = F_X^{-1}(u)$ is the unique solution of $F_X(x) = u, 0 < u < 1$. For distributions with nonconnected support or jumps, $F_X$ is not strictly increasing, and more care is needed to find its inverse. We choose the left-continuous version

$$F_X^{-1}(u) \triangleq \inf\{x : F_X(x) \ge u\}. \qquad (2.7)$$

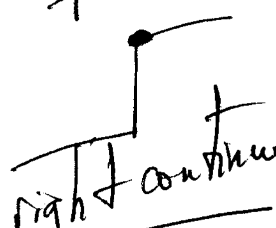
Discontinuity?!

left continuous

We state here the following results that justify the inversion method for generating nonuniform random numbers.

**Proposition 2.11.** *Let $F_X(x) = P(X \le x)$ be the distribution function of X. Then the following results hold.*

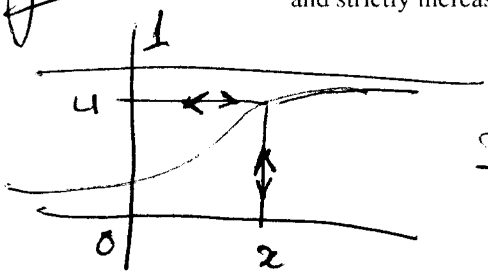
- *$u \le F_X(x) \iff F_X^{-1}(u) \le x$.* Due to monotonicity of $F_X$
- *If U is uniform in (0, 1), then $F_X^{-1}(U)$ has distribution function $F_X$.*
- *If $F_X$ is continuous, then $F_X(x)$ is uniform in (0, 1).*



right continuous

? Assumption

The part that is mainly used in simulation is the second statement, which allows us to generate $X$ as $F_X^{-1}(U)$. The most common case is an $F_X$ that is continuous and strictly increasing on an interval.




$X = F_X^{-1}(U)$

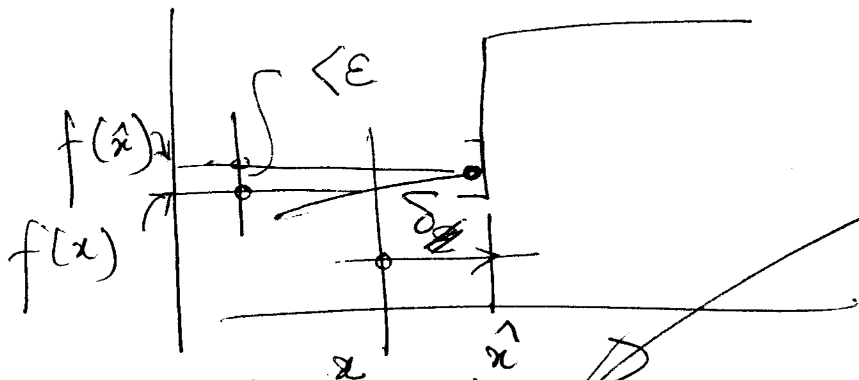
$\mathbb{P}(X \leqslant x) = \int_{-\infty}^{x} f_X(t)\,dt$

$F_X(x) = \mathbb{P}(X \leqslant x) = \mathbb{P}(\underbrace{F_X^{-1}(U) \leqslant x}_{\Updownarrow\; U \leqslant F_X(x)})$

**Continuity:** $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\|x - \hat{x}\| \leq \delta \Rightarrow \|f(x) - f(\hat{x})\| \leq \varepsilon$

$x \to \hat{x}$ from both sides.

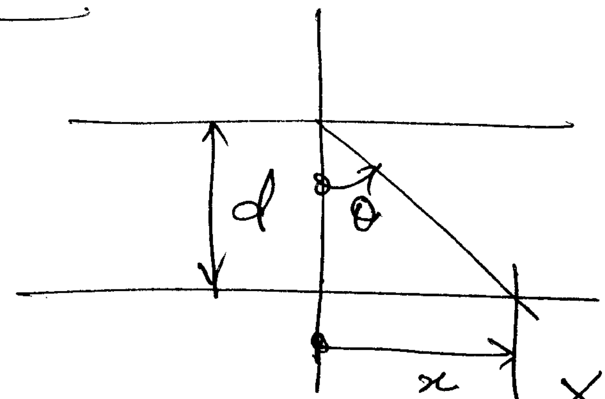
**Left cont.:** $x \uparrow \hat{x}$ ($x \to \hat{x}$ from below)




On Prop. 2.11

**Cauchy distr:**

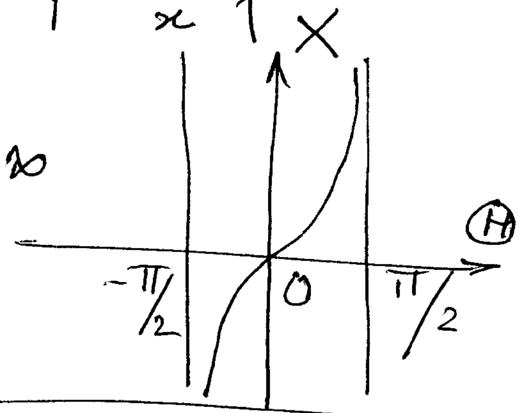
$x = d \tan \theta$

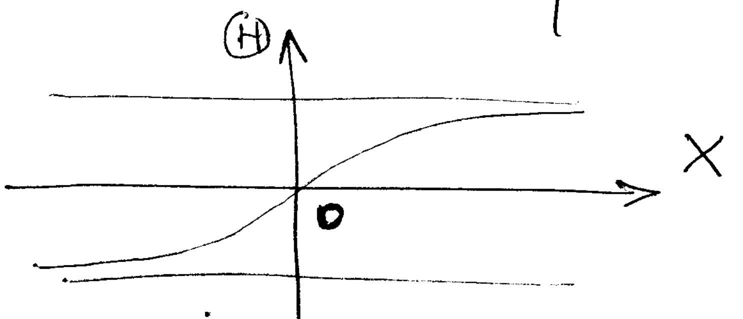



$\Theta \sim U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$X = d \tan \Theta$

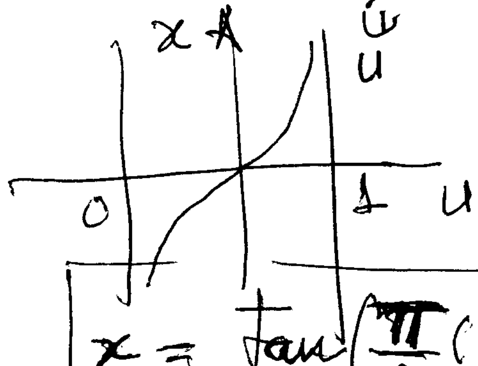
Can omit since $\left|\tan \pm \frac{\pi}{2}\right| = \infty$

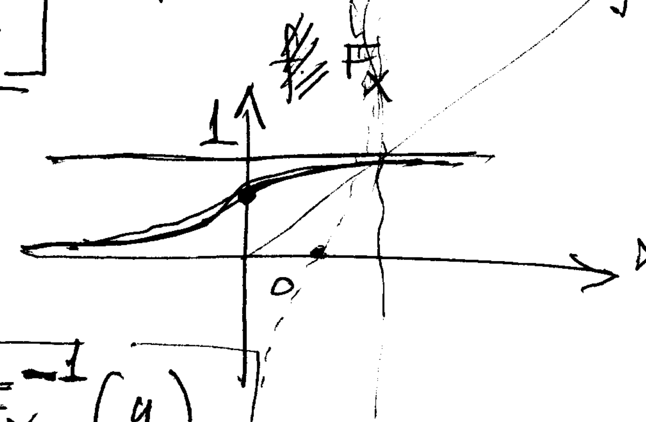



$\boxed{\theta = \frac{\pi}{2}(u-1)}$

$\theta(u) : \underbrace{[0,1]}_{u} \longrightarrow \underbrace{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}_{\theta}$




$\Rightarrow$

$\boxed{x = \tan\left(\frac{\pi}{2}(u-1)\right) = F_X^{-1}(u)}$

$x = \tan y \Rightarrow y = \tan^{-1} x = \frac{\pi}{2}(u-1)$

$\Rightarrow \boxed{u = \frac{2}{\pi}\tan^{-1} x + 1 = F_X(x) \in (0,1)}$

$\frac{dF_X(x)}{dx} = f_X(x) = \frac{2}{\pi}\left(\tan^{-1} x\right)$

$y = \tan^{-1} x \Rightarrow x = \tan y \Rightarrow dx = (1+\tan^2 y)\, dy$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

$\Rightarrow \boxed{f_X(x) = \frac{2}{\pi}\,\frac{1}{1+x^2}}$

Q: Consider $(f_Y \circ g_X)(x)$ or $(F_Y \circ G_X)(x)$

$\frac{d}{dx}(F_Y \circ G_X)(x) = \frac{d}{dy}F_Y(y)\,\frac{dG_X(x)}{dx}$ $\quad = F_Y(y) = F_Y(\underbrace{G_X(x)}_{y})$, $y = G_X(x)$

$= f_Y(y)\, g_X(x)$

$= (f_Y \circ g_X)(x)$

$u = h^{-1}(x)$

Now Consider $\left\{\begin{matrix} x \to u \\ y \to x \end{matrix}\right\}$ i.e, $x = h(u)$

and $u$ corresponds to $U \sim U(0,1)$ i.e, uniform distr. on $(0,1)$, i.e, $g_U(u) = 1$.

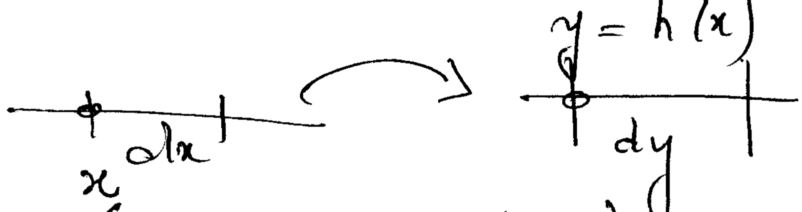
Q: What is distr. of $x$? what is $F_X(x)$?

$\boxed{Y = h(X)}$ Prop. 2.11 /3

Q: $y = h(x)$ and $F_X(x)$ is distri. of $X$; what is distr. of $Y$? Assuming $h^{-1}(y)$ exists.

$$dy = \frac{dh(x)}{dx}\,dx$$

$\cancel{F_X(x)}$




$$\underbrace{P(Y \in [y, y+dy])}_{\frac{dG}{dy}(y)\,\underbrace{dy}_{\frac{dh}{dx}dx}} = \underbrace{P(X \in (x, x+dx])}_{\frac{dF_X}{dx}\,dx}$$

$$\Rightarrow \boxed{\frac{dG_Y(y)}{dy} = \frac{dF_X(x)}{dx}\underbrace{\frac{dx}{dh}}}$$

$$\frac{dx}{dh} = \frac{dh^{-1}(y)}{dy} \quad \text{since } x = h^{-1}(y)$$

Application: ~~...~~ $X = U \sim \mathcal{U}(0,1)$ uniform distr. on $(0,1)$ $\Rightarrow \frac{dF_X(x)}{dx} = f_X(x) = 1$

$$\Rightarrow \frac{dG_Y(y)}{dy} = \frac{dh^{-1}(y)}{dy} \Rightarrow \boxed{G_Y(y) = h^{-1}(y)} + k \text{ const}$$

So if $h = F$ $\quad k = 0 \Rightarrow \boxed{h = G_Y^{-1}}$

$\left\{\begin{array}{l}\text{since } G_Y(y) \to 0 \text{ as } y \to -\infty \\ \text{and } h^{-1}(y) \to 0 \text{ as } y \to -\infty\end{array}\right\} \Rightarrow \boxed{k = 0}$

$\Rightarrow$ Prop 2.11 b: If $U \sim \mathcal{U}(0,1)$ and $X = F_X^{-1}(U)$

Prop 2.11/4

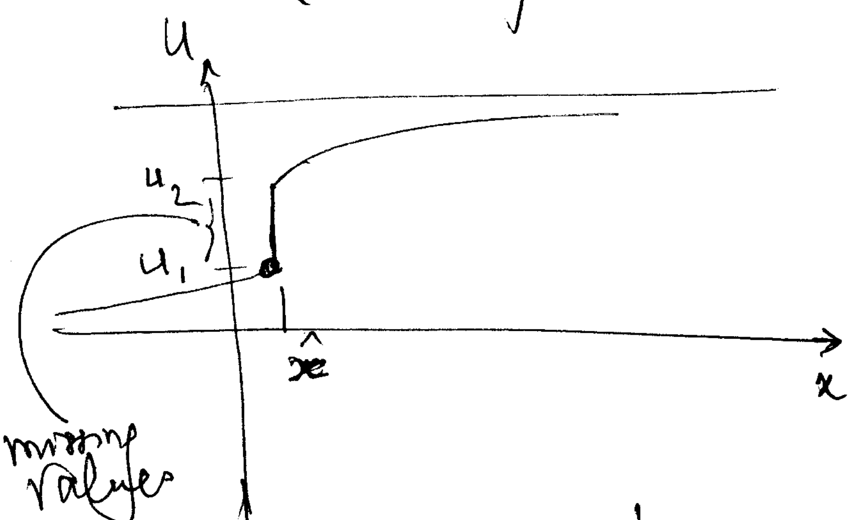
then $X$ has distribution $F_X$.

Prop 2.11c : $F_X$ continuous, $Y = F_X(X)$ has uniform distr. $U(0,1)$.

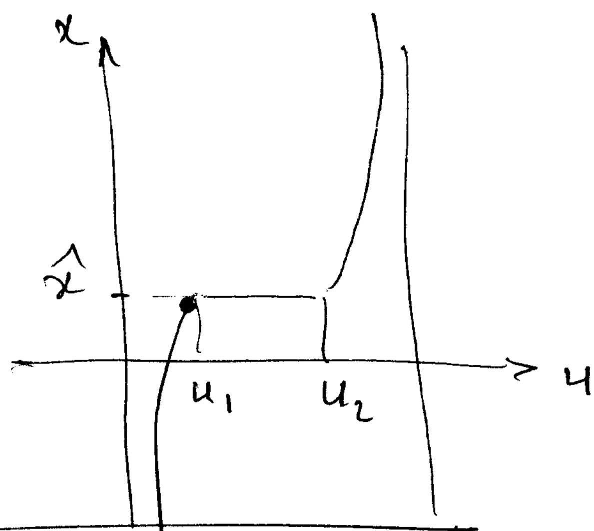
$$\frac{d G_Y(y)}{dy} = \frac{d F_X(x)}{dx} \, \frac{d F_X^{-1}(y)}{dy} = 1$$

$$\frac{d F_X^{-1}(y)}{dy} = \frac{dx}{dF_X}$$

$\Rightarrow$ $Y$ is uniform.




$\Rightarrow$ $U$ cannot be uniform since $\nexists\, U \in (u_1, u_2)$




It's ok for $F_X$ to have kinks, but cont. discont. in slope