CORDIC Background (4A)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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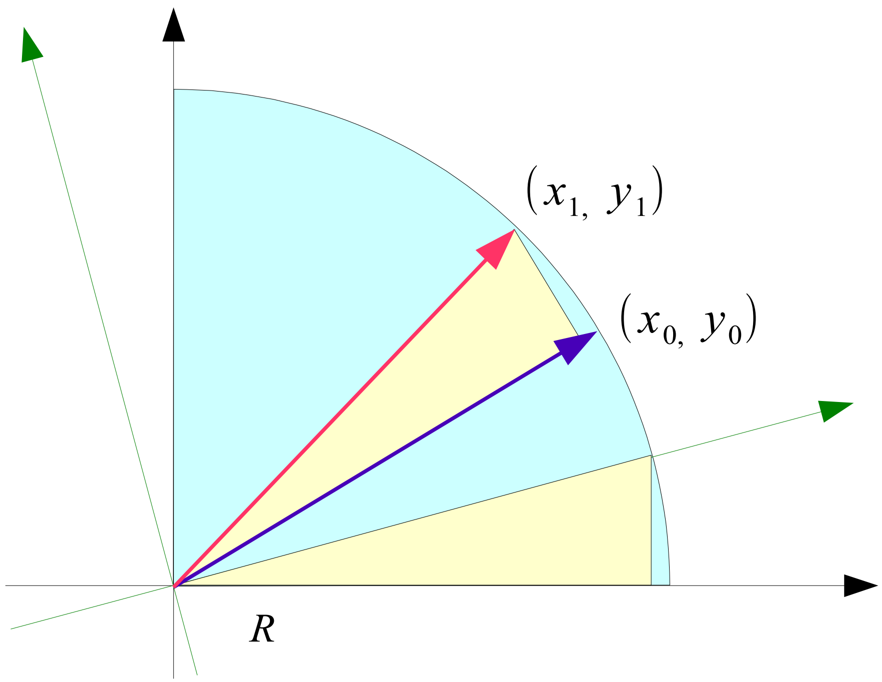
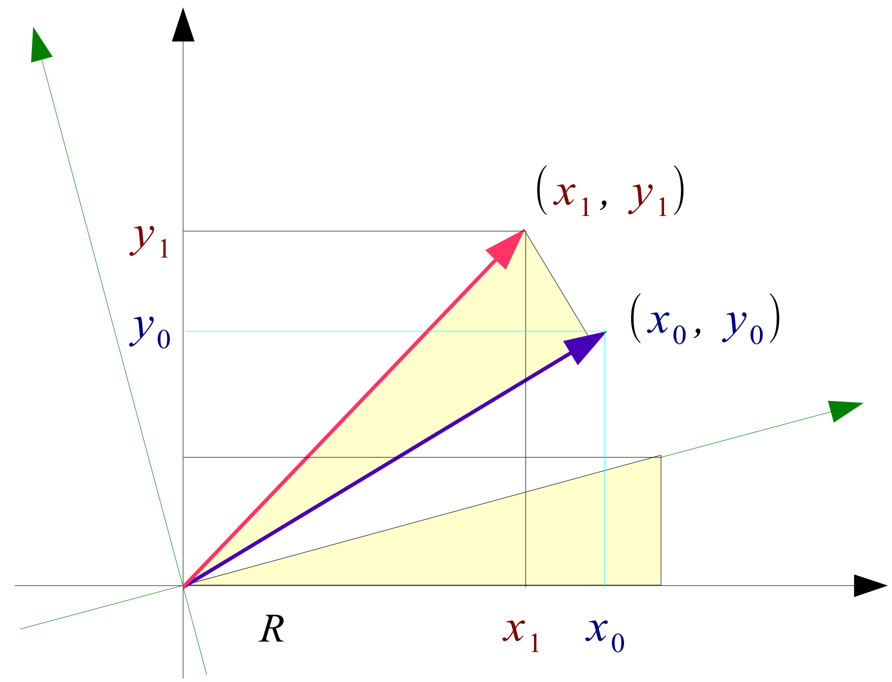
CORDIC Background

J. P. Deschamps, G. A. Bioul, G.D. Sutter, Synthesis of Arithmetic Circuits

Vector Rotation (1)

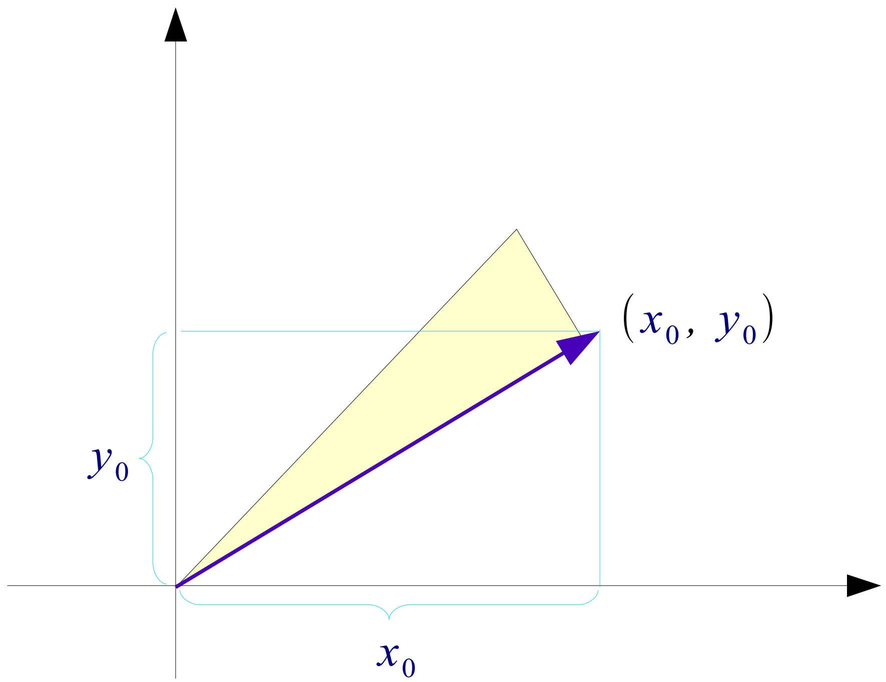
$$y_1 = x_0 \sin\alpha + y_0 \cos\alpha$$

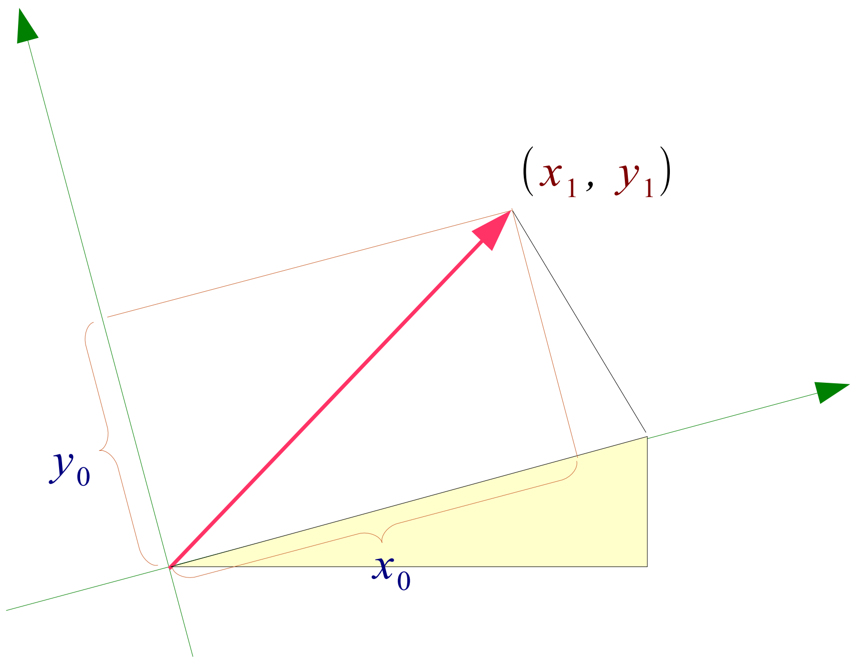




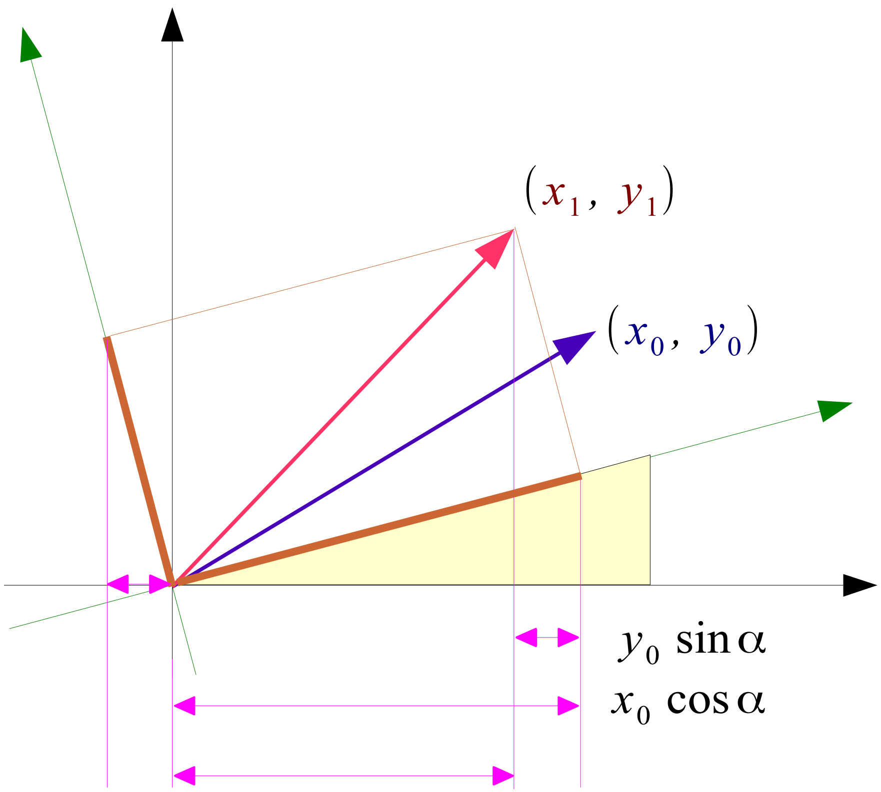
$$x_1 = x_0 \cos\alpha - y_0 \sin\alpha$$

Vector Rotation (2)

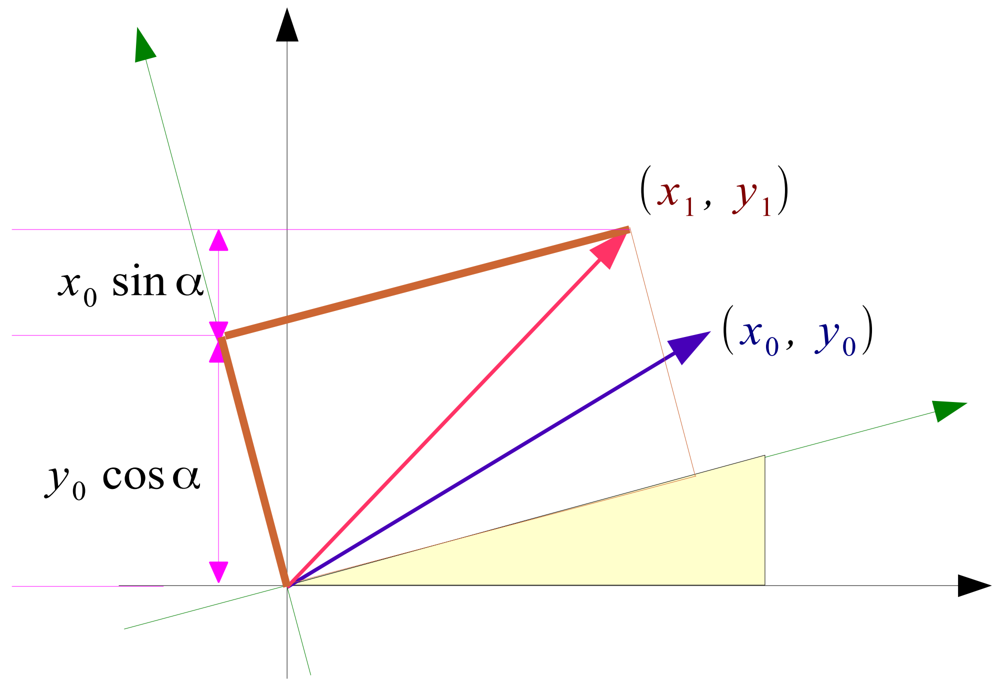




Vector Rotation (3)



$$x_1 = x_0 \cos\alpha - y_0 \sin\alpha$$



$$y_1 = x_0 \sin\alpha + y_0 \cos\alpha$$

Iterative Rotation

Given Vector
Given Angle

Rotate →

Computing
new coordinates

Given Unit Vector
Given Angle α

Rotate →

$$x = \cos\alpha$$
$$y = \sin\alpha$$

Given Vector $(x_{0,}\ \ y_0)$
Given Angle α

Rotate →

$$x_n = x_0 \cos\alpha - y_0 \sin\alpha$$
$$y_n = x_0 \sin\alpha + y_0 \cos\alpha$$

$$\alpha_0 \rightarrow \cdots \rightarrow \alpha_i \rightarrow \alpha_{i+1} \rightarrow \cdots \rightarrow \alpha_n \to 0$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} \rightarrow \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

CORDIC Rotation

$$\alpha_0 \rightarrow \cdots \rightarrow \alpha_i \rightarrow \alpha_{i+1} \rightarrow \cdots \rightarrow \alpha_n \to 0$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} \rightarrow \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{cases} x_{i+1} = x_i \cos\alpha_i - y_i \sin\alpha_i \\ y_{i+1} = x_i \sin\alpha_i + y_i \cos\alpha_i \end{cases}$$

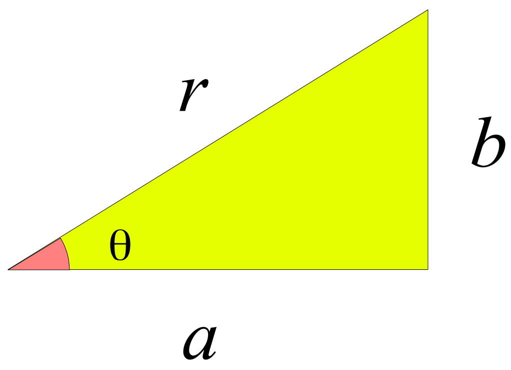
$$\begin{cases} x_{i+1} = \cos\alpha_i \left(x_i - y_i \tan\alpha_i\right) \\ y_{i+1} = \cos\alpha_i \left(x_i \tan\alpha_i + y_i\right) \end{cases}$$

$$\begin{cases} x_{i+1} = \dfrac{1}{\sqrt{1 + \tan^2\alpha_i}} \left(x_i - y_i \tan\alpha_i\right) \\ y_{i+1} = \dfrac{1}{\sqrt{1 + \tan^2\alpha_i}} \left(x_i \tan\alpha_i + y_i\right) \end{cases}$$

Pseudo-rotation

$$\begin{cases} x'_{i+1} = \left(x_i - y_i \tan\alpha_i\right) \\ y'_{i+1} = \left(x_i \tan\alpha_i + y_i\right) \end{cases}$$

cos θ



$$r = \sqrt{a^2 + b^2}$$

$$\cos\theta = \frac{a}{r} = \frac{a}{\sqrt{a^2 + b^2}}$$

$$= \frac{1}{\sqrt{1 + (b/a)^2}}$$

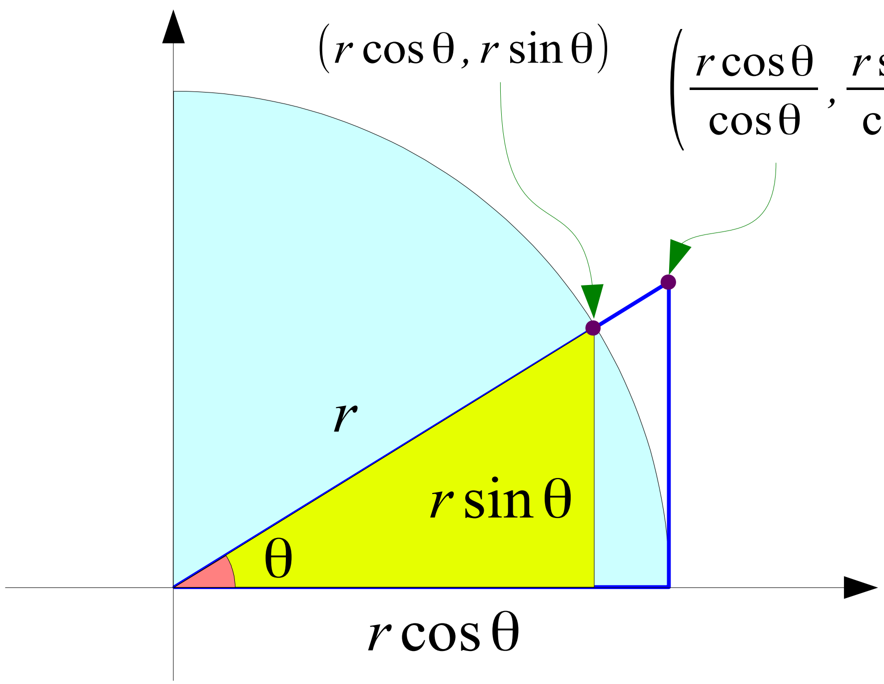
$$= \frac{1}{\sqrt{1 + \tan^2\theta}}$$

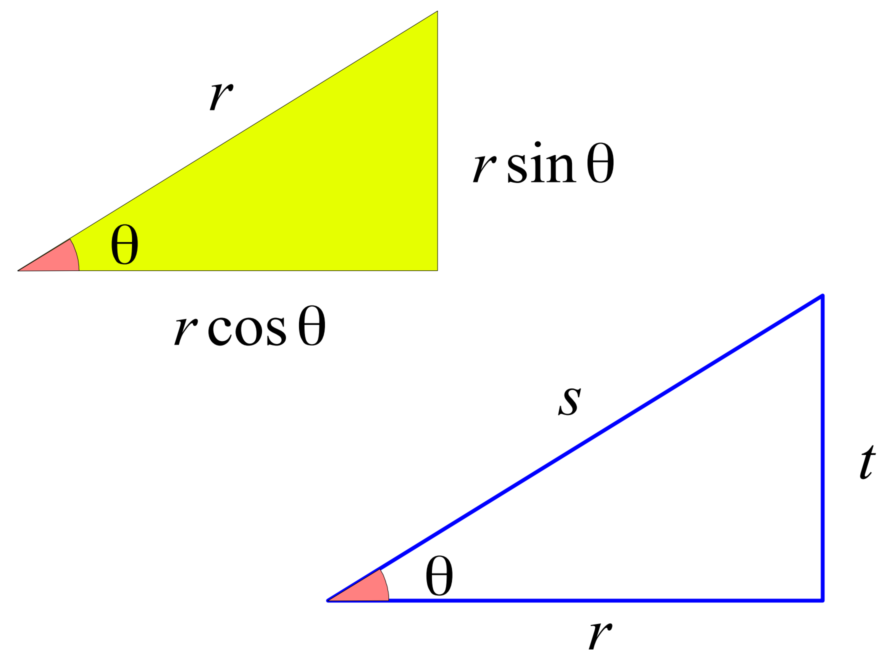
$$\cos\theta = \frac{a}{r}$$

$$\sin\theta = \frac{b}{r}$$

$$\tan\theta = \frac{b}{a}$$

Pseudo-rotation – factor of 1/cos θ





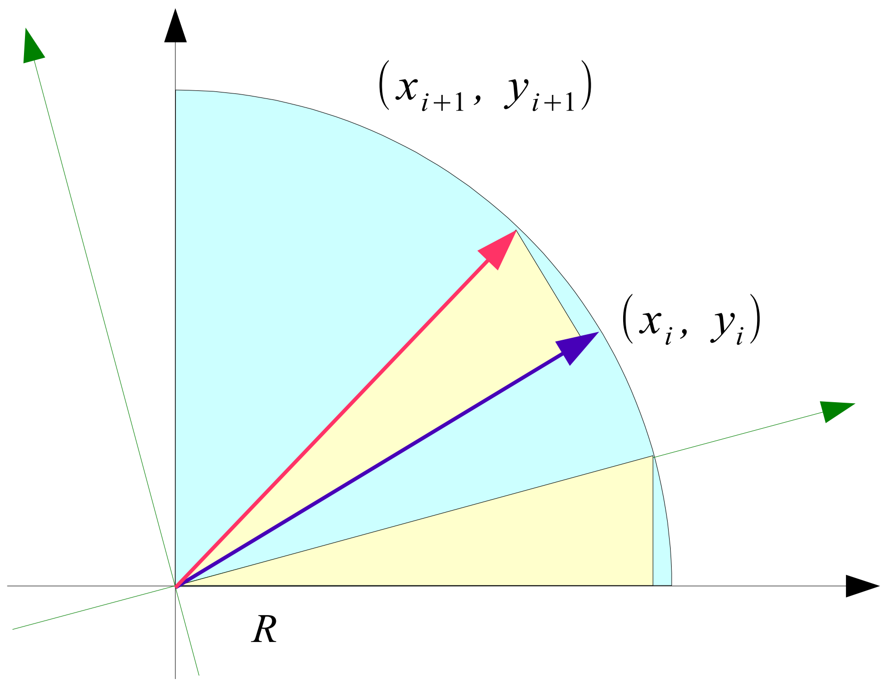
$$r : r\cos\theta = s : r$$

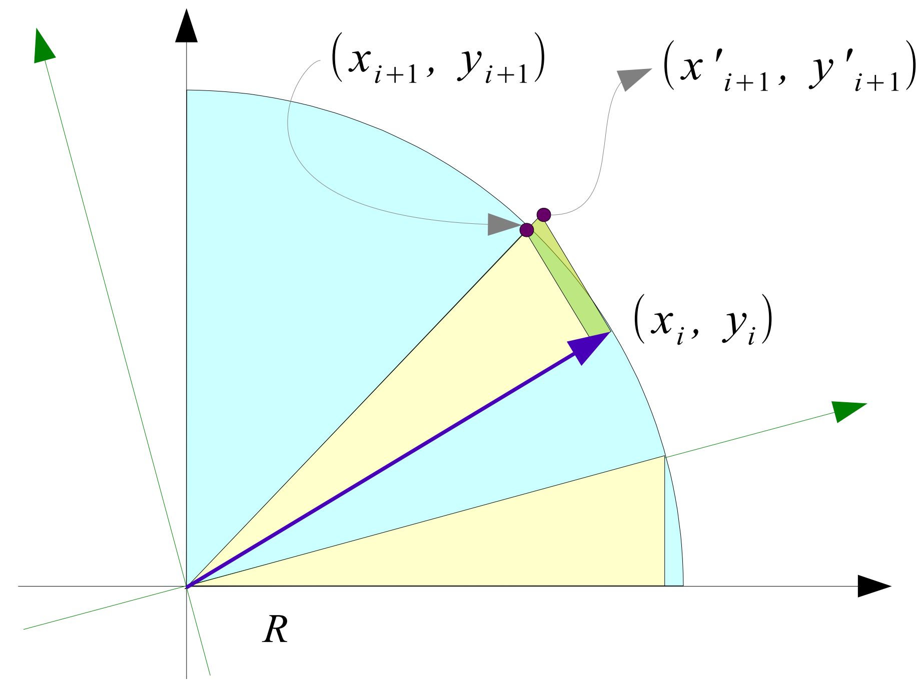
$$r\cos\theta : r\sin\theta = r : t$$

$$s = \frac{r}{\cos\theta}$$

$$s = \frac{r\sin\theta}{\cos\theta}$$

Pseudo-rotation (1)

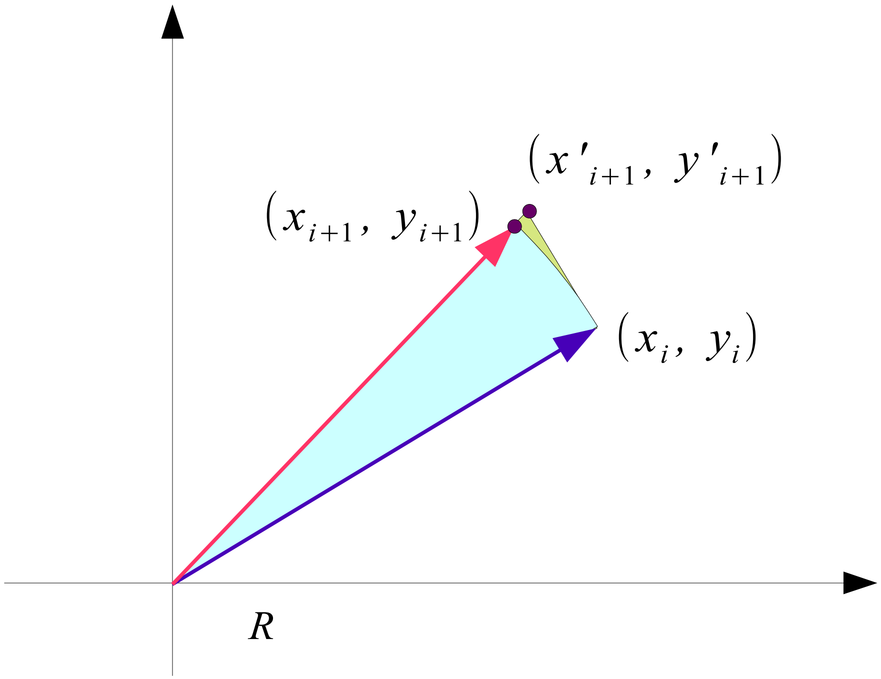


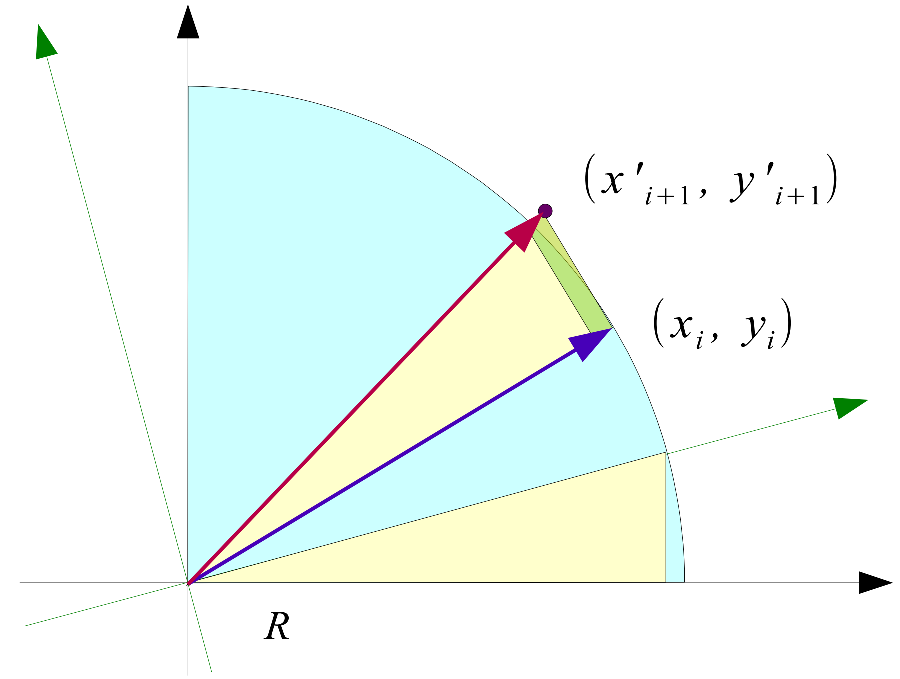


$$\begin{cases} x_{i+1} = \cos\alpha_i \left(x_i - y_i \tan\alpha_i\right) \\ y_{i+1} = \cos\alpha_i \left(x_i \tan\alpha_i + y_i\right) \end{cases}$$

$$\begin{cases} x'_{i+1} = \left(x_i - y_i \tan\alpha_i\right) \\ y'_{i+1} = \left(x_i \tan\alpha_i + y_i\right) \end{cases}$$

Pseudo-rotation (2)





$$x'_{i+1} = x_{i+1} / \cos\alpha_i$$
$$y'_{i+1} = y_{i+1} / \cos\alpha_i$$

$$x'_{i+1} > x_{i+1}$$
$$y'_{i+1} > y_{i+1}$$

CORDIC Iteration Equations (1)

$$\alpha_0 \rightarrow \cdots \rightarrow \alpha_i \rightarrow \alpha_{i+1} \rightarrow \cdots \rightarrow \alpha_n \rightarrow 0$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} \rightarrow \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$x_{i+1} = x_i \cos\alpha_i - y_i \sin\alpha_i = \cos\alpha_i \, (x_i - y_i \tan\alpha_i) = \left(1/\sqrt{1 + \tan^2\alpha_i}\right) (x_i - y_i \tan\alpha_i)$$

$$y_{i+1} = x_i \sin\alpha_i + y_i \cos\alpha_i = \cos\alpha_i \, (x_i \tan\alpha_i + y_i) = \left(1/\sqrt{1 + \tan^2\alpha_i}\right) (x_i \tan\alpha_i + y_i)$$

Pseudo-rotation

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_i \\ y'_i \end{pmatrix} \rightarrow \begin{pmatrix} x'_{i+1} \\ y'_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}$$

$$x'_{i+1} = (x'_i - y'_i \tan\alpha_i) = (x'_i \cos\alpha_i - y'_i \sin\alpha_i) \sqrt{1 + \tan^2\alpha_i}$$

$$y'_{i+1} = (x'_i \tan\alpha_i + y'_i) = (x'_i \sin\alpha_i + y'_i \cos\alpha_i) \sqrt{1 + \tan^2\alpha_i}$$

$$x'_n = \{x_0 \cos(\Sigma\alpha_i) - y_0 \sin(\Sigma\alpha_i)\} \cdot \Pi\{\sqrt{1 + \tan^2\alpha_i}\}$$

$$y'_n = \{x_0 \sin(\Sigma\alpha_i) + y_0 \cos(\Sigma\alpha_i)\} \cdot \Pi\{\sqrt{1 + \tan^2\alpha_i}\}$$

$$\alpha_n = \alpha - \Sigma\alpha_i$$

CORDIC Iteration Equations (2)

Pseudo-rotation

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_i \\ y'_i \end{pmatrix} \rightarrow \begin{pmatrix} x'_{i+1} \\ y'_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}$$

$$x'_{i+1} = (x'_i - y'_i \tan\alpha_i) = (x'_i \cos\alpha_i - y'_i \sin\alpha_i)\sqrt{1+\tan^2\alpha_i}$$
$$y'_{i+1} = (x'_i \tan\alpha_i + y'_i) = (x'_i \sin\alpha_i + y'_i \cos\alpha_i)\sqrt{1+\tan^2\alpha_i}$$

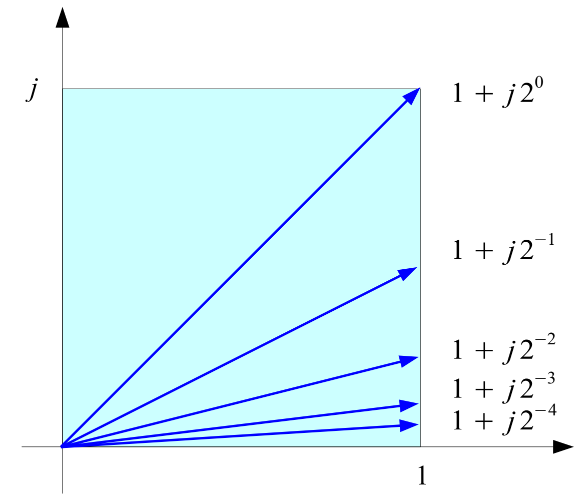
$$x'_n = \{x_0 \cos(\Sigma\alpha_i) - y_0 \sin(\Sigma\alpha_i)\} \cdot \prod\{\sqrt{1+\tan^2\alpha_i}\}$$
$$y'_n = \{x_0 \sin(\Sigma\alpha_i) + y_0 \cos(\Sigma\alpha_i)\} \cdot \prod\{\sqrt{1+\tan^2\alpha_i}\}$$

$$\alpha_n = \alpha - \Sigma\alpha_i$$

Choose α_i such that $\tan\alpha_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

$$\tan\alpha_i = \sigma_i 2^{-i} \qquad \sigma_i \in \{+1, -1\}$$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$
$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$
$$\alpha_{i+1} = \alpha_i - \tan(\sigma_i 2^{-i})$$

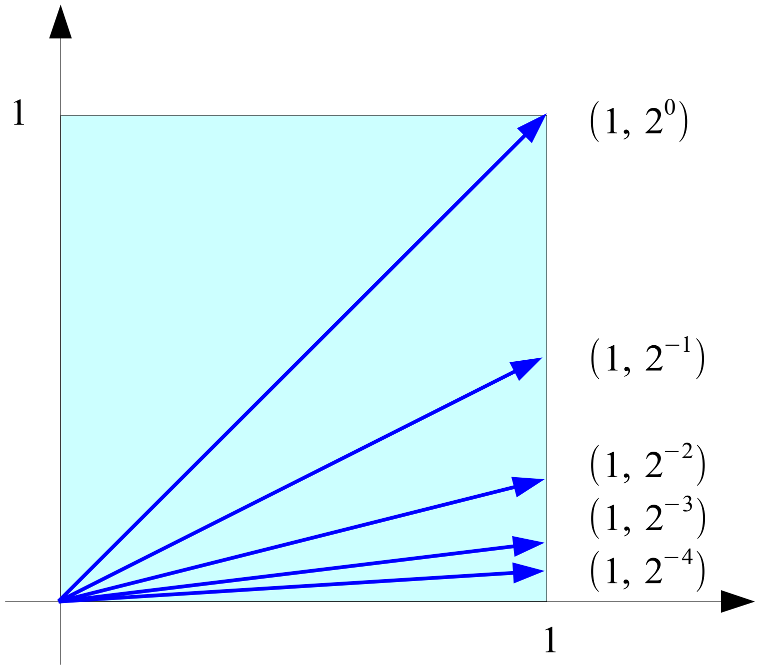


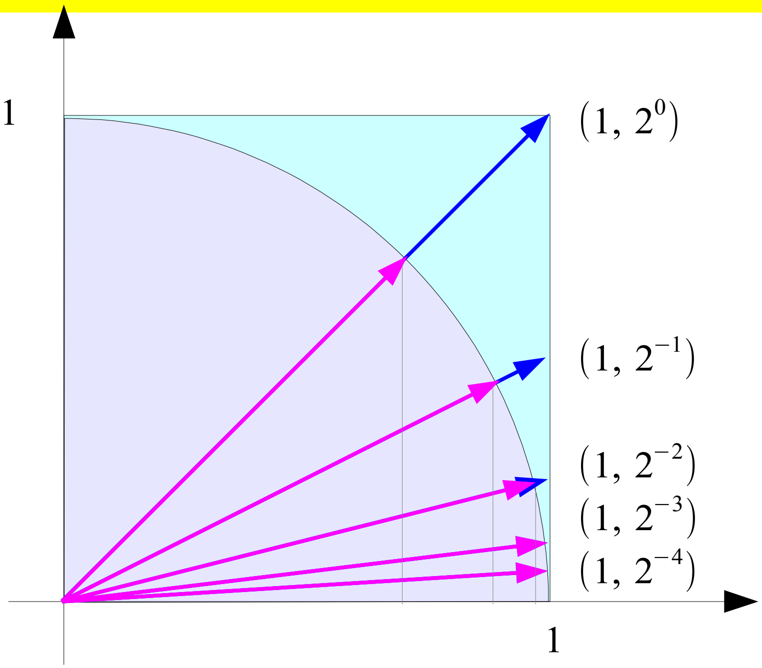
CORDIC Iteration Equations (2)

Choose α_i such that $\tan\alpha_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

$$\tan\alpha_i = \sigma_i 2^{-i} \qquad \sigma_i \in \{+1, -1\}$$

$$x'_{i+1} = (x'_i - y'_i\,\sigma_i 2^{-i})$$
$$y'_{i+1} = (x'_i\,\sigma_i 2^{-i} + y'_i)$$
$$\alpha_{i+1} = \alpha_i - \tan(\sigma_i 2^{-i})$$



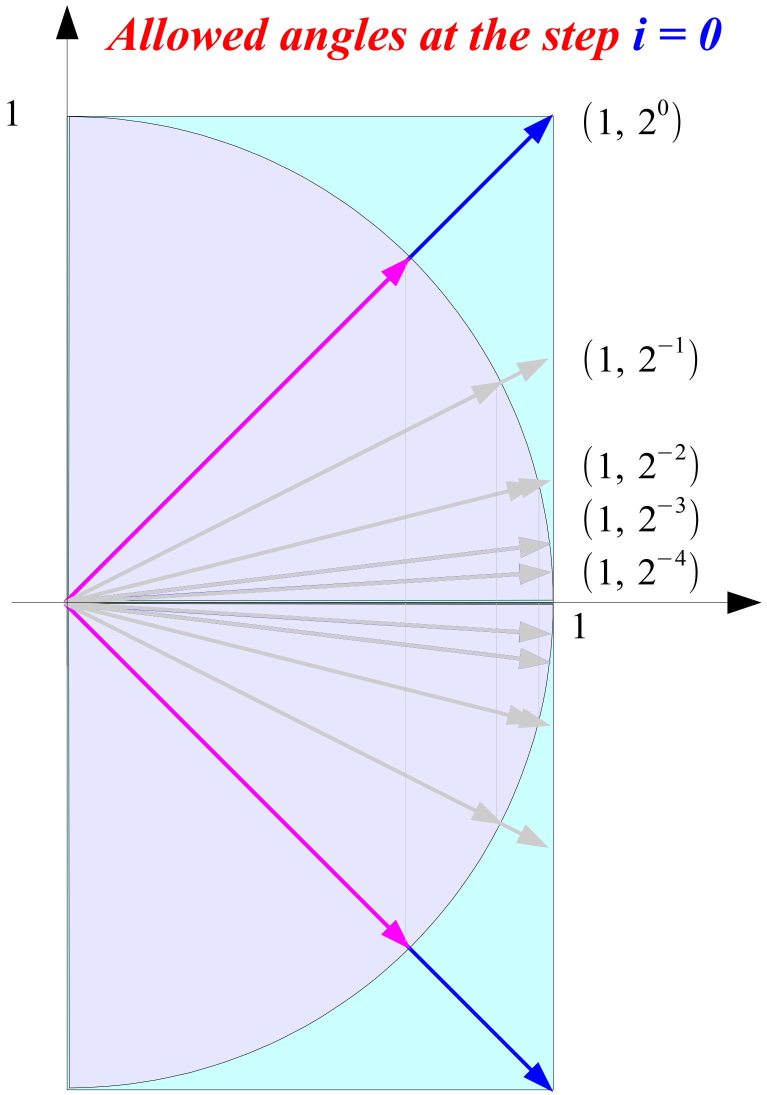


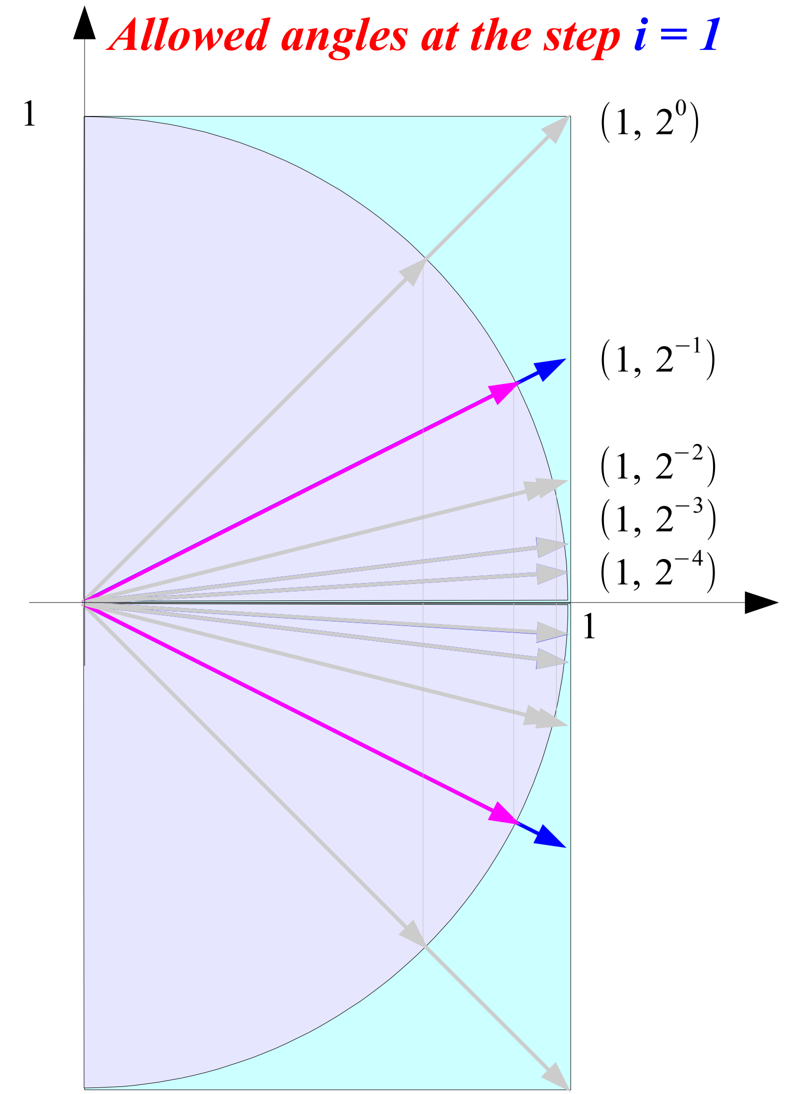
$$\tan\alpha_i = \pm 2^{-i} \qquad \cos\alpha_i = \frac{+1}{\sqrt{1 + 2^{-2i}}}$$

$$\sin\alpha_i = \frac{\pm 2^{-i}}{\sqrt{1 + 2^{-2i}}}$$

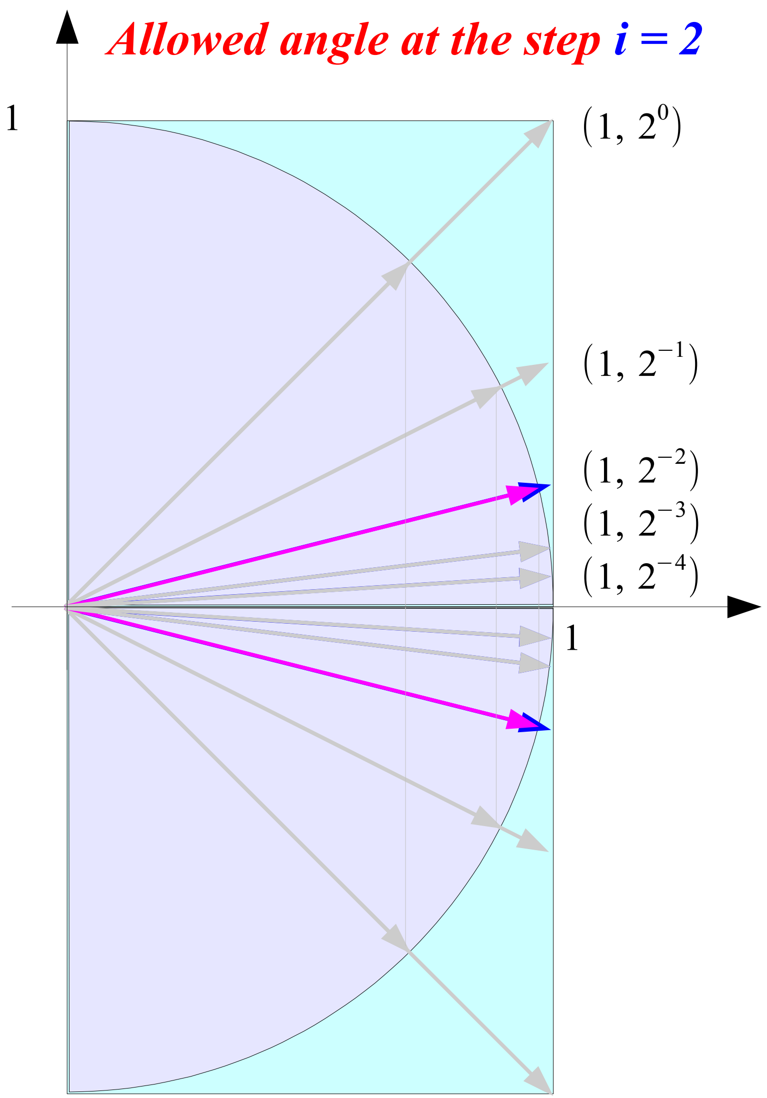
$$\begin{pmatrix} +\cos\alpha_i & -\sin\alpha_i \\ +\sin\alpha_i & +\cos\alpha_i \end{pmatrix} = \frac{1}{\sqrt{1 + 2^{-2i}}} \begin{pmatrix} +1 & \mp 2^{-i} \\ \pm 2^{-i} & +1 \end{pmatrix}$$

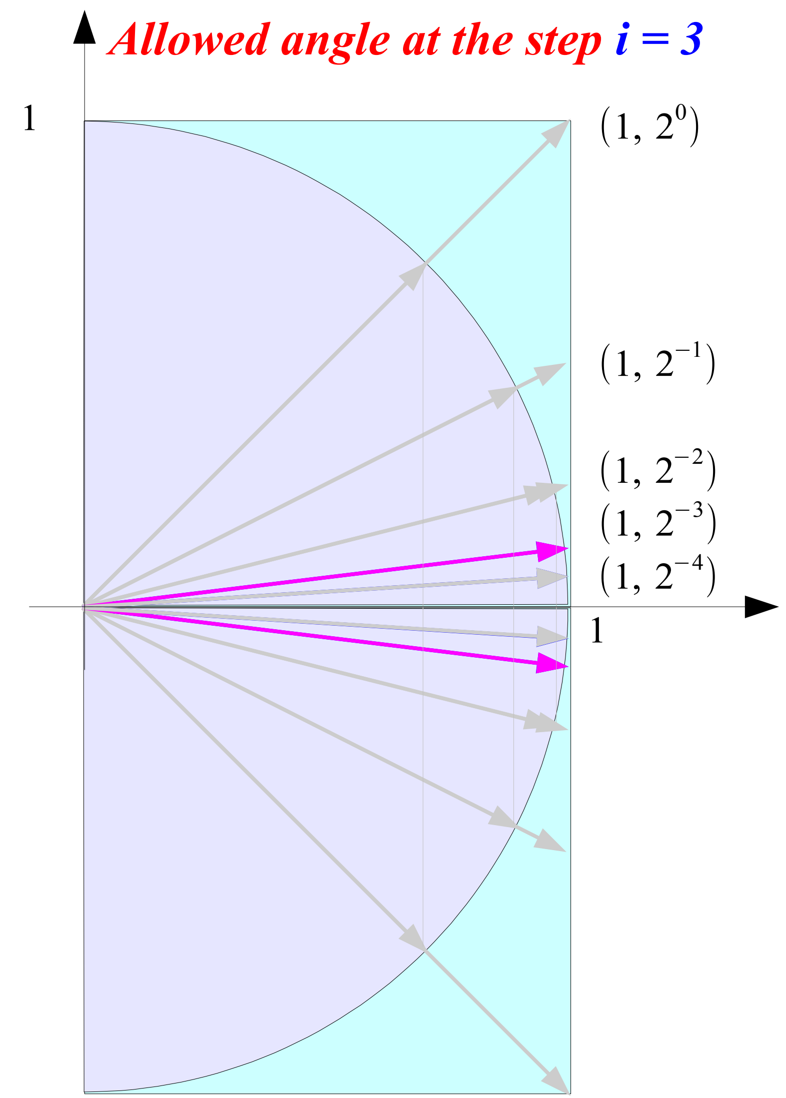
CORDIC Iteration Equations (3)





CORDIC Iteration Equations (4)





CORDIC Iteration Equations (2)

Choose α_i such that $\tan\alpha_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

$$\tan\alpha_i = \sigma_i 2^{-i} \qquad \sigma_i \in \{+1, -1\}$$

$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i\,\sigma_i 2^{-i}) \\ y'_{i+1} &= (x'_i\,\sigma_i 2^{-i} + y'_i) \\ \alpha_{i+1} &= \alpha_i - \tan(\sigma_i 2^{-i}) \end{aligned}$$

$$\begin{pmatrix} +\cos\alpha & -\sin\alpha \\ +\sin\alpha & +\cos\alpha \end{pmatrix}$$

$$= \frac{1}{\sqrt{1+2^{-2\cdot 0}}}\begin{pmatrix} +1 & \mp 2^{-0} \\ \pm 2^{-0} & +1 \end{pmatrix} \cdot \frac{1}{\sqrt{1+2^{-2\cdot 1}}}\begin{pmatrix} +1 & \mp 2^{-1} \\ \pm 2^{-1} & +1 \end{pmatrix} \cdots \frac{1}{\sqrt{1+2^{-2\cdot n}}}\begin{pmatrix} +1 & \mp 2^{-n} \\ \pm 2^{-n} & +1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{1+2^{-2\cdot 0}}} \cdot \frac{1}{\sqrt{1+2^{-2\cdot 1}}} \cdots \frac{1}{\sqrt{1+2^{-2\cdot n}}} \begin{pmatrix} +1 & \mp 2^{-0} \\ \pm 2^{-0} & +1 \end{pmatrix}\begin{pmatrix} +1 & \mp 2^{-1} \\ \pm 2^{-1} & +1 \end{pmatrix} \cdots \begin{pmatrix} +1 & \mp 2^{-n} \\ \pm 2^{-n} & +1 \end{pmatrix}$$

➡ $1 / K = 0.607$

➡ $\begin{pmatrix} +\cos(\sum\alpha_i) & -\sin(\sum\alpha_i) \\ +\sin(\sum\alpha_i) & +\cos(\sum\alpha_i) \end{pmatrix}$

$$\begin{pmatrix} +\cos(\sum\alpha_i) & -\sin(\sum\alpha_i) \\ +\sin(\sum\alpha_i) & +\cos(\sum\alpha_i) \end{pmatrix} = K \cdot \begin{pmatrix} +\cos\alpha & -\sin\alpha \\ +\sin\alpha & +\cos\alpha \end{pmatrix}$$

$$K = \prod\left\{\sqrt{1+\tan^2\alpha_i}\right\} = 1.647$$

CORDIC Iteration Equations (5)

Pseudo-rotation

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_i \\ y'_i \end{pmatrix} \rightarrow \begin{pmatrix} x'_{i+1} \\ y'_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}$$

$$x'_{i+1} = (x'_i - y'_i \tan\alpha_i) = (x'_i \cos\alpha_i - y'_i \sin\alpha_i)\sqrt{1 + \tan^2\alpha_i}$$

$$y'_{i+1} = (x'_i \tan\alpha_i + y'_i) = (x'_i \sin\alpha_i + y'_i \cos\alpha_i)\sqrt{1 + \tan^2\alpha_i}$$

$$x'_n = \{x_0 \cos(\Sigma\alpha_i) - y_0 \sin(\Sigma\alpha_i)\} \cdot \Pi\{\sqrt{1 + \tan^2\alpha_i}\}$$

$$y'_n = \{x_0 \sin(\Sigma\alpha_i) + y_0 \cos(\Sigma\alpha_i)\} \cdot \Pi\{\sqrt{1 + \tan^2\alpha_i}\}$$

$$\alpha_n = \alpha - \Sigma\alpha_i$$

Choose α_i such that $\tan\alpha_i = \sigma_i 2^{-i}$ $\quad \sigma_i \in \{+1, -1\}$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_n = K(x_0 \cos\alpha - y_0 \sin\alpha)$$

$$y'_n = K(x_0 \sin\alpha + y_0 \cos\alpha)$$

$$\alpha_n = \alpha - \Sigma \tan^{-1}(\sigma_i 2^{-i})$$

$$K = \Pi\{\sqrt{1 + \tan^2\alpha_i}\}$$

CORDIC Iteration Equations (4)

Pseudo-rotation

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_i \\ y'_i \end{pmatrix} \rightarrow \begin{pmatrix} x'_{i+1} \\ y'_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}$$

Choose α_i such that $\tan\alpha_i = \sigma_i 2^{-i}$ $\quad \sigma_i \in \{+1, -1\}$

$$x'_{i+1} = (x'_i - y'_i\,\sigma_i 2^{-i})$$
$$y'_{i+1} = (x'_i\,\sigma_i 2^{-i} + y'_i)$$
$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_n = K(x_0 \cos\alpha - y_0 \sin\alpha)$$
$$y'_n = K(x_0 \sin\alpha + y_0 \cos\alpha)$$
$$\alpha_n = \alpha - \sum \tan^{-1}(\sigma_i 2^{-i})$$

pre-compute $K = \prod \{\sqrt{1 + \tan^2\alpha_i}\} = 1.647$

set $x_0 = 1/K = 0.607$

$y_0 = 0$

then $x'_n = K(1/K \cos\alpha - 0 \cdot \sin\alpha) = \cos\alpha$

$y'_n = K(1/K \sin\alpha + 0 \cdot \cos\alpha) = \sin\alpha$

Example: cos75°, sin75° - Step 0

i = 0

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0.607 \\ 0 \end{pmatrix} \qquad \alpha_0 = 75^\circ$$

$$(\alpha_i > 0) \;?\; \sigma_i = +1 \;:\; \sigma_i = -1$$

$$(\alpha_0 = 75)^\circ > 0^\circ \quad \Rightarrow \quad \sigma_0 = +1$$

$$2^{-i} = 2^0 = 1 \quad \Rightarrow \quad \tan^{-1} 2^{-i} = \tan^{-1} 1 = 45^\circ$$

$$x'_{i+1} = (x'_i - y'_i \, \sigma_i 2^{-i})$$
$$y'_{i+1} = (x'_i \, \sigma_i 2^{-i} + y'_i)$$
$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_1 = 0.607 - 0 \cdot (+1) \cdot 1 = 0.607$$
$$y'_1 = 0.607 \cdot (+1) \cdot 1 + 0 = 0.607$$
$$\alpha_1 = \alpha_0 - 45^\circ = 30^\circ$$

Example: cos75°, sin75° - Step 1

i = 1

$$\begin{pmatrix} x'_1 \\ y'_1 \end{pmatrix} = \begin{pmatrix} 0.607 \\ 0.607 \end{pmatrix} \qquad \alpha_1 = 30^\circ$$

$$(\alpha_i > 0) \; ? \; \sigma_i = +1 \; : \; \sigma_i = -1$$

$$(\alpha_1 = 30)^\circ > 0^\circ \quad \Rightarrow \quad \sigma_1 = +1$$

$$2^{-i} = 2^{-1} = 0.5 \quad \Rightarrow \quad \tan^{-1} 2^{-i} = \tan^{-1} 0.5 = 26.565^\circ$$

$$x'_{i+1} = (x'_i - y'_i \, \sigma_i 2^{-i})$$

$$y'_{i+1} = (x'_i \, \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_2 = 0.607 - 0.607 \cdot (+1) \cdot 0.5 = 0.3035$$

$$y'_2 = 0.607 \cdot (+1) \cdot 0.5 + 0.607 = 0.9105$$

$$\alpha_2 = \alpha_1 - 26.565^\circ = 3.435^\circ$$

Example: cos75°, sin75° - Step 2

$i = 2$

$$\begin{pmatrix} x'_2 \\ y'_2 \end{pmatrix} = \begin{pmatrix} 0.3035 \\ 0.9105 \end{pmatrix} \qquad \alpha_2 = 3.435^\circ$$

$$(\alpha_i > 0) \; ? \; \sigma_i = +1 \; : \; \sigma_i = -1$$

$$(\alpha_2 = 3.435)^\circ > 0^\circ \quad \Rightarrow \quad \sigma_2 = +1$$

$$2^{-i} = 2^{-2} = 0.25 \quad \Rightarrow \quad \tan^{-1} 2^{-2} = \tan^{-1} 0.25 = 14.036^\circ$$

$$x'_{i+1} = (x'_i - y'_i\, \sigma_i 2^{-i})$$
$$y'_{i+1} = (x'_i\, \sigma_i 2^{-i} + y'_i)$$
$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_3 = 0.3035 - 0.9105 \cdot (+1) \cdot 0.25 = 0.0759$$
$$y'_3 = 0.3035 \cdot (+1) \cdot 0.25 + 0.9105 = 0.9864$$
$$\alpha_3 = \alpha_2 - 14.036^\circ = -10.601^\circ$$

Example: cos75°, sin75° - Step 3

i = 3 $\begin{pmatrix} x'_3 \\ y'_3 \end{pmatrix} = \begin{pmatrix} 0.0759 \\ 0.9864 \end{pmatrix}$ $\alpha_3 = -10.601^\circ$

$(\alpha_i > 0)$? $\sigma_i = +1$: $\sigma_i = -1$

$(\alpha_3 = -10.601)^\circ < 0^\circ \Rightarrow \sigma_3 = -1$

$2^{-i} = 2^{-3} = 0.125 \Rightarrow \tan^{-1} 2^{-3} = \tan^{-1} 0.125 = 7.125^\circ$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$
$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$
$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_4 = 0.0759 - 0.9864 \cdot (-1) \cdot 0.125 = 0.1992$$
$$y'_4 = 0.0759 \cdot (-1) \cdot 0.125 + 0.9864 = 0.9373$$
$$\alpha_4 = \alpha_3 + 7.125^\circ = -3.476^\circ$$

Example: cos75°, sin75° - Step 4

i = 4

$$\begin{pmatrix} x'_4 \\ y'_4 \end{pmatrix} = \begin{pmatrix} 0.1992 \\ 0.9373 \end{pmatrix} \qquad \alpha_4 = -3.476^\circ$$

$$(\alpha_i > 0) \; ? \; \sigma_i = +1 \; : \; \sigma_i = -1$$

$$(\alpha_4 = -3.476)^\circ < 0^\circ \Rightarrow \sigma_4 = -1$$

$$2^{-i} = 2^{-4} = 0.0625 \Rightarrow \tan^{-1} 2^{-4} = \tan^{-1} 0.0625 = 3.576^\circ$$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_5 = 0.1992 - 0.9373 \cdot (-1) \cdot 0.0625 = 0.2578$$

$$y'_5 = 0.1992 \cdot (-1) \cdot 0.0625 + 0.9373 = 0.9249$$

$$\alpha_5 = \alpha_4 + 3.576^\circ = 0.1^\circ$$

References

[1] http://en.wikipedia.org/
[2] CORDIC FAQ, www.dspguru.com
[3] R. Andraka, A survey of CORDIC algorithms for FPGA based computers
[4] J. S. Walther, A Unified Algorithm for Elementary Functions
[5] J. P. Deschamps, G. A. Bioul, G.D. Sutter, Synthesis of Arithmetic Circuits