

Signals and Spectra (1A)

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Energy and Power

Instantaneous Power

$$p(t) = x^2(t)$$

Energy dissipated
during $(-T/2, +T/2)$

$$E_x^T = \int_{-T/2}^{+T/2} x^2(t) dt$$

Affects the performance
of a communication system

Real signal

Average power dissipated
during $(-T/2, +T/2)$

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt$$

The rate at which energy is
dissipated

Determines the voltage

Energy and Power Signals (1)

Energy dissipated
during $(-T/2, +T/2)$

$$E_x^T = \int_{-T/2}^{+T/2} x^2(t) dt$$

Energy Signal

Nonzero but finite energy

For all time

$$0 < E_x < +\infty$$

$$\begin{aligned} E_x &= \lim_{T \rightarrow +\infty} \int_{-T/2}^{+T/2} x^2(t) dt \\ &= \int_{-\infty}^{+\infty} x^2(t) dt \end{aligned}$$

Average power dissipated
during $(-T/2, +T/2)$

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt$$

Power Signal

Nonzero but finite power

For all time

$$0 < P_x < +\infty$$

$$P_x = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt$$

Energy and Power Signals (2)

Energy Signal

$$0 < E_x < +\infty$$

$$E_x = \lim_{T \rightarrow +\infty} \int_{-T/2}^{+T/2} x^2(t) dt$$
$$= \int_{-\infty}^{+\infty} x^2(t) dt$$

$$P_x = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt$$
$$= \lim_{T \rightarrow +\infty} \frac{B}{T} \rightarrow 0$$

Non-periodic signals
Deterministic signals

Power Signal

$$0 < P_x < +\infty$$

$$P_x = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt$$

$$E_x = \lim_{T \rightarrow +\infty} \int_{-T/2}^{+T/2} x^2(t) dt$$
$$= \lim_{T \rightarrow +\infty} B \cdot T \rightarrow +\infty$$

Periodic signals
Random signals

Energy and Power Spectral Densities (1)

Energy Spectral Density

$$\begin{aligned} E_x &= \int_{-\infty}^{+\infty} x^2(t) dt \\ &= \int_{-\infty}^{+\infty} |X(f)|^2 df \\ &= \int_{-\infty}^{+\infty} \Psi(f) df \\ &= 2 \int_0^{+\infty} \Psi(f) df \end{aligned}$$

$$\Psi(f) = |X(f)|^2$$

Power Spectral Density

$$\begin{aligned} P_x &= \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x^2(t) dt \\ &= \sum_{n=-\infty}^{+\infty} |c_n|^2 \\ &= \int_{-\infty}^{+\infty} G_x(f) df \\ &= 2 \int_0^{+\infty} G_x(f) df \end{aligned}$$

$$G_x(f) = \sum_{n=-\infty}^{+\infty} |c_n|^2 \delta(f - n f_0)$$

Energy and Power Spectral Densities (2)

Energy Spectral Density

$$\begin{aligned} E_x &= \int_{-\infty}^{+\infty} x^2(t) dt \\ &= \int_{-\infty}^{+\infty} \Psi(f) df \end{aligned}$$

$$\Psi(f) = |X(f)|^2$$

Power Spectral Density

$$\begin{aligned} P_x &= \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x^2(t) dt \\ &= \int_{-\infty}^{+\infty} G_x(f) df \end{aligned}$$

$$G_x(f) = \sum_{n=-\infty}^{+\infty} |c_n|^2 \delta(f - n f_0)$$

$$G_x(f) = \lim_{T \rightarrow +\infty} \frac{1}{T} |X_T(f)|^2$$

Autocorrelation of Energy and Power Signals

Autocorrelation of an Energy Signal

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau) dt$$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \leq R_x(0)$$

$$R_x(\tau) \Leftrightarrow \Psi(f)$$

$$R_x(0) = \int_{-\infty}^{+\infty} x^2(t) dt$$

Autocorrelation of a Power Signal

$$R_x(\tau) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)x(t+\tau) dt$$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \leq R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x^2(t) dt$$

Ensemble Average

Random Variable

$$\begin{aligned} m_x &= \mathbf{E}\{X\} \\ &= \int_{-\infty}^{+\infty} x p_X(x) dx \end{aligned}$$

$$\begin{aligned} \mathbf{E}\{X^2\} \\ &= \int_{-\infty}^{+\infty} x^2 p_X(x) dx \end{aligned}$$

Random Process

$$\begin{aligned} m_x(t_k) &= \mathbf{E}\{X(t_k)\} \\ &= \int_{-\infty}^{+\infty} x p_{X_k}(x) dx \end{aligned}$$

$$\begin{aligned} R_x(t_1, t_2) &= \mathbf{E}\{X(t_1) X(t_2)\} \\ &= p_{X_1}(x) p_{X_2}(x) \end{aligned}$$

WSS (Wide Sense Stationary)

Random Process

$$\begin{aligned} m_x(t_k) &= \mathbf{E}\{X(t_k)\} \\ &= \int_{-\infty}^{+\infty} x p_{X_k}(x) dx \end{aligned}$$

$$R_x(t_1, t_2) = \mathbf{E}\{X(t_1) X(t_2)\}$$

WSS Process

 m_x  $R_x(t_1 - t_2)$

Autocorrelation of Random and Power Signals

Autocorrelation of an Random Signal

$$R_x(\tau) = \mathbf{E}\{X(t) X(t + \tau)\}$$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \leq R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \mathbf{E}\{X^2(t)\}$$

Autocorrelation of a Power Signal

$$R_x(\tau) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t) x(t + \tau) dt$$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \leq R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x^2(t) dt$$

Time Averaging and Ergodicity

Random Process

$$\begin{aligned} m_x(t_k) &= \mathbf{E}\{X(t_k)\} \\ &= \int_{-\infty}^{+\infty} x p_{X_k}(x) dx \end{aligned}$$


$$R_x(t_1, t_2) = \mathbf{E}\{X(t_1) X(t_2)\}$$


WSS Process

 m_x

 $R_x(t_1 - t_2)$

Ergodic Process

 $\lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} X(t) dt$

 $\lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} X(t) X(t+\tau) dt$

Autocorrelation of Random and Power Signals

Autocorrelation of a Random Signal

$$R_x(\tau) = \mathbf{E}\{X(t) X(t + \tau)\}$$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \leq R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \mathbf{E}\{X^2(t)\}$$

Power Spectral Density of a Random Signal

$$G_x(f) = \lim_{T \rightarrow +\infty} \frac{1}{T} |X_T(f)|^2$$

$$G_x(f) = G_x(-f)$$

$$G_x(f) \geq 0$$

$$G_x(f) \Leftrightarrow R_x(\tau)$$

$$P_x(0) = \int_{-\infty}^{+\infty} G_x(f) df$$

Time Averaging and Ergodicity

References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”