Problem 5 (mtg 31)

Statement: Show that the following equation

$$P_n(x) = \sum_{i=0}^{[n/2]} (-1)^i \frac{(2n-2i)!\, x^{n-2i}}{2^n i!\,(n-i)!\,(n-2i)!} \qquad (1)$$

is the same as

$$P_n(x) = \sum_{i=0}^{[n/2]} \frac{1 \cdot 3 \cdot \ldots \cdot (2n-2i-1)}{2^i i!\,(n-2i)!} (-1)^i x^{n-2i} \qquad (2)$$

Solution: $$P_n(x) = \sum \frac{(2n-2i)!}{2^n (n-i)!} \underbrace{\frac{(-1)^i x^{n-2i}}{i!\,(n-2i)!}}_{\text{this is the same in both (1) and (2)}} \qquad (3)$$

So, we must show

$$\frac{(2n-2i)!}{2^n (n-1)!} = \frac{1 \cdot 3 \cdot \ldots \cdot (2n-2i-1)}{2^i} \qquad (4)$$

Problem 6 (mtg 31)

Statement: Verify that the following equations

$$P_0(x) = 1 \qquad (1)$$

$$P_1(x) = x \qquad (2)$$

$$P_2(x) = 1/2\,(3x^2 - 1) \qquad (3)$$

$$P_3(x) = 1/2\,(5x^3 - 3x) \qquad (4)$$

$$P_4(x) = 35/8\, x^4 - 15/4\, x^2 + 3/8 \qquad (5)$$

can be written as

$$P_n(x) = \sum_{i=0}^{[n/2]} \frac{(-1)^i (2n-2i)!\, x^{n-2i}}{2^n i!\,(n-i)!\,(n-2i)!} \qquad (6)$$

Solution:

for $n=0$: $$P_0(x) = \sum_{i=0}^{[0/2]=0} \frac{(-1)^0 (2\cdot 0 - 2\cdot 0)!\, x^{0-2\cdot 0}}{2^0\, 0!\,(0-0)!\,(0-2\cdot 0)!} \qquad (7)$$

$$= \frac{1\cdot 1\cdot x^0}{1\cdot 1\cdot 1\cdot 1} \qquad (8)$$

$$\boxed{P_0(x) = 1}$$

for $n=1$:

$$P_1(x) = \sum_{i=0}^{[1/2]=.5\Rightarrow 0} \frac{(-1)^0(2\cdot1-2\cdot0)!\,x^{1-2\cdot0}}{2^1\,0!\,(1-0)!\,(1-2\cdot0)!} \quad (9)$$

$$= \frac{1\cdot 2\cdot x}{2\cdot1\cdot1\cdot1} \quad (10)$$

$$\boxed{P_1(x) = x}$$

for $n=2$:

$$P_2(x) = \sum_{i=0}^{[2/2]=1} \frac{(-1)^0(2\cdot2-2\cdot0)!\,x^{2-2\cdot0}}{2^2\,0!\,(2-0)!\,(2-2\cdot0)!} \quad (11)$$

$$+ \frac{(-1)^1(2\cdot2-2\cdot1)!\,x^{2-2\cdot1}}{2^2\,1!\,(2-1)!\,(2-2\cdot1)!} \quad (12)$$

$$= \frac{1\cdot 24\cdot x^2}{4\cdot1\cdot2\cdot2} + \frac{(-1)\cdot2\cdot x^0}{4\cdot1\cdot1\cdot1} \quad (13)$$

$$= \frac{3}{2}x^2 - \frac{1}{2} \quad (14)$$

$$\boxed{P_2(x) = \frac{1}{2}(3x^2-1)}$$

for $n=3$:

$$P_3(x) = \sum_{i=0}^{[3/2]=1.5\Rightarrow 1} \frac{(-1)^0(2\cdot3-2\cdot0)!\,x^{3-2\cdot0}}{2^3\,0!\,(3-0)!\,(3-2\cdot0)!} \tag{15}$$

$$+ \frac{(-1)^1(2\cdot3-2\cdot1)!\,x^{3-2\cdot1}}{2^3\,1!\,(3-1)!\,(3-2\cdot1)!}$$

$$= \frac{1\cdot720\cdot x^3}{8\cdot1\cdot6\cdot6} - \frac{1\cdot24\cdot x^1}{8\cdot1\cdot2\cdot1} \tag{16}$$

$$= \frac{5}{2}x^3 - \frac{3}{2}x \tag{17}$$

$$\boxed{P_3(x) = \frac{1}{2}(5x^3-3x)}$$

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for $n=4$:

$$P_4(x) = \sum_{i=0}^{[4/2]=2} \frac{(-1)^0(2\cdot4-2\cdot0)!\,x^{4-2\cdot0}}{2^4\,0!\,(4-0)!\,(4-2\cdot0)!} \tag{18}$$

$$+ \frac{(-1)^1(2\cdot4-2\cdot1)!\,x^{4-2\cdot1}}{2^4\,1!\,(4-1)!\,(4-2\cdot1)!} + \frac{(-1)^2(2\cdot4-2\cdot2)\,x^{4-2\cdot2}}{2^4\,2!\,(4-2)!\,(4-2\cdot2)!}$$

$$= \frac{1\cdot40320\cdot x^4}{16\cdot1\cdot24\cdot24} + \frac{(-1)\cdot720\cdot x^2}{16\cdot1\cdot6\cdot2} + \frac{1\cdot24\cdot x^0}{16\cdot2\cdot2\cdot1} \tag{19}$$

$$\boxed{P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}}$$

Problem 7 (mtg 32)

Statement: Verify that the following equations

$$P_0(x) = 1 \qquad (1)$$

$$P_1(x) = x \qquad (2)$$

$$P_2(x) = 1/2(3x^2 - 1) \qquad (3)$$

$$P_3(x) = 1/2(5x^3 - 3x) \qquad (4)$$

$$P_4(x) = 35/8\,x^4 - 15/4\,x^2 + 3/8 \qquad (5)$$

are solutions of the Legendre equation:

$$F = (1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (6)$$

Solution:

$n=0$ $\qquad P_0(x) = y_n = 1, \quad y' = 0 \quad y'' = 0 \qquad (7)$

$$F_0 = (1-x^2)y'' - 2xy' = 0 \qquad (8)$$

substitute (7) into (8)

$$F_0 = (1-x^2)0 - (2x)0 = 0$$

$$0 = 0$$

$\underline{n=1}$ $\quad P_1(x) = y_n = X \quad , \quad y' = 1 \quad , \quad y'' = 0$ (9)

$$F_1 = (1-x^2)y'' - 2xy' + 2y = 0 \qquad (10)$$

Substitute (9) into (10)

$$F_1 = (1-x^2)0 - 2x(1) + 2x = 0$$

$$-2x + 2x = 0$$

$$0 = 0$$

$\underline{n=2}$ $\quad P_2(x) = y_n = 3/2\,x^2 - 1/2 \quad , \quad y' = 3x \quad y'' = 3$ (11)

$$F_2 = (1-x^2)y'' - 2xy' + 6y = 0 \qquad (12)$$

~~Substitute (11) into (12)~~

$$F_2 = (1-x^2)3 - 2x(3x) + 6(3/2\,x^2 - 1/2) = 0$$

$$3 - 3x^2 - 6x^2 + 9x^2 - 3 = 0$$

$$0 = 0$$

$\underline{n=3}$ $P_3(x) = y_n = 5/2\,x^3 - 3/2\,x, \quad y' = 15/2\,x^2 - 3/2$ (13)

$$y'' = 15x$$

$$F_3 = (1-x^2)y'' - 2xy' + 12y = 0 \qquad (14)$$

substitute (13) into (14)

$$(1-x^2)15x - 2x(15/2\,x^2 - 3/2) + 12(5/2\,x^3 - 3/2\,x) = 0$$

$$15x - 15x^3 - 15x^3 - 3x + 30x^3 - 18x = 0$$

$$0 = 0$$

$\underline{n=4}$ $P_4(x) = 35/8\,x^4 - 15/4\,x^2 + 3/8 = y_n$

$$y' = 35/2\,x^3 - 15/2\,x, \quad y'' = 105/2\,x^2 - 15/2 \qquad (15)$$

$$F_4 = (1-x^2)y'' - 2xy' + 20y = 0 \qquad (16)$$

substitute (15) into (16)

$$(1-x^2)(105/2\,x^2 - 15/2) - 2x(35/2\,x^3 - 15/2\,x) + 20(35/8\,x^4 - 15/4\,x^2 + 3/8) = 0$$

$$\frac{105}{2}x^2 - \frac{15}{2} - \frac{105}{2}x^4 + \frac{15}{2}x^2 - 35x^4 + 15x^2 + \frac{175}{2}x^4 - 75x^2 + \frac{15}{2} = 0$$

$$0 = 0$$