CLTI Correlation (2A)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Correlation

How signals move
relative to each other

Positively correlated — the same direction

Average of product > product of averages

Negatively correlated — the opposite direction

Average of product < product of averages

Uncorrelated

Correlation Function for Energy Signals

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)\, y^*(t+\tau)\, dt = \int_{-\infty}^{+\infty} x(t-\tau)\, y^*(t)\, dt$$

Both real

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)\, y(t+\tau)\, dt = \int_{-\infty}^{+\infty} x(t-\tau)\, y(t)\, dt$$

Correlation and Convolution

Both real

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)\, y(t+\tau)\, dt = \int_{-\infty}^{+\infty} x(t-\tau)\, y(t)\, dt$$

Convolution

$$x(t) * y(t) = \int_{-\infty}^{+\infty} x(t-\tau)\, y(\tau)\, d\tau$$

$$R_{xy}(\tau) = x(-\tau) * y(\tau)$$

$$x(-t) \Longleftrightarrow X^*(f)$$

$$R_{xy}(\tau) \Longleftrightarrow X^*(f)\, Y(f)$$

Correlation for Power Signals

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)\, y^*(t+\tau)\, dt = \int_{-\infty}^{+\infty} x(t-\tau)\, y^*(t)\, dt$$ Energy Signal

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_T x(t)\, y^*(t+\tau)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_T x(t-\tau)\, y^*(t)\, dt$$ Power Signal

Both real

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)\, y(t+\tau)\, dt = \int_{-\infty}^{+\infty} x(t-\tau)\, y(t)\, dt$$ Energy Signal

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_T x(t)\, y(t+\tau)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_T x(t-\tau)\, y(t)\, dt$$ Power Signal

$$R_{xy}(\tau) = \frac{1}{T} \int_T x(t)\, y(t+\tau)\, dt$$ Periodic Power Signal

Correlation for Periodic Power Signals

$$R_{xy}(\tau) = \frac{1}{T}\int_T x(t)\,y(t+\tau)\,dt$$

Periodic Power Signal

$$R_{xy}(\tau) = \frac{1}{T}[x(-\tau) \circledast y(\tau)]$$

$$R_{xy}(\tau) \overset{\text{CTFS}}{\Longleftrightarrow} X^*[k]\,Y[k]$$

Circular Convolution

$$x(t) * y(t) \overset{\text{CTFS}}{\Longleftrightarrow} T\,X[k]\,Y[k]$$

$$x[n] * y[n] \overset{\text{CTFS}}{\Longleftrightarrow} N_0\,Y[k]\,X[k]$$

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)\,y(t+\tau)\,dt$$

$$R_{xy}(\tau) = \frac{1}{T}\int_T x(t)\,y(t+\tau)\,dt$$

Correlation for Power & Energy Signals

One signal – a power signal
The other – an energy signal

Use the Energy Signal Version

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)\,y(t+\tau)\,dt$$

Autocorrelation

$$R_{xx}(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau)\, dt$$

Energy Signal

$$R_{xx}(0) = \int_{-\infty}^{+\infty} x^2(t)\, dt$$

total signal energy

$$R_{xx}(\tau) = \frac{1}{T}\int_{T} x(t)x(t+\tau)\, dt$$

Power Signal

$$R_{xx}(0) = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{+\infty} x^2(t)\, dt$$

average signal power

Autocorrelation Property

$$R_{xx}(0) \geq R_{xx}(\tau)$$

total signal energy

$$R_{xx}(\tau) = \int_{-\infty}^{+\infty} x(t)\,x(t+\tau)\,dt$$

Energy Signal

$$R_{xx}(-\tau) = \int_{-\infty}^{+\infty} x(t)\,x(t-\tau)\,dt$$

$$R_{xx}(\tau) = R_{xx}(-\tau)$$

$$y(t) = x(t-t_0)$$

$$R_{yy}(\tau) = \int_{-\infty}^{+\infty} y(t)\,y(t+\tau)\,dt = \int_{-\infty}^{+\infty} x(t-t_0)\,x(t-t_0+\tau)\,dt$$

$$= \int_{-\infty}^{+\infty} x(s)\,x(s+\tau)\,ds = R_{xx}(\tau)$$

Autocorrelation of Random Signals

$$x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \theta_k)$$

$$R_x(\tau) = \sum_{k=1}^{N} R_k(\tau)$$

$R_k(\tau)$ Autocorrelation of $a_k \cos(2\pi f_k t + \theta_k)$

Independent of choice of θ_k

Assume random phase shift, but the same amplitudes and frequencies

Different appearance, but the identical autocorrelation function

Similar look but not exactly the same

The common characteristics (the amplitudes and frequencies) are described by the autocorrelation function

Describes a signal generally, but not exactly – suitable for a random signal

CrossCorrelation

$$R_{xy}(\tau) = R_{xy}(-\tau)$$

The largest peak occurs at a shift which is exactly the amount of shift
Between x(t) and y(t)

The signal power of the sum depends strongly on whether two signals are correlated

Positively correlated vs. uncorrelated

ESD (Energy Spectral Density)

Parseval's theorem

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |X(f)|^2 \, df$$

$$|X(f)|^2 = \Psi_x(f)$$ **Energy Spectral Density**

Real x(t) Even, Non-negative, Real $\Psi_x(f)$

$$E_x = 2\int_{0}^{+\infty} \Psi_x(f) \, df$$

Positively correlated vs. uncorrelated

ESD and Band-pass Filtering

$$E_y = 2\int_0^{+\infty} \Psi_y(f)\, df = 2\int_0^{+\infty} |Y(f)|^2\, df = 2\int_0^{+\infty} |H(f)X(f)|^2\, df$$

$$E_y = 2\int_0^{+\infty} |H(f)|^2 \Psi_x(f)\, df = 2\int_{f_L}^{f_H} \Psi_x(f)\, df$$

$$\Psi_y(f) = |H(f)|^2 \Psi_x(f) = H(f)H^*(f)\Psi_x(f)$$

A description of the signal energy versus frequency
How the signal energy is distributed in frequency

ESD and Autocorrelation

$$R_x(t) \Longleftrightarrow \Psi_x(f)$$

$$\Psi_x(f) = |X(f)|^2$$

$$R_x(t) \Longleftrightarrow X^*(f)\,X(f)$$

$$R_x(t) = x(-t) * x(t) = \int_{-\infty}^{+\infty} x(-\tau)\,x(t-\tau)\,d\tau$$

$$R_x(t) = \int_{-\infty}^{+\infty} x(\tau)\,x(\tau+t)\,d\tau$$

Power Spectral Density (PSD)

The ESD of a truncated version of x(t)

$$x_T(t) \qquad x(t) \qquad |t| < \frac{T}{2} \qquad\qquad rect\left(\frac{t}{T}\right)x(t)$$

$$0$$

$$\Psi_{x_T}(f) = |X_T(f)|^2 \qquad X_T(f) = \int_{-\infty}^{+\infty} x_T(\tau)e^{-2\pi f t}\,dt = \int_{-T/2}^{+T/2} x_T(\tau)e^{-2\pi f t}\,dt$$

Average Signal Power

$$G_{X_T}(f) = \frac{\Psi_{X_T}}{T} = \frac{1}{T}|X_T(f)|^2$$

$$G_x(f) = \lim_{T\to\infty} G_{X_T}(f) = \lim_{T\to\infty} \frac{1}{T}|X_T(f)|^2$$

The power of a finite signal power signal in a bandwidth f_L f_H

$$2\int_{f_L}^{f_H} G(f)\,df$$

PSD and Band-pass Filtering

$$G_y(f) = |H(f)|^2 G_x(f) = H(f)H^*(f)G_x(f)$$

A description of the signal energy versus frequency
How the signal energy is distributed in frequency

References

[1] http://en.wikipedia.org/
[2] M.J. Roberts, Signals and Systems,