DFT Analysis (5B)

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Frequency View of a DFT Matrix

	$row\ \ 0$	$f\ =\ 0$	$(0\,cycle)$	0
$\frac{N}{2}-1$	$row\ \ 1$	$f\ =\ -1 * f_o$	$(-1\,cycle)$	$\frac{-1}{N}$
	$row\ \ 2$	$f\ =\ -2 * f_o$	$(-2\,cycles)$	$\frac{-2}{N}$
	$row\ \ (\frac{N}{2}-1)$	$f\ =\ -(\frac{N}{2}-1) * f_o$	$(-(\frac{N}{2}-1)\,cycles)$	$\frac{-1}{2}+\frac{1}{N}$
	$row\ \ (\frac{N}{2})$	$f\ =\ -(\frac{N}{2}) * f_o$	$(\frac{N}{2}\ \ cycles)$	$\frac{-1}{2}$
$\frac{N}{2}-1$	$row\ \ (\frac{N}{2}+1)$	$f\ =\ +(\frac{N}{2}-1) * f_o$	$(\frac{N}{2}-1\,cycles)$	$\frac{1}{2}-\frac{1}{N}$
	$row\ \ N-2$	$f\ =\ +2 * f_o$	$(2\,cycles)$	$\frac{2}{N}$
	$row\ \ N-1$	$f\ =\ +1 * f_o$	$(1\,cycle)$	$\frac{1}{N}$

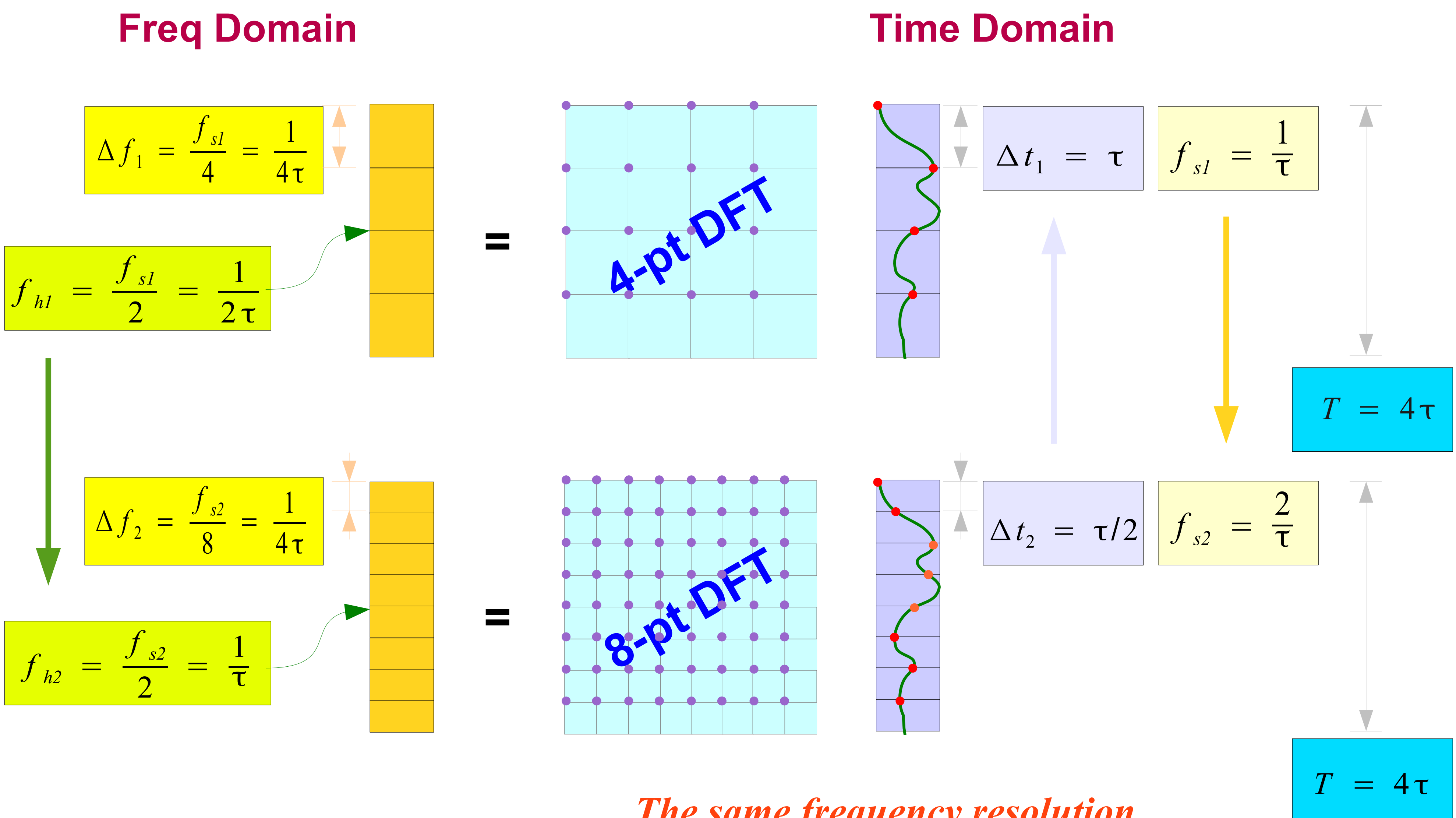
Normalized Frequency

$$f_o\ =\ \frac{f_s}{N}$$

Frequency View of a X[i] Vector

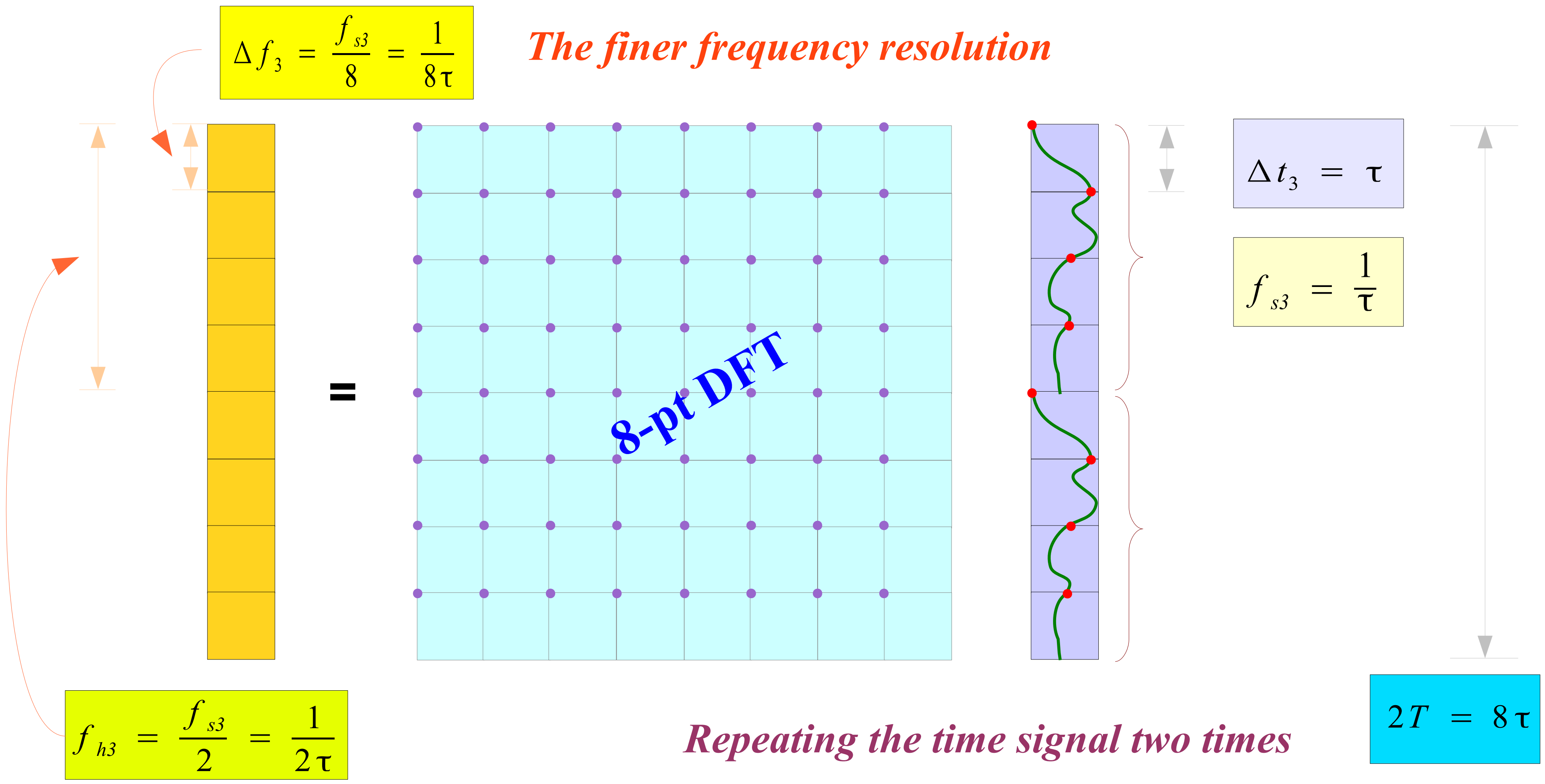
				Normalized Frequency	
	$X[0]$	$f = 0$	0	0	$(0\,cycle)$
	$X[1]$	$f = +1 * f_o$	$\frac{1}{N} f_s$	$\frac{1}{N}$	$(1\,cycle)$
	$X[2]$	$f = +2 * f_o$	$\frac{2}{N} f_s$	$\frac{2}{N}$	$(2\,cycles)$
$\frac{N}{2} - 1$					
	$X[\frac{N}{2}-1]$	$f = +(\frac{N}{2} - 1) * f_o$	$\frac{(\frac{N}{2} - 1)}{N} f_s$	$\frac{1}{2} - \frac{1}{N}$	$(\frac{N}{2}-1\,cycles)$
	$X[\frac{N}{2}]$	$f = +(\frac{N}{2}) * f_o$	$\frac{1}{2} f_s$	$\frac{1}{2}$	$(\frac{N}{2}\quad cycles)$
	$X[\frac{N}{2}+1]$	$f = -(\frac{N}{2} - 1) * f_o$	$-\frac{(\frac{N}{2} - 1)}{N} f_s$	$\frac{-1}{2} + \frac{1}{N}$	$(-(\frac{N}{2}-1)\,cycles)$
$\frac{N}{2} - 1$					
	$X[N-2]$	$f = -2 * f_o$	$-\frac{2}{N} f_s$	$\frac{-2}{N}$	$(-2\,cycles)$
	$X[N-1]$	$f = -1 * f_o$	$-\frac{1}{N} f_s$	$\frac{-1}{N}$	$(-1\,cycle)$

Frequency and Time Interval (1)



The same frequency resolution

Frequency and Time Interval (2)



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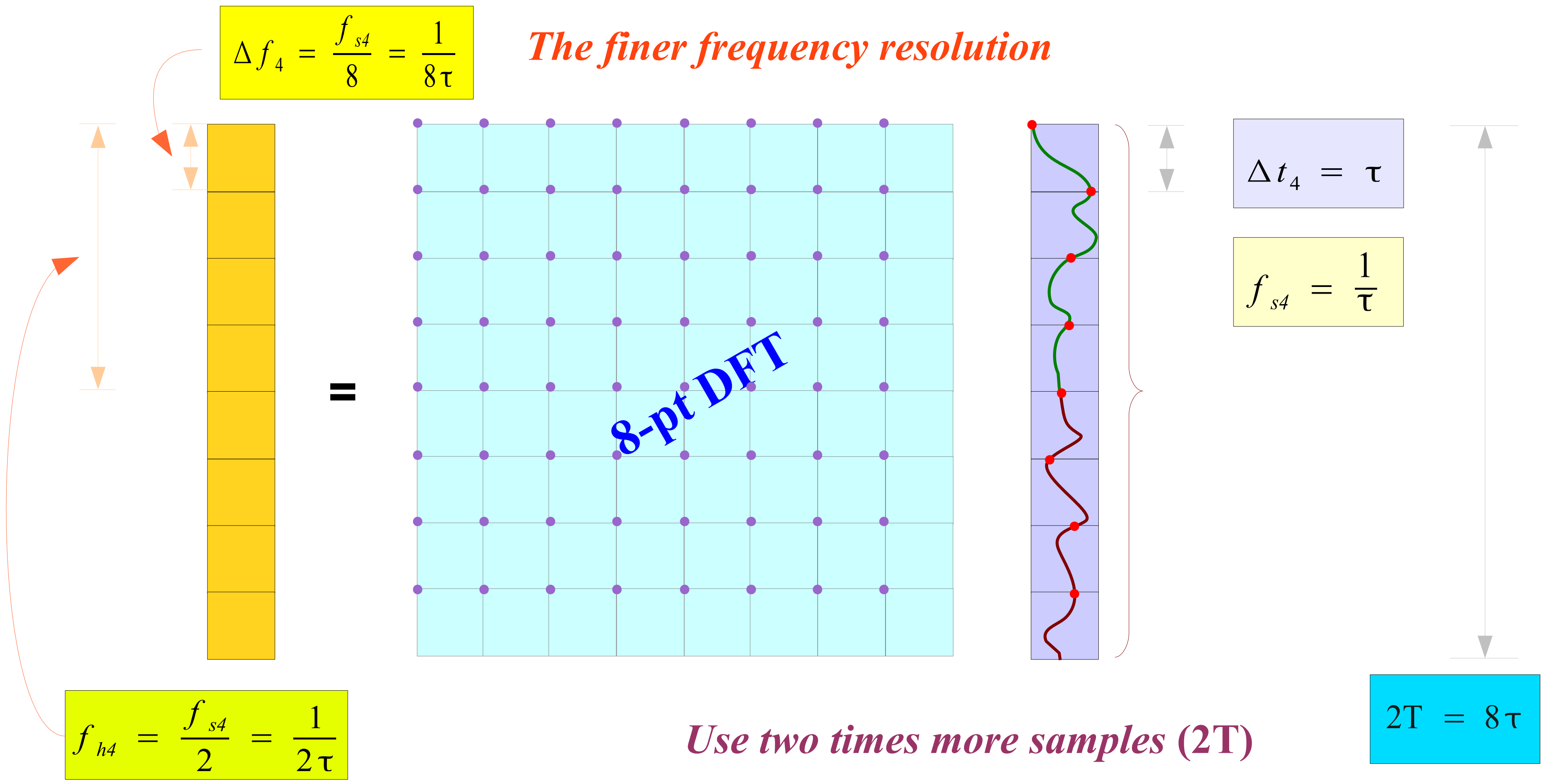
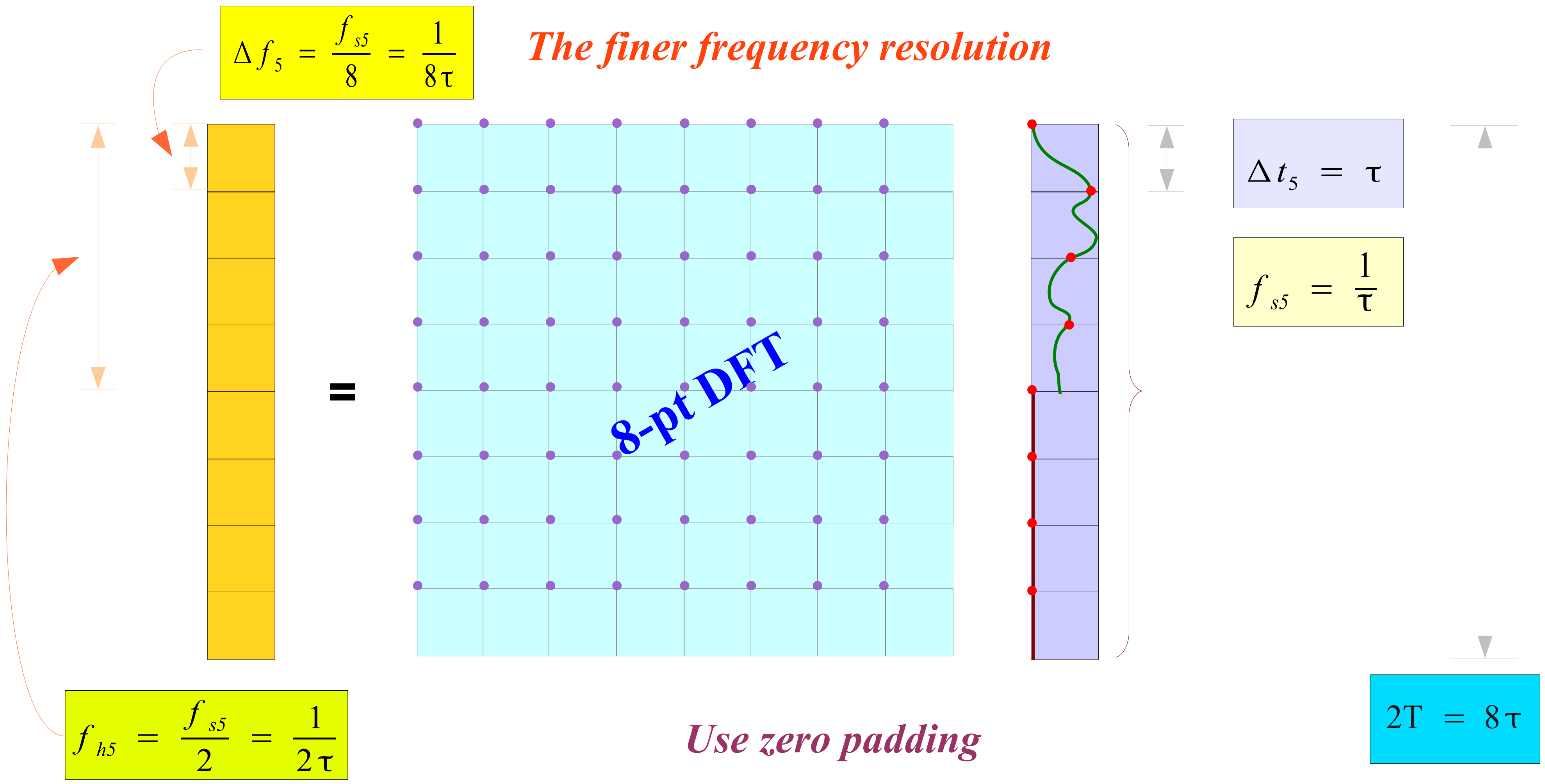
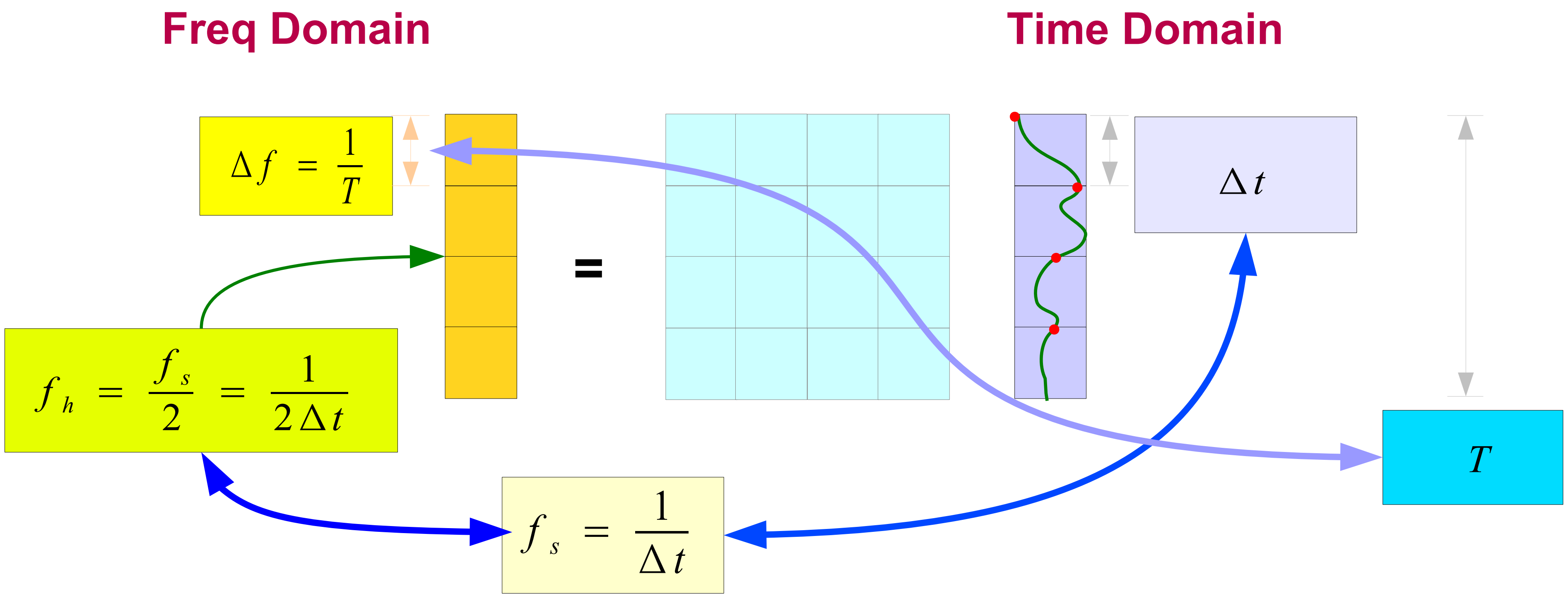
Frequency and Time Interval (3)



Frequency and Time Interval (4)



Frequency and Time Interval (5)



Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$\boldsymbol{k} = +1, +2, \ldots$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k} \right)$$

$\boldsymbol{k} = +1, +2, \ldots$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$g_k \cos(k\omega_0 t + \phi_k) = g_k \cos(\phi_k) \cos(k\omega_0 t) - g_k \sin(\phi_k) \sin(k\omega_0 t)$$

$$a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$a_k = g_k \cos(\phi_k)$$

$$-b_k = g_k \sin(\phi_k)$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

Two-Sided Spectrum

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$k = \ldots, -2, -1, 0, +1, +2, \ldots$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} \tan^{-1}(-b_k/a_k) & (k > 0) \\ \tan^{-1}(+b_k/a_k) & (k < 0) \end{cases}$$

Power Spectrum *Two-Sided*

$$|C_k|^2 + |C_{-k}|^2 = \frac{1}{2} g_k^2 = \frac{1}{2}(a_k^2 + b_k^2)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$k = \ldots, -2, -1, 0, +1, +2, \ldots$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2} g_k \, e^{+j\phi_k} & (k > 0) \\ \frac{1}{2} g_k \, e^{-j\phi_k} & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} +\phi_k & (k > 0) \\ -\phi_k & (k < 0) \end{cases}$$

Periodogram *One-Sided*

$$2 \cdot |C_k| = g_k = \sqrt{a_k^2 + b_k^2}$$

CTFS and DTFS (1)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$ **CTFS**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$\boldsymbol{k} = \ldots,\ -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$x(t) \approx \sum_{k=-M}^{+M} C_k \, e^{+jk\omega_0 t} \qquad N = 2M + 1$$

Truncate coefficeints

$$\boldsymbol{k} = -M,\ \ldots,\ -1,\ 0,\ +1,\ \ldots,\ +M$$

N Time Samples: N equations

$$t \rightarrow \boldsymbol{n}\, Ts = \boldsymbol{n}\left(\tfrac{T}{N}\right)$$

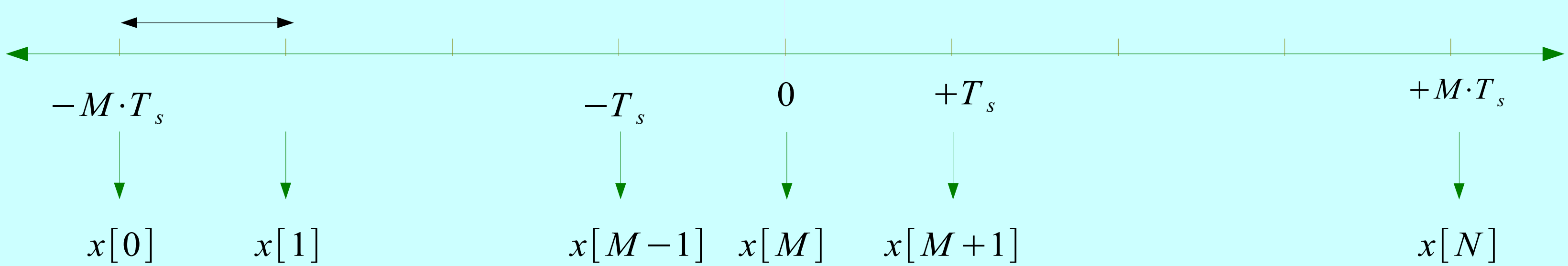
$$x[\boldsymbol{n}] = \sum_{k=-M}^{+M} \gamma_k \, e^{+jk\omega_0 n\left(\frac{T}{N}\right)}$$

$$jk\omega_0 t \rightarrow \boldsymbol{k}\left(\frac{2\pi}{T}\right)\boldsymbol{n}\left(\frac{T}{N}\right) = \left(\frac{2\pi}{T}\right)\boldsymbol{n}\, k$$

$$x[\boldsymbol{n}] = \sum_{k=-M}^{+M} \gamma_k \, e^{+j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{n} = 0,\ 1,\ 2,\ \ldots,\ N-1,$$

$$T_s = \frac{T}{N}$$ **Time Samples**



$0 \le n < L$ $\qquad T_s \rightarrow 0$

CTFS and DTFS (2)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$ **CTFS**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$k = \ldots, -2, -1, 0, +1, +2, \ldots$

$$x[n] = \sum_{k=-M}^{+M} \gamma_k \, e^{+j\left(\frac{2\pi}{N}\right)nk}$$

$n = 0, 1, 2, \ldots, N-1,$

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)nk}$$

$k = -M, \ldots, 0, \ldots, +M$

Approximate Continuous Signal
With the truncated coefficients

$$x_{FS}(t) = \sum_{k=-M}^{+M} \gamma_k \, e^{+jk\omega_0 t}$$ **DTFS**

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$ **Approximated Fourier Coefficients**

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$ **DFT**

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$ **IDFT**

$0 \le n < L$ $\quad T_s \to 0$

CTFS and DTFS (3)

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$x[\boldsymbol{n}] = \sum_{\boldsymbol{k}=-M}^{+M} \gamma_{\boldsymbol{k}}\, e^{+j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{n} = 0,\ 1,\ 2,\ \ldots\ ,\ N-1,$$

$$\gamma_{\boldsymbol{k}} = \frac{1}{N} \sum_{\boldsymbol{n}=0}^{N-1} x[\boldsymbol{n}]\, e^{-j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{k} = -M,\ \ldots\ ,\ 0,\ \ldots\ ,\ +M$$

$$x(t) = \sum_{\boldsymbol{k}=-\infty}^{+\infty} C_{\boldsymbol{k}}\, e^{+j\boldsymbol{k}\omega_0 t}$$

CTFS

$$C_k \approx \gamma_{\boldsymbol{k}} = \frac{X[k]}{N}$$

Approximated Fourier Coefficients

Truncate Fourier Coefficients

$$x_{FS}(t) = \sum_{\boldsymbol{k}=-M}^{+M} \gamma_{\boldsymbol{k}}\, e^{+j\boldsymbol{k}\omega_0 t}$$

DTFS

Power Spectrum using FFT

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

`X = fft(x)`

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

`x = ifft(X)`

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$ CTFS

Approximated
Fourier Series Coefficients

$$x_{FS}(t) = \sum_{k=-M}^{+M} \gamma_k\, e^{+jk\omega_0 t}$$ DTFS

`fc = fft(x)/N = X/N`

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$

Approximated
Fourier Coefficients

`x = ifft(fc)*N`

$$|C_k|^2 \approx \frac{|X[k]|^2}{N^2}$$

Approximated
Power Spectrum

Periodogram using FFT

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$ **Approximated Fourier Coefficients**

$$|C_k|^2 \approx \frac{|X[k]|^2}{N^2}$$ **Approximated Power Spectrum**

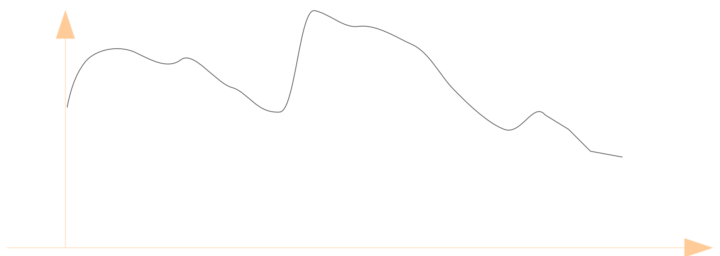
$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2$$ **Average Power**

$$\left(\sqrt{\frac{\sum_{k=0}^{N-1}\frac{|X[k]|^2}{N}}{N}}\right)^2$$ **RMS of sq root Periodogram**

$$\frac{|X[k]|^2}{N} \quad k = 0, 1, \ldots, N-1$$ **Approximated Periodogram**

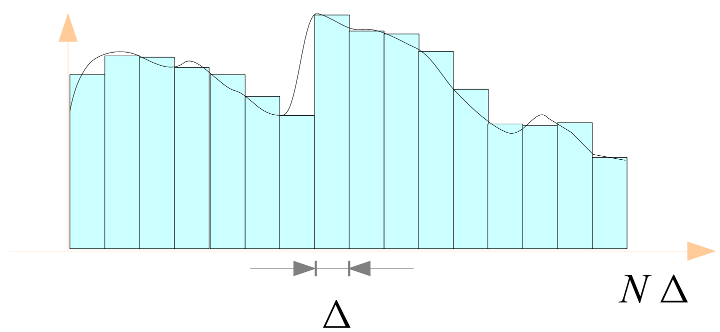
$$\frac{|X[k]|}{\sqrt{N}} \quad k = 0, 1, \ldots, N-1$$ **Square root Periodogram**

RMS in continuous time



$$\frac{1}{T}\int_0^T g^2(t)\, dt$$

RMS in discrete time



$$\frac{1}{N\Delta}\sum_{k=0}^{N-1} |g[k]|^2 \Delta = \frac{1}{N}\sum_{k=0}^{N-1} |g[k]|^2$$

From CTFS to CTFT

Continuous Time Fourier Series

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \qquad x_{T_0}(t) = \sum_{n=0}^{\infty} C_k\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{2\pi} \cdot \frac{T_0}{T_0}$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \qquad x_{T_0}(t) = \frac{1}{2\pi}\sum_{k=0}^{\infty} C_k T_0\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{T_0}$$

$$T_0 \to \infty \qquad \omega_0 = \frac{2\pi}{T_0} \to d\omega \qquad C_k T_0 \to X(j\omega) \qquad x_{T_0} \to x(t)$$

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

CTFS and CTFT

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$x[\boldsymbol{n}] = \sum_{\boldsymbol{k}=-M}^{+M} \gamma_{\boldsymbol{k}}\, e^{+j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{n} = 0,\ 1,\ 2,\ \ldots,\ N-1,$$

$$\gamma_{\boldsymbol{k}} = \frac{1}{N} \sum_{\boldsymbol{n}=0}^{N-1} x[\boldsymbol{n}]\, e^{-j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{k} = -M,\ \ldots,\ 0,\ \ldots,\ +M$$

$$C_k \approx \gamma_{\boldsymbol{k}} = \frac{X[k]}{N}$$

Approximated Fourier Coefficients

Continuous Time Fourier Series

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=0}^{\infty} C_k T_0\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{T_0}$$

$$T_0 \to \infty, \quad \omega_0 \to 0 \quad (\omega_0 \to d\omega)$$

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Fourier Series Coefficients

	Periodic Signals	Aperiodic Signals
Frequency Spacing	$\Delta f = \dfrac{1}{N \Delta t}$	$\Delta f = \dfrac{1}{N \Delta t}$
Two Sided F.S. Coefficient	$\dfrac{1}{N} X(k)$	$\dfrac{\Delta t}{N} X(k)$
One Sided F.S. Coefficient	$\dfrac{1}{N} X(k) \quad k=0, \frac{N}{2}$ $\dfrac{2}{N} X(k) \quad k=1,\cdots,\frac{N}{2}-1$	$\dfrac{\Delta t}{N} X(k) \quad k=0, \frac{N}{2}$ $\dfrac{2\Delta t}{N} X(k) \quad k=1,\cdots,\frac{N}{2}-1$
Frequency Spacing	$k \Delta f$	$k \Delta f$

	Periodic Signals	Aperiodic Signals	Random Signals	
Frequency Spacing	$\Delta f = \frac{1}{N \Delta t}$	$\Delta f = \frac{1}{N \Delta t}$	$\sum S \Delta f = \frac{1}{N \Delta t} \sum S$	$\frac{1}{N \Delta t} \sum x^2 \Delta t$
Two Sided FC Coeff	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$	$S(k) = \frac{\Delta t}{N} \lvert X(k) \rvert^2$	$P = \sum_{k=0}^{N-1} S(k) \Delta f$
One Sided				
$k = 0, \frac{N}{2}$	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$	$S_1(k) = 2\mathrm{S}(k)$	$P = \sum_{k=0}^{N/2} S_1(k) \Delta f$
$k = 1, \cdots, \frac{N}{2} - 1$	$\frac{2}{N} X(k)$	$\frac{2 \Delta t}{N} X(k)$	$S_1(k) = S(k)$	
Frequency Scale	$k \Delta f$	$k \Delta f$	$k \Delta f$	

Random Signals

One-sided Power Spectral Density

$$P = \sum_{k=0}^{N-1} S(k) \Delta f$$

One-sided Power Spectral Density

$$P = \sum_{k=0}^{N/2} S_1(k) \Delta f$$

$$S_1(k) = 2\mathrm{S}(k) \qquad k = 1, \ldots, \tfrac{N}{2} - 1$$

$$S_1(k) = S(k) \qquad k = 0, \tfrac{N}{2}$$

Two Sided Fourier Series Coefficient

$$\tfrac{1}{N \Delta t} \sum x^2 \Delta t$$

$$\sum S \Delta f = \tfrac{1}{N \Delta t} \sum S$$

$$S(k) = \tfrac{\Delta t}{N} |X(k)|^2$$

$$k \Delta f$$

Amplitude Spectrum

$$A_k = \frac{1}{N}|X(k)| = \frac{1}{N}\sqrt{\Re^2(X(k))+\Im^2(X(k))}$$

$$k = 0,1,2,\cdots,N-1$$

Power Spectrum

$$P_k = \frac{1}{N^2}|X(k)|^2 = \frac{1}{N^2}\{\Re^2(X(k))+\Im^2(X(k))\}$$

$$k = 0,1,2,\cdots,N-1$$

One Sided Amplitude Spectrum

$$\bar{A}_k = \frac{1}{N}|X(0)| \qquad k=0$$

$$\bar{A}_k = \frac{2}{N}|X(0)| \qquad k=1,2,\cdots,N/2$$

One Sided Power Spectrum

$$\bar{P}_k = \frac{1}{N^2}|X(0)|^2 \qquad k=0$$

$$\bar{P}_k = \frac{2}{N^2}|X(0)|^2 \qquad k=1,2,\cdots,N/2$$

Frequency Bin

$$f=\frac{k f_s}{N}$$

Frequency Bin

$$f=\frac{k f_s}{N}$$

Phase Spectrum

$$\phi_k = \tan^{-1}\left(\frac{\Im(X(k))}{\Re(X(k))}\right) \qquad k=0,1,2,\cdots,N-1$$

Data Truncation
Frequency Resolution
Zero Padding
Periodogram
Spectral Plot

Amplitude spectrum in quantity peak
Phase spectrum in radians
Amplitude spectrum in volts rms
Phase spectrum in degrees
Power spectrum

Signals without discontinuity
Signals with discontinuity

Sampling frequency is not an integer multiple of the FFT length

Leakage

$\left[0, \quad \frac{f_s}{2}\right]$

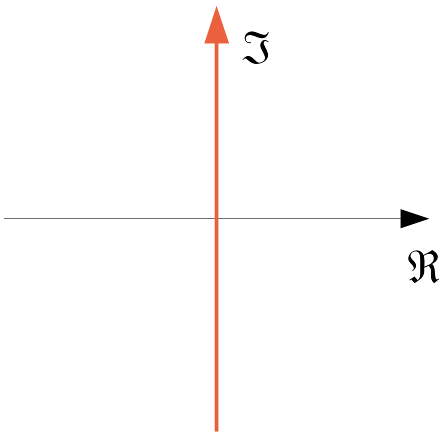
Fourier Transform

$f(t)$ A continuous sum of weighted exponential functions :

$$f(t)\, e^{-j\omega t}$$

$$-\infty \;<\; \omega \;<\; +\infty$$

Not so useful in transient analysis

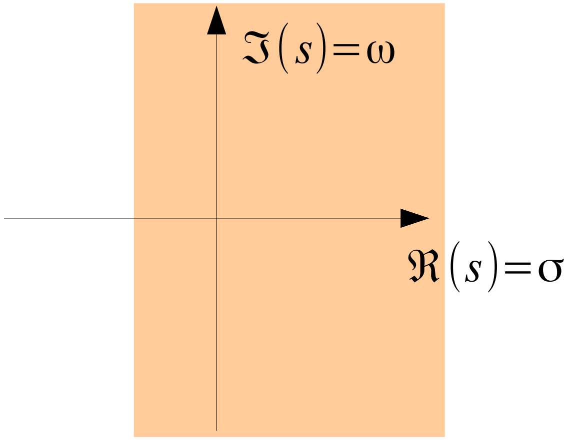


Laplace Transform

$$f(t)\, e^{-st} \;=\; f(t) e^{-(\sigma + j\omega)t}$$

Linear Time Domain Analysis

Initial Condition



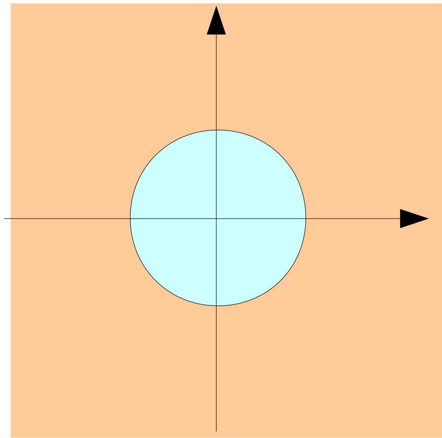
z Transform

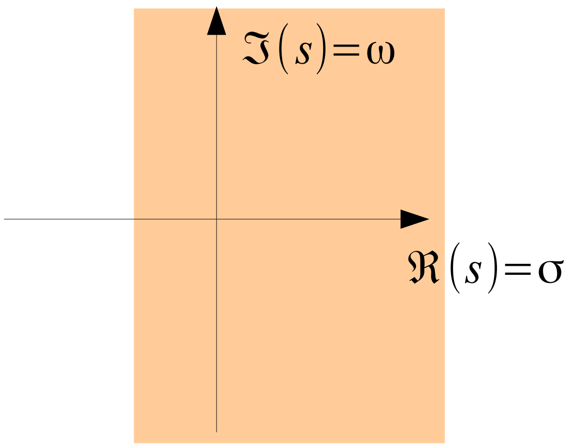
$$f[n]\, z^{-n}$$

Discrete Time System

Difference Equation

$$z \;=\; e^{sT} \;=\; e^{\sigma T}\, e^{j\omega T}$$





References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A "graphical interpretation" of the DFT and FFT, by Steve Mann