Euclidean Vector Space (1A)

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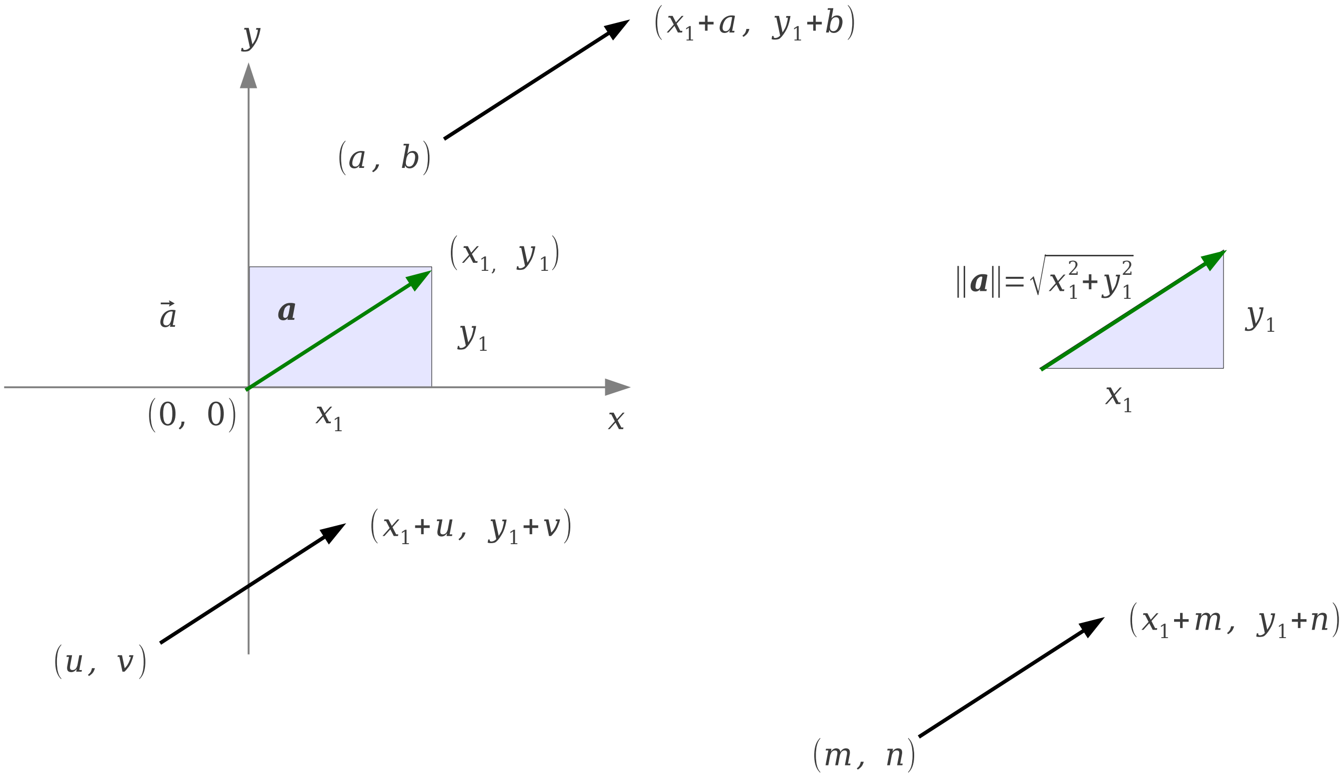
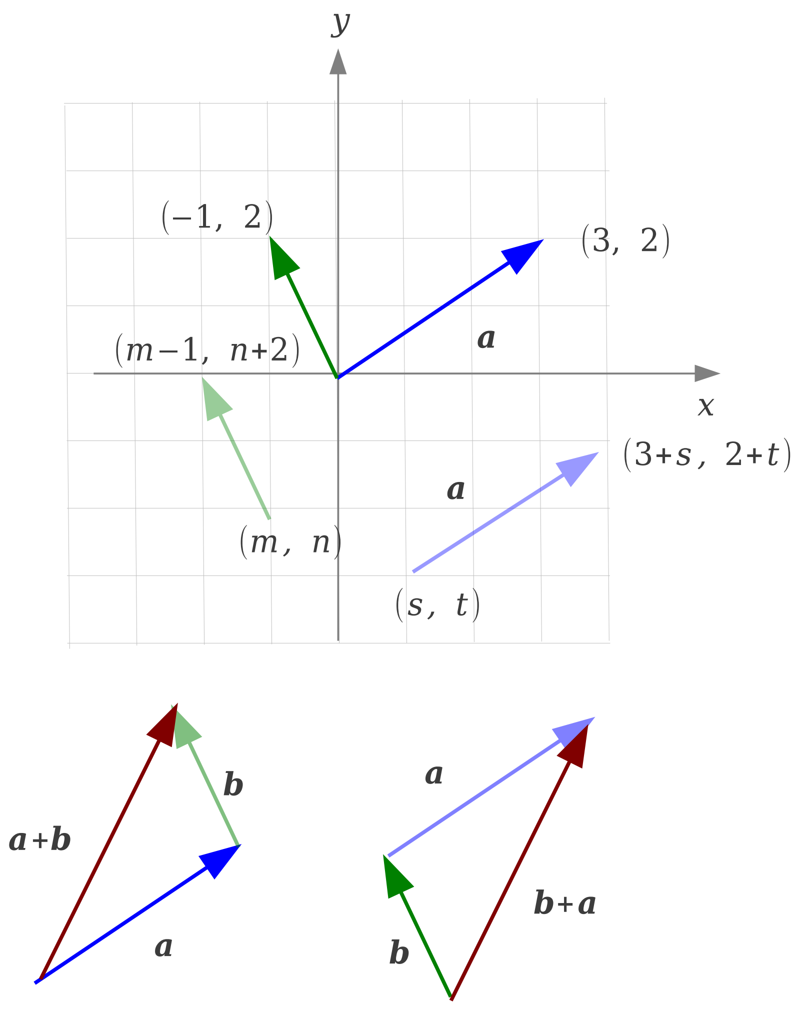
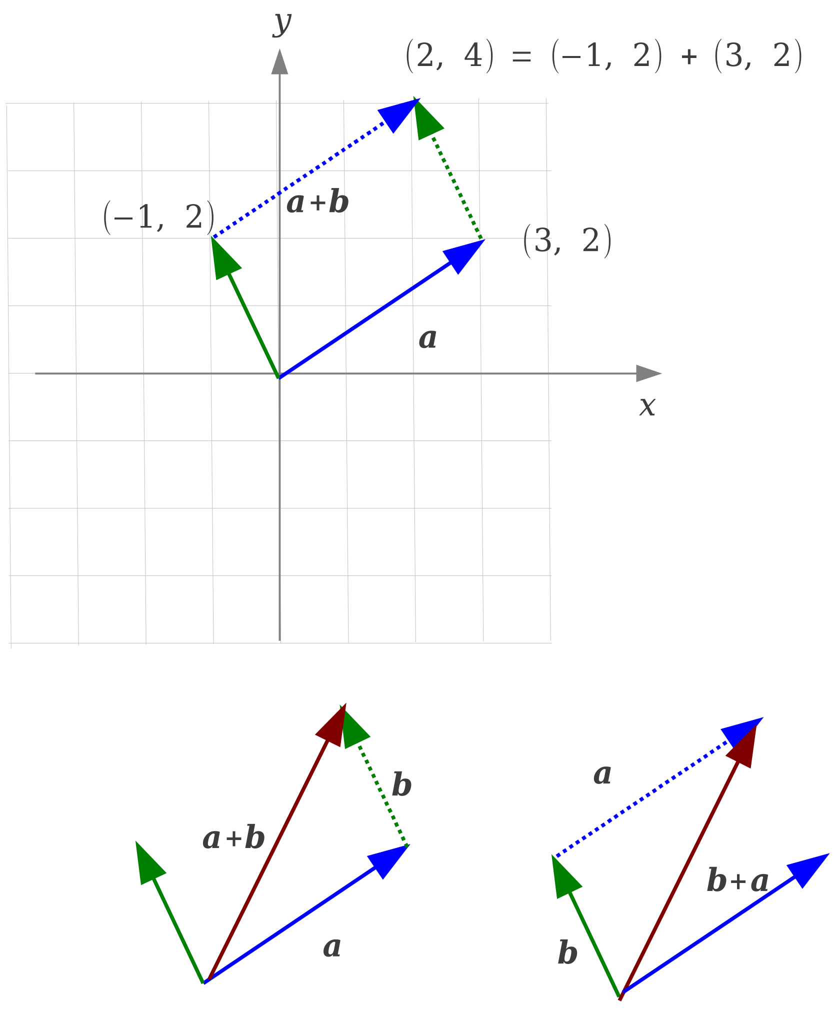
Vectors



$(x_1+a,\ y_1+b)$
y
$(a,\ b)$
$(x_1,\ y_1)$
$\|\boldsymbol{a}\| = \sqrt{x_1^2 + y_1^2}$
$\vec{a}$
$\boldsymbol{a}$
y_1
y_1
$(0,\ 0)$
x_1
x
x_1
$(x_1+u,\ y_1+v)$
$(x_1+m,\ y_1+n)$
$(u,\ v)$
$(m,\ n)$

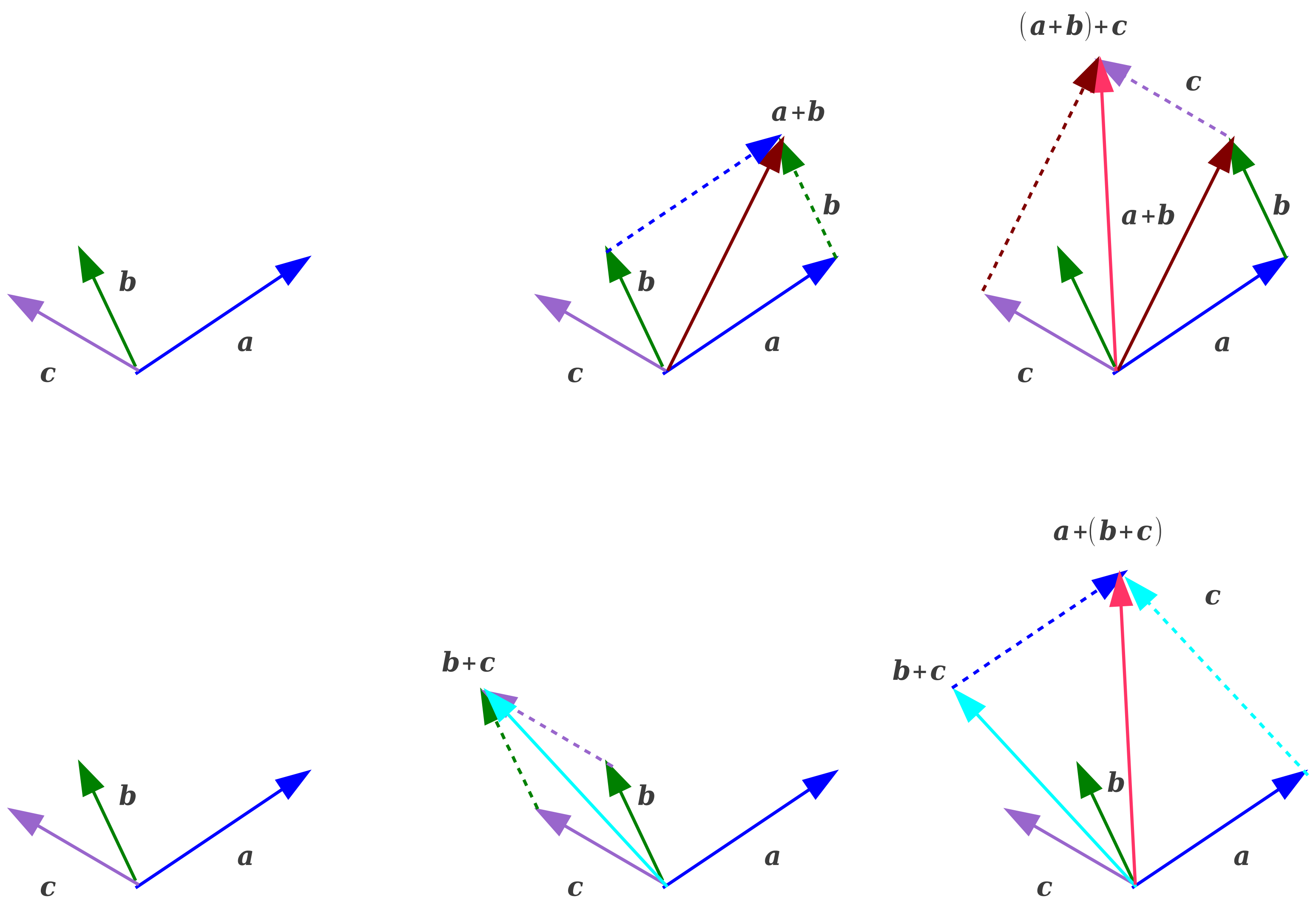
Vector Addition





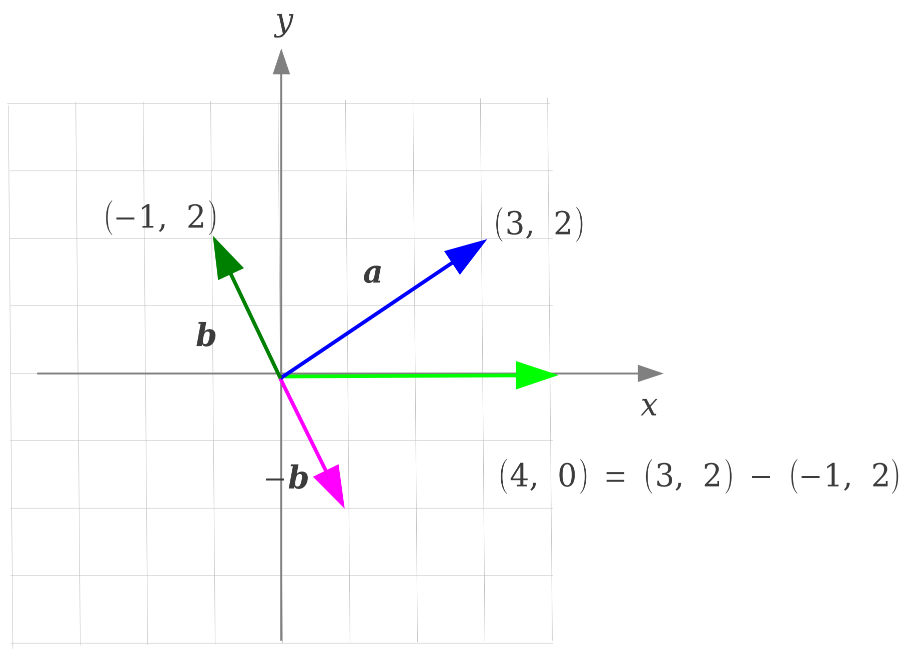
$$\boldsymbol{a}+\boldsymbol{b} = \boldsymbol{b}+\boldsymbol{a}$$

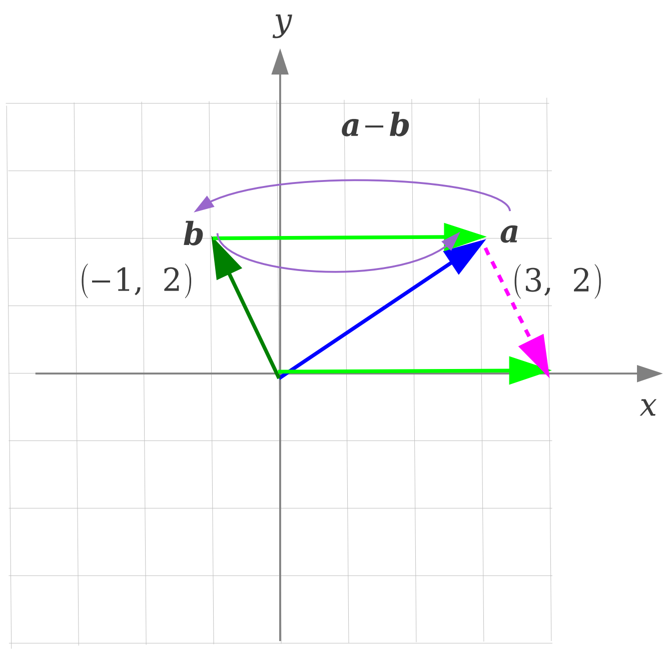
Vector Addition

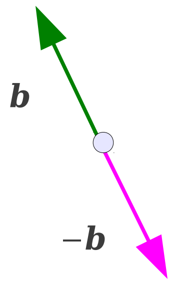


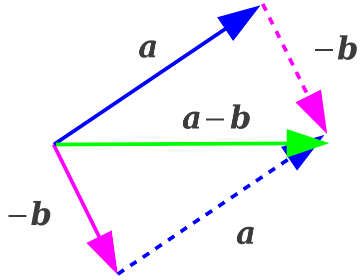
$$(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c} = \boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$$

Vector Subtraction







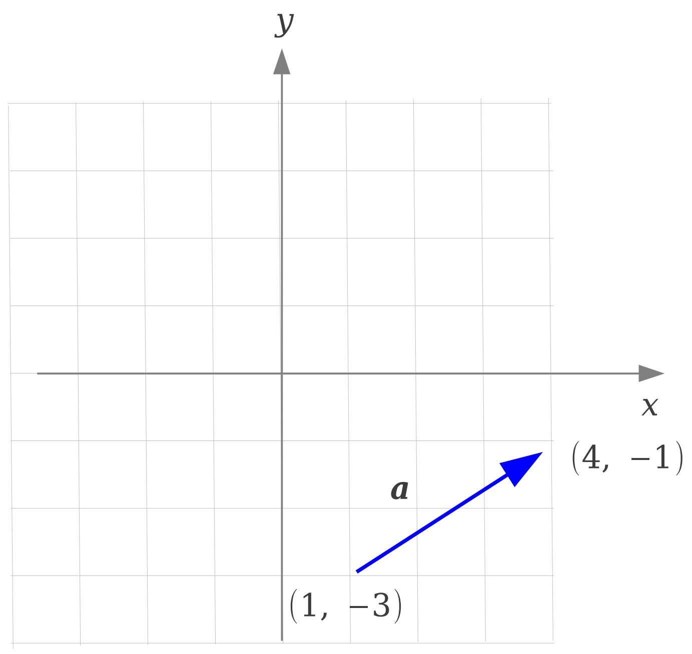


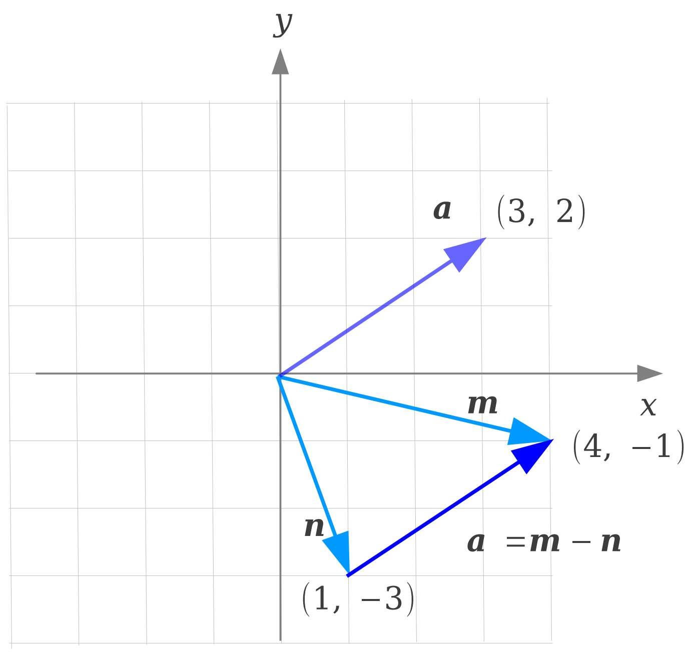
a* − *b

subtract **a** from **b**

arrow from **b** to **a**

Component Form





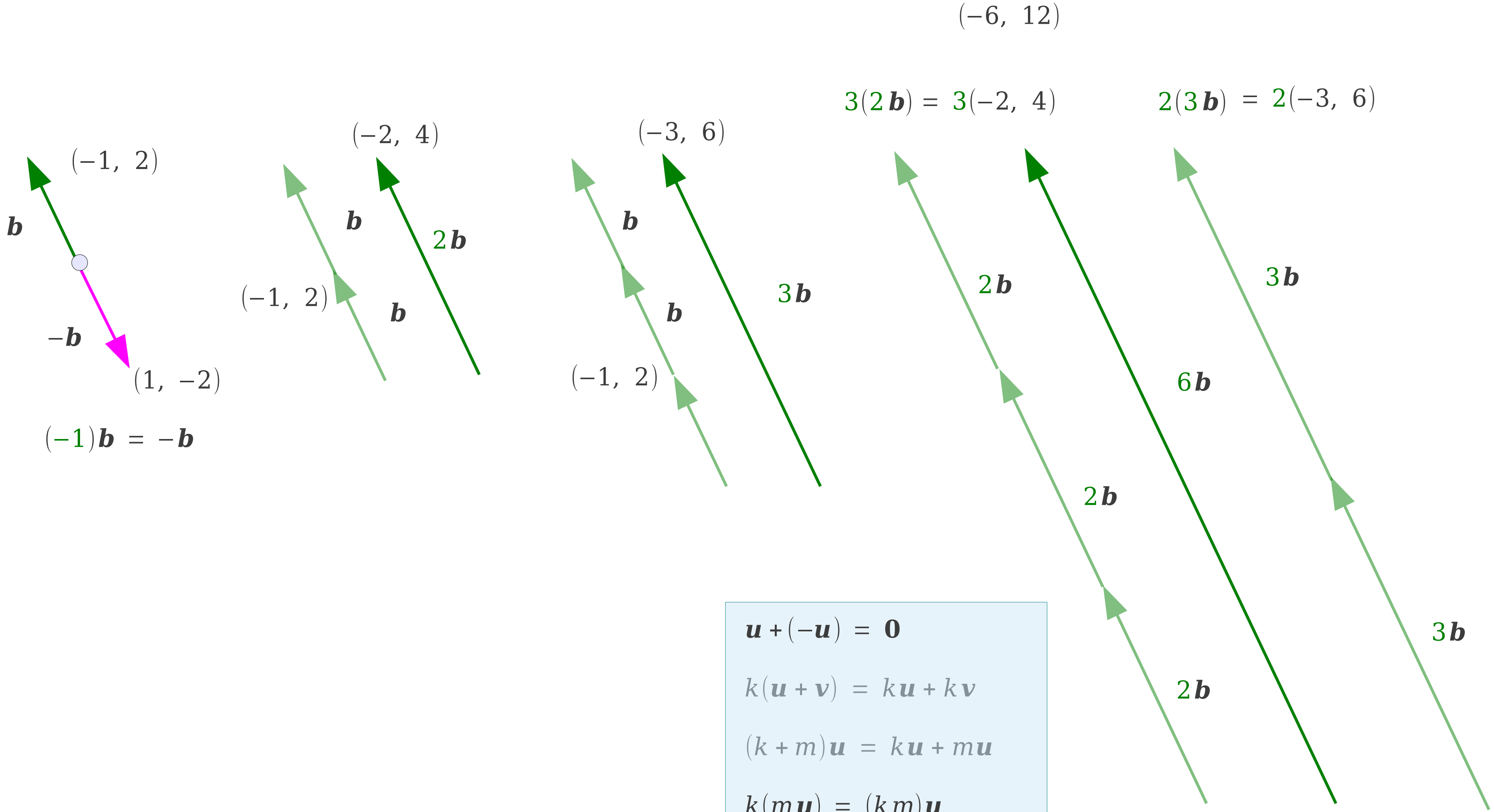
$$\boldsymbol{m} = (4,\ -1)$$

$$\boldsymbol{n} = (1,\ -3)$$

$$\boldsymbol{a} = \boldsymbol{m} - \boldsymbol{n} = (4,\ -1) - (1,\ -3) = (3,\ 2)$$

Finding the **Component Form** of a vector

Scalar Multiplication (1)



$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

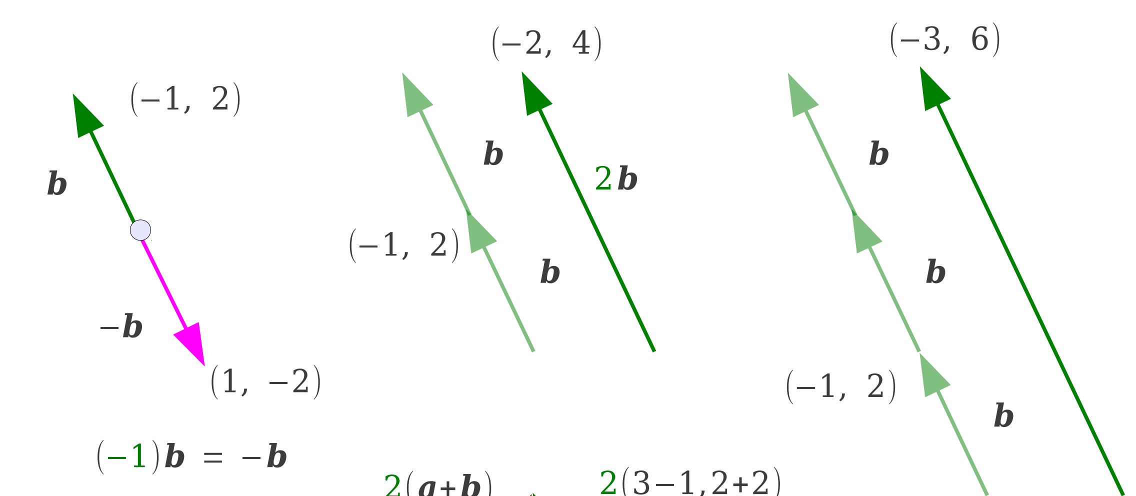
$k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

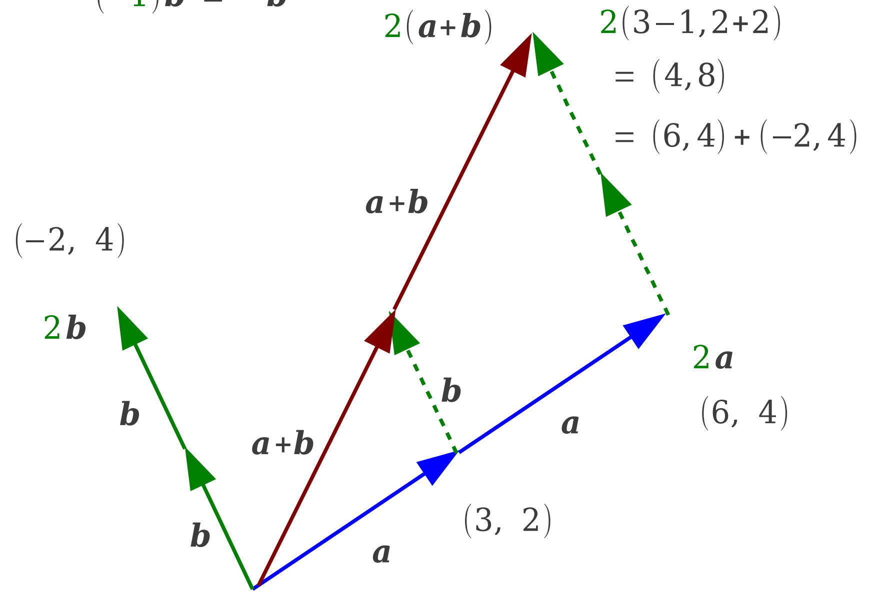
$(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

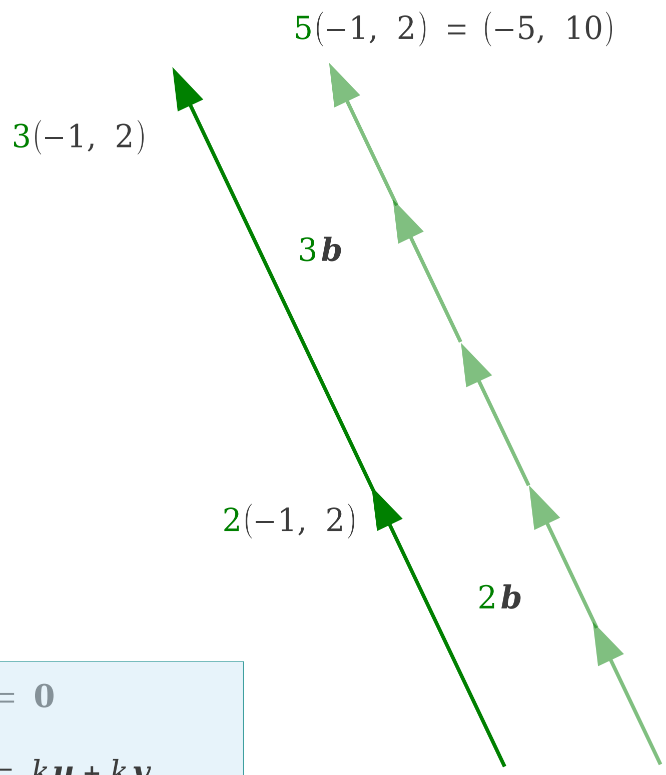
$k(m\mathbf{u}) = (km)\mathbf{u}$

$1\mathbf{u} = \mathbf{u}$

Scalar Multiplication (2)







$$\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$$

$$k(\boldsymbol{u} + \boldsymbol{v}) = k\boldsymbol{u} + k\boldsymbol{v}$$

$$(k + m)\boldsymbol{u} = k\boldsymbol{u} + m\boldsymbol{u}$$

$$k(m\boldsymbol{u}) = (km)\boldsymbol{u}$$

$$1\boldsymbol{u} = \boldsymbol{u}$$

n-Space

Ordered 2-tuples $(v_1,\ v_2)$ $v_1,\ v_2 \in R$

R^2 ⬌ 2-space ⬌ $\{\,all\ ordered\ 2{-}tuples\ (v_1,\ v_2)\,\}$

Ordered 3-tuples $(v_1,\ v_2,\ v_3)$ $v_1,\ v_2,\ v_3 \in R$

R^3 ⬌ 3-space ⬌ $\{\,all\ ordered\ 3{-}tuples\ (v_1,\ v_2,\ v_3)\,\}$

Ordered n-tuples $(v_1,\ v_2,\ \cdots\ ,\ v_n)$ $v_1,\ v_2,\ \cdots\ ,\ v_n \in R$

R^n ⬌ n-space ⬌ $\{\,all\ ordered\ n{-}tuples\ (v_1,\ v_2,\ \cdots\ ,\ v_n)\,\}$

set

Properties of Real Vector Spaces

$\mathbf{u}, \mathbf{v}, \mathbf{w}$ vectors in R^n

k, m scalars

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$

$$k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$$

$$(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$$

$$k(m\mathbf{u}) = (km)\mathbf{u}$$

$$1\mathbf{u} = \mathbf{u}$$

Linear Combination

$\boldsymbol{w}, \quad \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r$ vectors in R^n

$k_1, k_2, \cdots, k_r$ scalars

$\boldsymbol{w}$ is a linear combination of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r$

$$\boldsymbol{w} = k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= m_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + m_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= n_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + n_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + n_3\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Vector Magnitude

- Norm
- Length
- Magnitude

$\|\boldsymbol{v}\|$

R^2 $\quad \boldsymbol{v} = (v_1, v_2)$ $\quad \|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2} \geq 0$

R^3 $\quad \boldsymbol{v} = (v_1, v_2, v_3)$ $\quad \|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \geq 0$

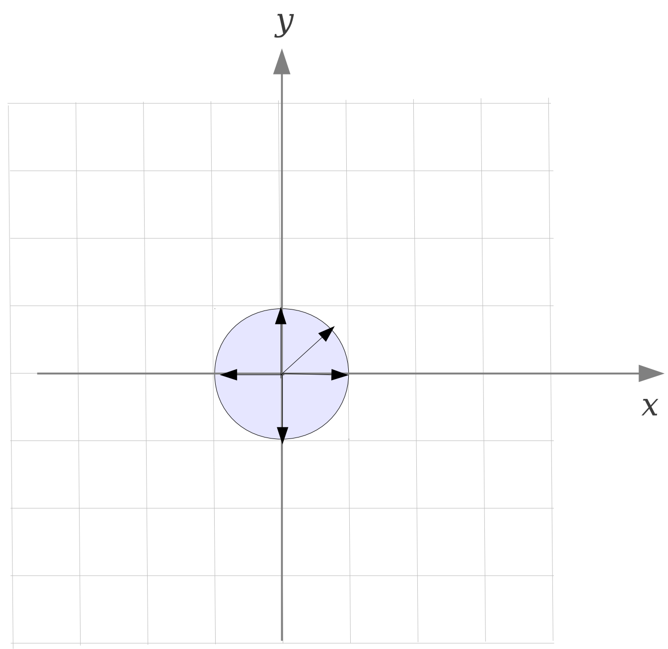
R^n $\quad \boldsymbol{v} = (v_1, v_2, \cdots, v_n)$ $\quad \|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \geq 0$

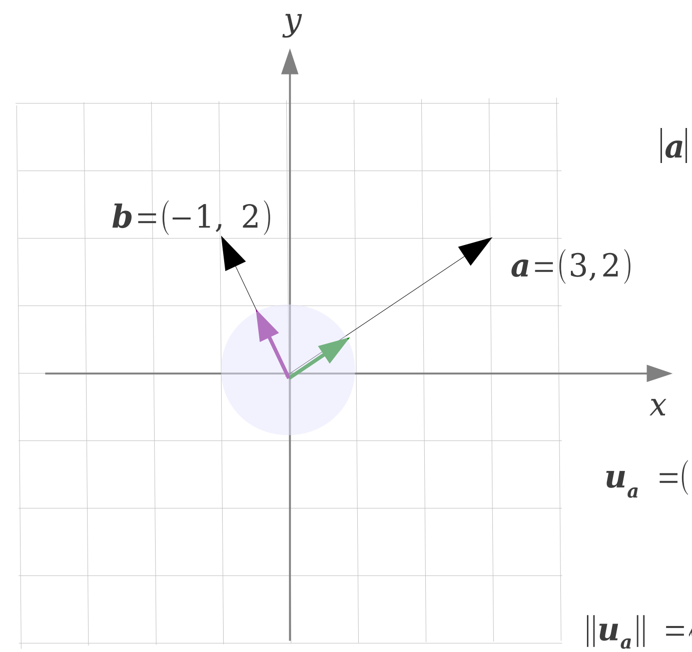
$$\|\boldsymbol{v}\| \geq 0$$

$$\|\boldsymbol{v}\| = 0 \qquad \boldsymbol{v} = \boldsymbol{0}$$

$$\|k\,\boldsymbol{v}\| = |k|\,\|\boldsymbol{v}\| \geq 0$$

Unit Vector





$$|\boldsymbol{a}| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$\boldsymbol{u}_a = \left(\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right)$$

$$\|\boldsymbol{u}_a\| = \sqrt{\frac{3^2}{13} + \frac{2^2}{13}} = \sqrt{\frac{13}{13}} = 1$$

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$$

$$\|\boldsymbol{u}\| = \left\|\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\right\|$$

$$k = \|\boldsymbol{v}\| \geq 0$$

$$= \left\|\frac{\boldsymbol{v}}{k}\right\| = \frac{\|\boldsymbol{v}\|}{k}$$

$$= \frac{\|\boldsymbol{v}\|}{\|\boldsymbol{v}\|} = 1$$

Standard Unit Vectors

$R^2 \Leftrightarrow$ 2-space $\Leftrightarrow \{ \textit{all ordered } 2\textit{-tuples } (v_1, v_2) \}$

$\mathbf{i} = (1, 0)$

$\mathbf{j} = (0, 1)$

$\mathbf{v} = (v_1, v_2) = v_1(1, 0) + v_2(0, 1)$

$R^3 \Leftrightarrow$ 3-space $\Leftrightarrow \{ \textit{all ordered } 3\textit{-tuples } (v_1, v_2, v_3) \}$

$\mathbf{i} = (1, 0, 0)$

$\mathbf{j} = (0, 1, 0)$

$\mathbf{k} = (0, 0, 1)$

$\mathbf{v} = (v_1, v_2, v_3) = v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1)$

$R^n \Leftrightarrow$ n-space $\Leftrightarrow \{ \textit{all ordered } n\textit{-tuples } (v_1, v_2, \cdots, v_n) \}$

$\mathbf{e_1} = (1, 0, \cdots, 0)$

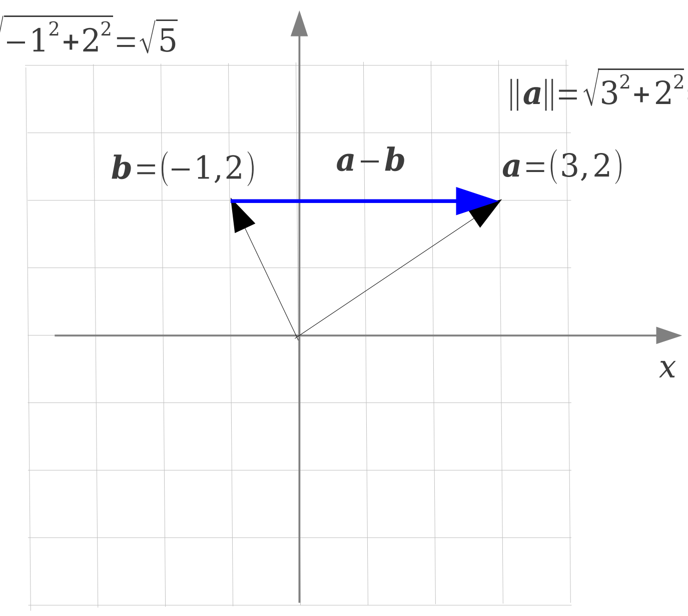
$\mathbf{e_2} = (0, 1, \cdots, 0)$

$\cdots$

$\mathbf{e_n} = (0, 0, \cdots, 1)$

$\mathbf{v} = (v_1, v_2, \cdots, v_n) = v_1(1, 0, \cdots, 0) + v_2(0, 1, \cdots, 0) + \cdots + v_n(0, 0, \cdots, 1)$

Distance between Vectors



R^2

$$d(\boldsymbol{a}, \boldsymbol{b}) = \|\boldsymbol{a} - \boldsymbol{b}\|$$

$$= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \geq 0$$

R^3

$$d(\boldsymbol{a}, \boldsymbol{b}) = \|\boldsymbol{a} - \boldsymbol{b}\|$$

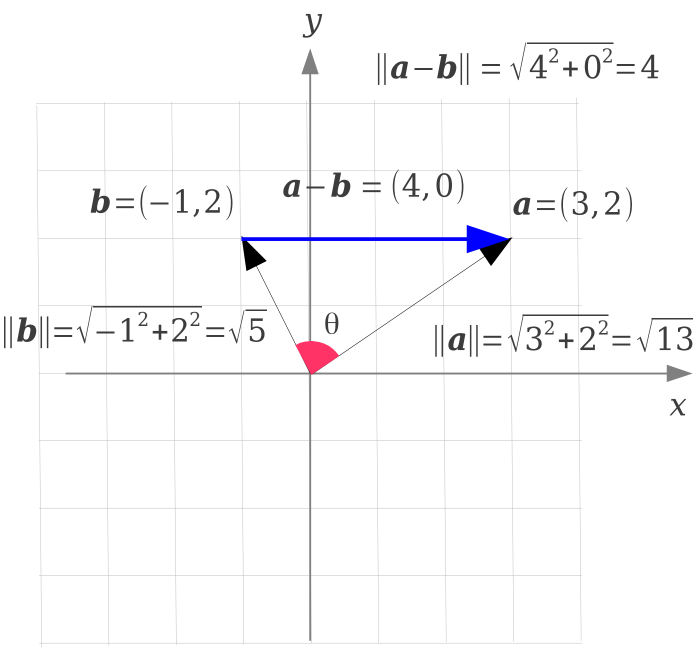
$$= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2} \geq 0$$

R^n

$$d(\boldsymbol{a}, \boldsymbol{b}) = \|\boldsymbol{a} - \boldsymbol{b}\|$$

$$= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2} \geq 0$$

Law of Cosine



$4^2=\sqrt{13}^2+\sqrt{5}^2-2\sqrt{13}\sqrt{5}\cos\theta$

$2=2\sqrt{13}\sqrt{5}\cos\theta \qquad \cos\theta=\frac{1}{\sqrt{65}}$

$\boldsymbol{a}\cdot\boldsymbol{b}=3\cdot-1+2\cdot2=-3+4=1$

$R^2 \quad d(\boldsymbol{a},\boldsymbol{b}) = \|\boldsymbol{a}-\boldsymbol{b}\| = \sqrt{(a_1-b_1)^2+(a_2-b_2)^2} \geq 0$

Law of Cosine

$$\|\boldsymbol{a}-\boldsymbol{b}\|^2 = \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 - 2\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta$$

$\boldsymbol{a}-\boldsymbol{b}$

$= (a_1-b_1,\ a_2-b_2)$

$\boldsymbol{a} = (a_1,\ a_2)$

$\boldsymbol{b} = (b_1,\ b_2)$

$(a_1-b_1)^2 + (a_2-b_2)^2$

$a_1^2+a_2^2$

$b_1^2+b_2^2$

$-2(a_1b_1+a_2b_2)$

$a_1^2+a_2^2 \qquad \sqrt{a_1^2+a_2^2}$

$b_1^2+b_2^2 \qquad \sqrt{b_1^2+b_2^2}$

$-2 \qquad \cos\theta$

Inner Product

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

$$\boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta$$

Dot Product

Dot Product
Euclidean Inner Product

$\boldsymbol{a}, \boldsymbol{b}$ Vectors in R^n

θ Angle between $\boldsymbol{a}, \boldsymbol{b}$

$$\boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta$$

$\boldsymbol{b} \leftarrow \boldsymbol{a}$

$$\boldsymbol{a}\cdot\boldsymbol{a} = \|\boldsymbol{a}\|\,\|\boldsymbol{a}\|\cos 0°$$

$$\boldsymbol{a}\cdot\boldsymbol{a} = \|\boldsymbol{a}\|^2$$

$$\|\boldsymbol{a}\| = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$$

R^2

$$\boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta$$
$$= a_1 b_1 + a_2 b_2$$

R^3

$$\boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta$$
$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

R^n

$$\boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta$$
$$= a_1 b_1 + a_2 b_2 + \cdots\ a_n b_n$$

Properties of Dot Products

$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$

$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$

$k(\mathbf{a}\cdot\mathbf{b}) = (k\,\mathbf{a})\cdot\mathbf{b}$

$\mathbf{a}\cdot\mathbf{a} \geq 0 \qquad \mathbf{a}\cdot\mathbf{a} = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$

$\mathbf{0}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{0} = 0$

$(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c}$

$\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{c}$

$(\mathbf{a}-\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{c}$

$k(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(k\,\mathbf{v})$

$R^n \qquad \mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta$

$= a_1 b_1 + a_2 b_2 + \cdots\ a_n b_n$

Cauchy-Schwarz Inequality (1)

$$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|}$$

$$R^n \qquad \boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta$$

$$= a_1 b_1 + a_2 b_2 + \cdots a_n b_n$$

$$-1 \le \cos\theta \le +1$$

$$-1 \le \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|} \le +1$$

$$\frac{|\boldsymbol{a}\cdot\boldsymbol{b}|}{\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|} \le 1$$

$$\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$$

$$\boldsymbol{b} = (b_1, b_2, \cdots, b_n)$$

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$\|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

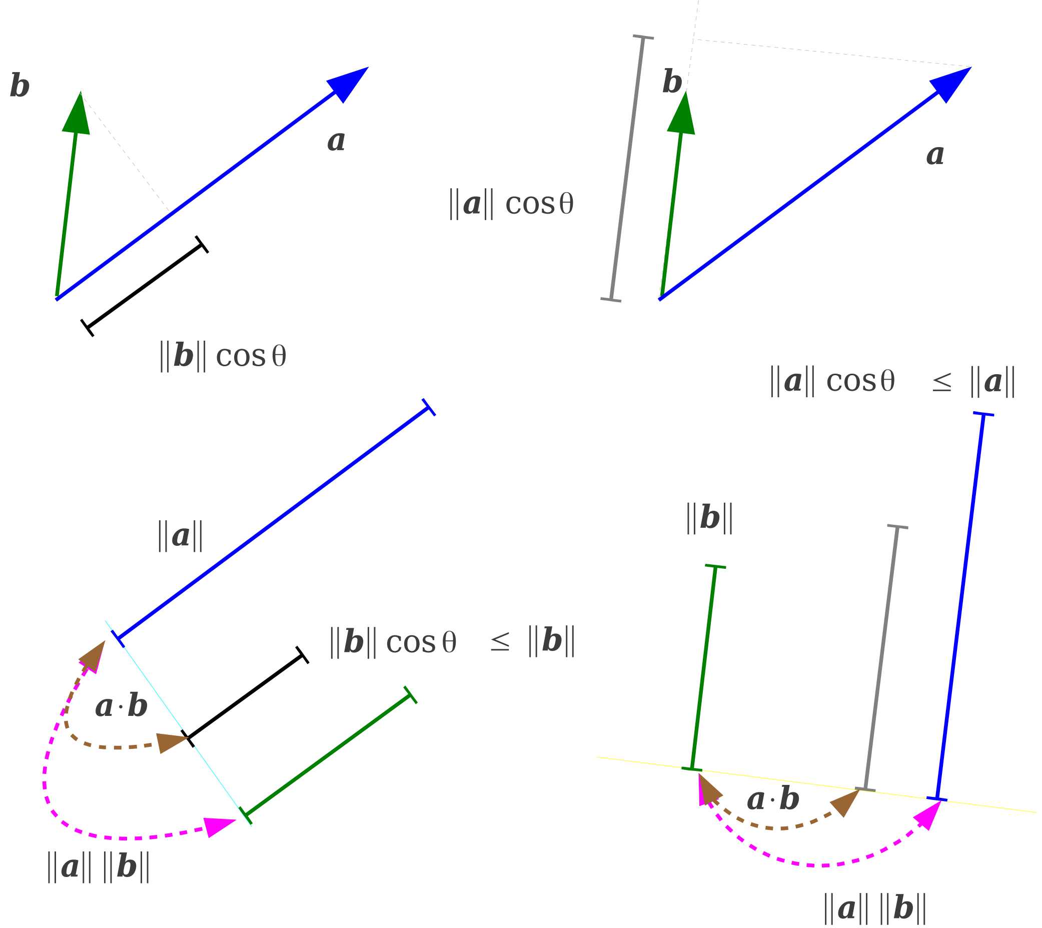
$$\|\boldsymbol{b}\| = \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

$$|\boldsymbol{a}\cdot\boldsymbol{b}| \le \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|$$

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \le \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

Cauchy-Schwarz Inequality (2)

Assume $0 \le \theta \le 90^\circ$



R^n $\quad \boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta$

$$= a_1 b_1 + a_2 b_2 + \cdots a_n b_n$$

$$\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$$

$$\boldsymbol{b} = (b_1, b_2, \cdots, b_n)$$

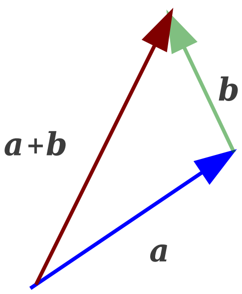
$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$\|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

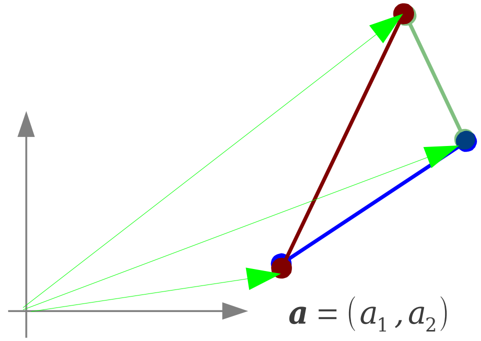
$$\|\boldsymbol{b}\| = \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

$$|\boldsymbol{a}\cdot\boldsymbol{b}| \le \|\boldsymbol{a}\|\,\|\boldsymbol{b}\| \quad \textit{for } 0 \le \theta \le 180^\circ$$

Triangular Inequalities



$c = (c_1, c_2)$



$b = (b_1, b_2)$

$a = (a_1, a_2)$

$$\|a+b\| \leq \|a\|\,\|b\|$$

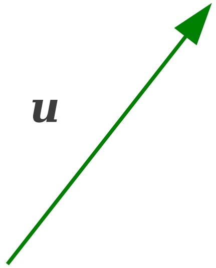
$$\|a+b\|^2 = (a+b)\cdot(a+b)$$
$$= (a\cdot a) + 2(a\cdot b) + (b\cdot b)$$
$$\leq \|a\|^2 + 2|a\cdot b| + \|b\|^2$$
$$\leq \|a\|^2 + 2\|a\|\,\|b\| + \|b\|^2$$
$$\leq (\|a\| + \|b\|)^2$$

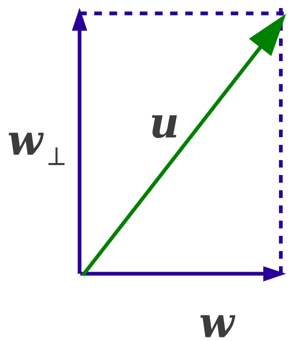
$$a\cdot a = \|a\|^2$$
$$b\cdot b = \|b\|^2$$
$$a\cdot b \leq |a\cdot b|$$
$$|a\cdot b| \leq \|a\|\,\|b\|$$

$$d(a,b) \leq d(a,c) + d(c,b)$$

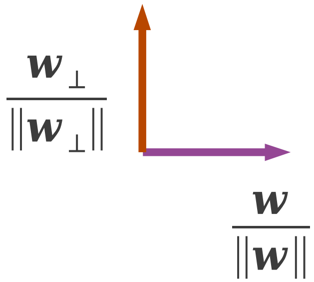
$$d(a,b) = \|a-b\|$$
$$= \|a-c+c-b\|$$
$$= \|(a-c)+(c-b)\|$$
$$\leq \|a-c\| + \|c-b\|$$
$$\leq d(a,c) + d(c,b)$$

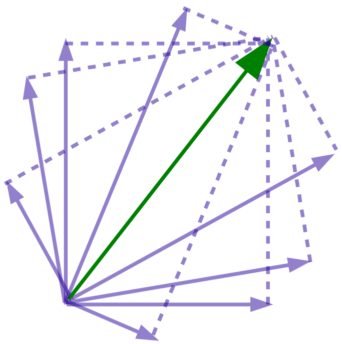
Vector Decomposition (1)





unit vectors

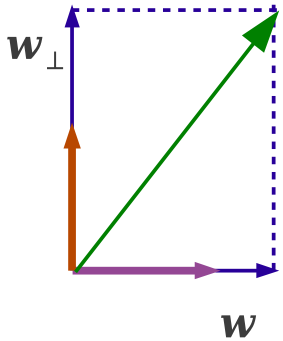


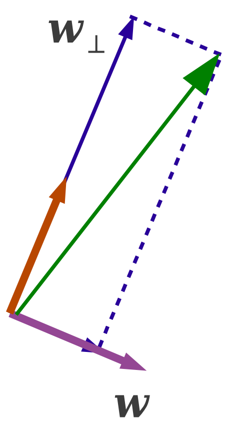


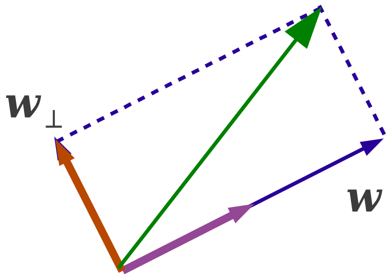
$$\mathbf{u} = \mathbf{w} + \mathbf{w}_{\perp}$$

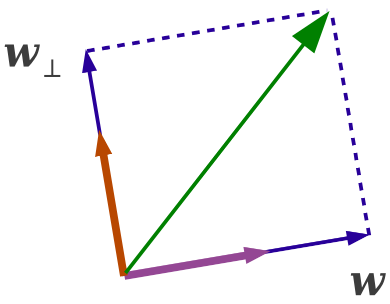
$$\mathbf{u} - \mathbf{w} = \mathbf{w}_{\perp}$$

Various Decompositions

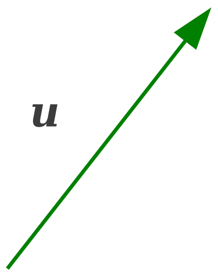


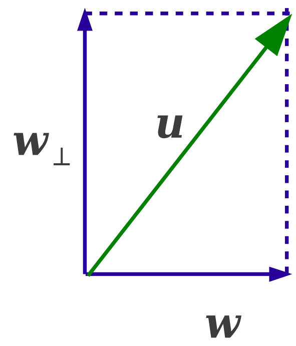






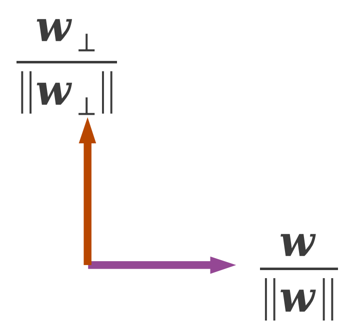
Vector Decomposition (2)

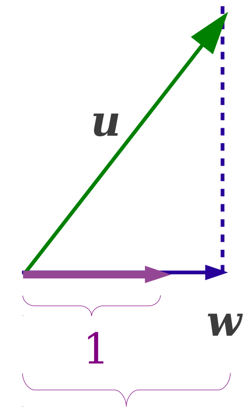




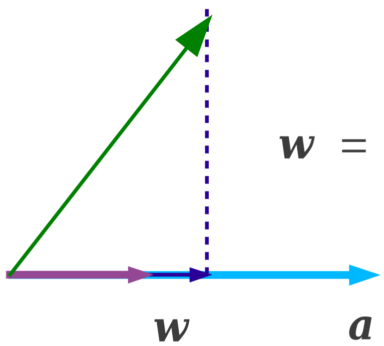
$$\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{w}_\perp$$

$$\boldsymbol{u} - \boldsymbol{w} = \boldsymbol{w}_\perp$$



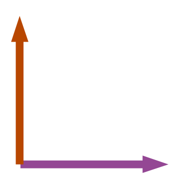


$$\boldsymbol{u} \cdot \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \|\boldsymbol{u}\| \frac{\|\boldsymbol{w}\|}{\|\boldsymbol{w}\|} \cos\theta = \|\boldsymbol{u}\| \cos\theta$$

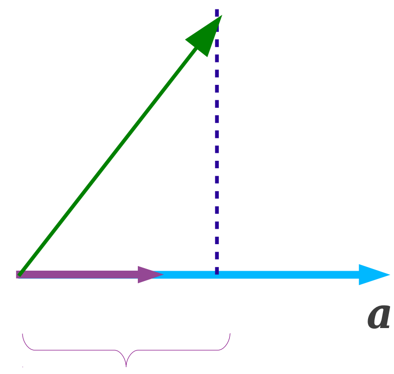


$$\boldsymbol{w} = k\,\boldsymbol{a}$$

unit vectors

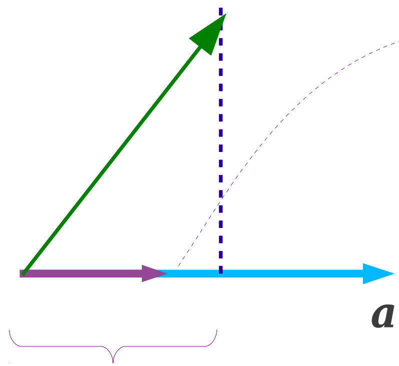


$$\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$$



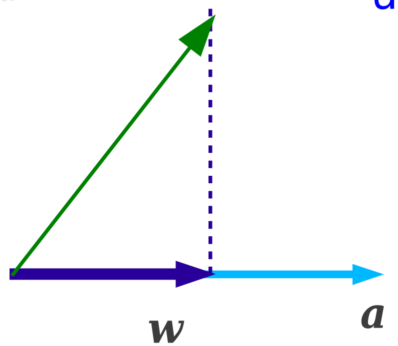
$$\boldsymbol{u} \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \|\boldsymbol{u}\| \frac{\|\boldsymbol{a}\|}{\|\boldsymbol{a}\|} \cos\theta = \|\boldsymbol{u}\| \cos\theta$$

Projection



$$\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$$

unit vectors

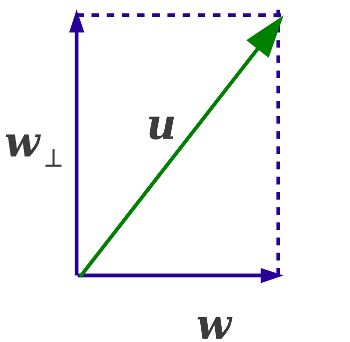


$$\boldsymbol{u} \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \|\boldsymbol{u}\| \cos\theta$$

$$\left(\boldsymbol{u} \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}\right) \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \left(\frac{\boldsymbol{u} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|}\right) \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \frac{\boldsymbol{u} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2} \boldsymbol{a}$$

scale factor

another scale factor w.r.t. $\boldsymbol{a}$



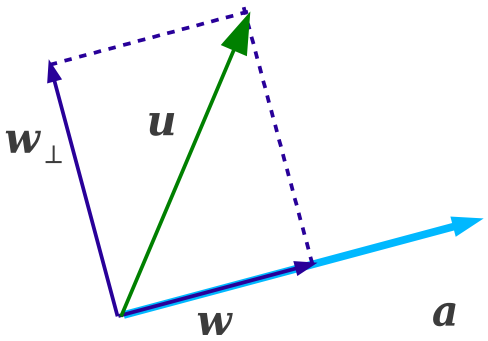
$$\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{w}_{\perp}$$

$$\boldsymbol{u} - \boldsymbol{w} = \boldsymbol{w}_{\perp}$$

$$\boldsymbol{w} = k\,\boldsymbol{a} = \frac{\boldsymbol{u} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2} \boldsymbol{a}$$

$$\boldsymbol{w}_{\perp} = \boldsymbol{u} - \boldsymbol{w} = \boldsymbol{u} - \frac{\boldsymbol{u} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2} \boldsymbol{a}$$

Projection (2)



vector component of **u** along **a**

$$\boldsymbol{w} = k\,\boldsymbol{a}$$

$$proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u}\cdot\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\,\boldsymbol{a}$$

vector component of **u** orthogonal **a**

$$\boldsymbol{w}_{\perp} = \boldsymbol{u} - \boldsymbol{w}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = \boldsymbol{u} - \frac{\boldsymbol{u}\cdot\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\,\boldsymbol{a}$$

$$\|proj_{\boldsymbol{a}}\boldsymbol{u}\| = \frac{|\boldsymbol{u}\cdot\boldsymbol{a}|}{\|\boldsymbol{a}\|} = \|\boldsymbol{u}\|\cos\theta$$

Orthogonality

$$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|\ \|\boldsymbol{b}\|}$$

$$-1 \le \cos\theta \le +1$$

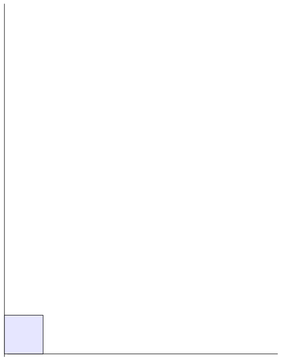
$$-1 \le \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|\ \|\boldsymbol{b}\|} \le +1$$

$$\frac{|\boldsymbol{a}\cdot\boldsymbol{b}|}{\|\boldsymbol{a}\|\ \|\boldsymbol{b}\|} \le 1$$

$$\cos 90^\circ = 0$$

$$\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|\ \|\boldsymbol{b}\|} = 0$$

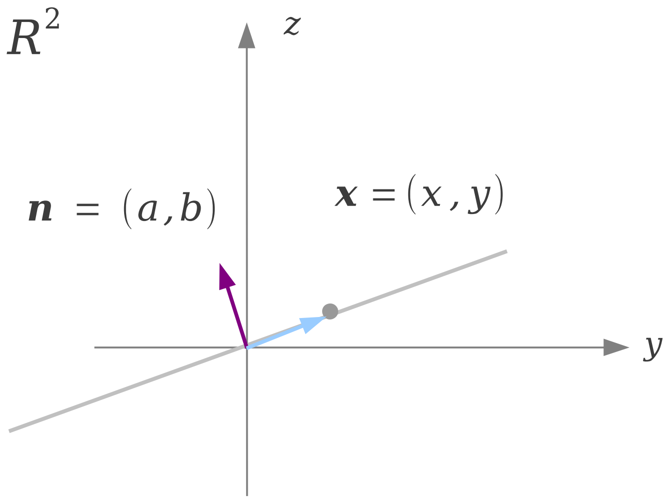
$$\boldsymbol{a}\cdot\boldsymbol{b} = 0$$

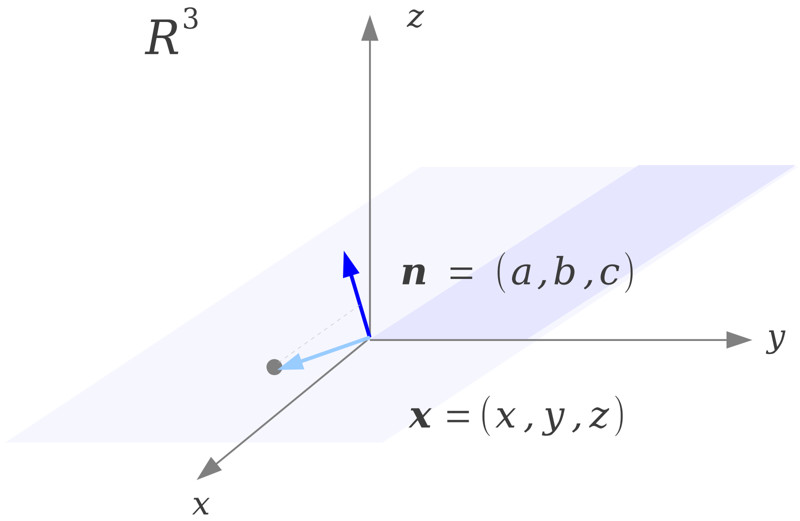


R^n $\quad \boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\ \|\boldsymbol{b}\|\cos\theta$

$$= a_1 b_1 + a_2 b_2 + \cdots a_n b_n$$

Point-Normal Equation (1)





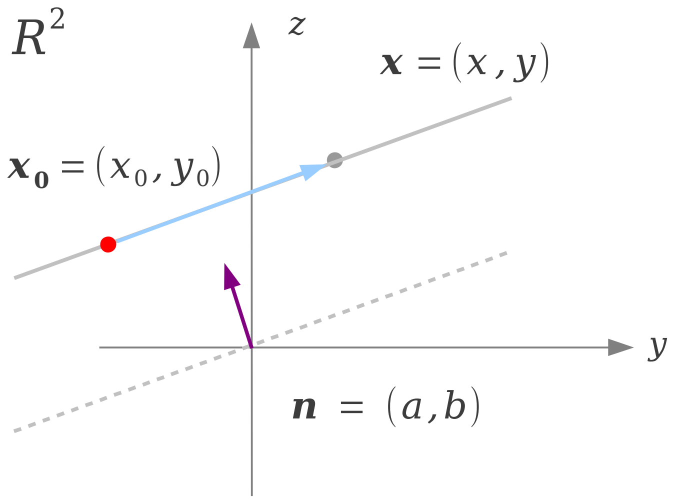
$$\boldsymbol{n} \cdot \boldsymbol{x} = 0$$

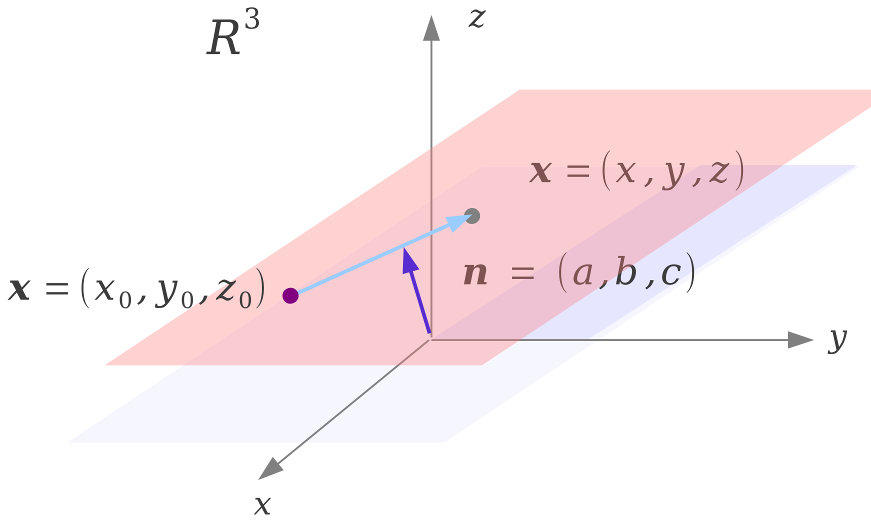
$$a x + b y = 0$$

$$\boldsymbol{n} \cdot \boldsymbol{x} = 0$$

$$a x + b y + c z = 0$$

Point-Normal Equation (2)





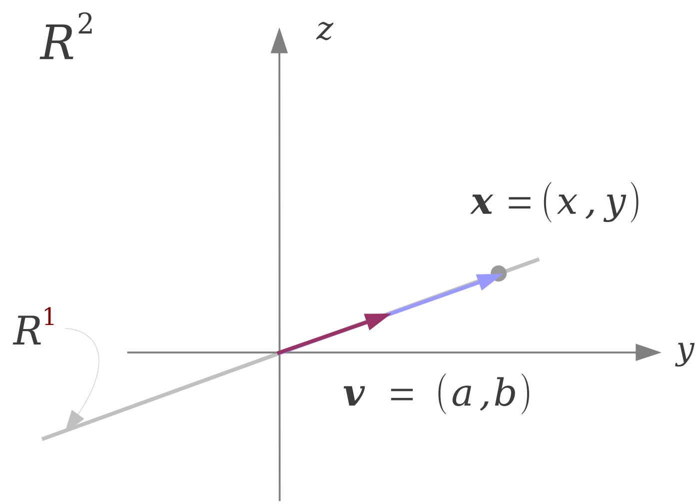
$$\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{x_0}) = 0$$

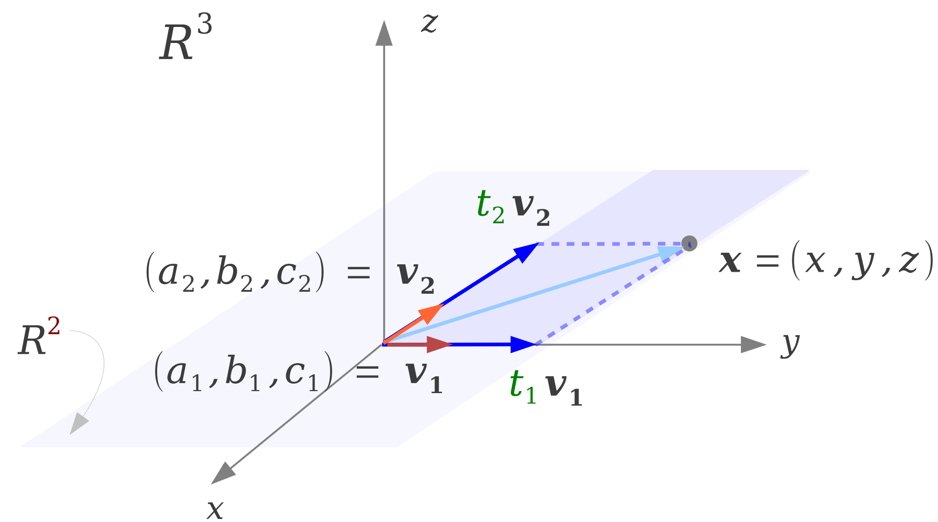
$$a(x - x_0) + b(y - y_0) = 0$$

$$\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{x_0}) = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Parametric Equation (1)





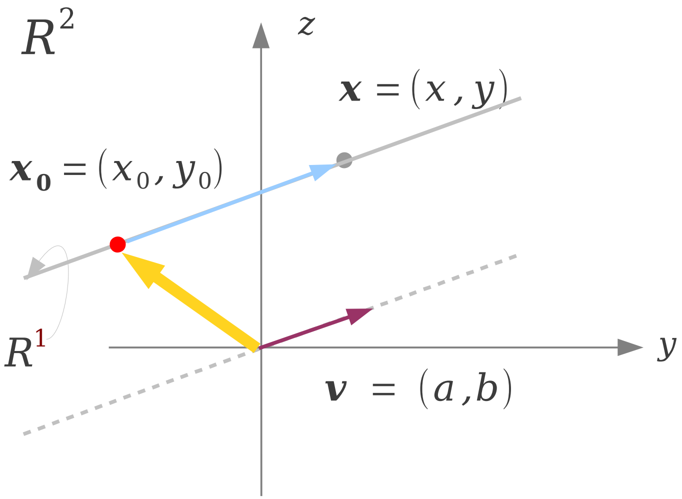
$$\mathbf{x} = t\,\mathbf{v}$$

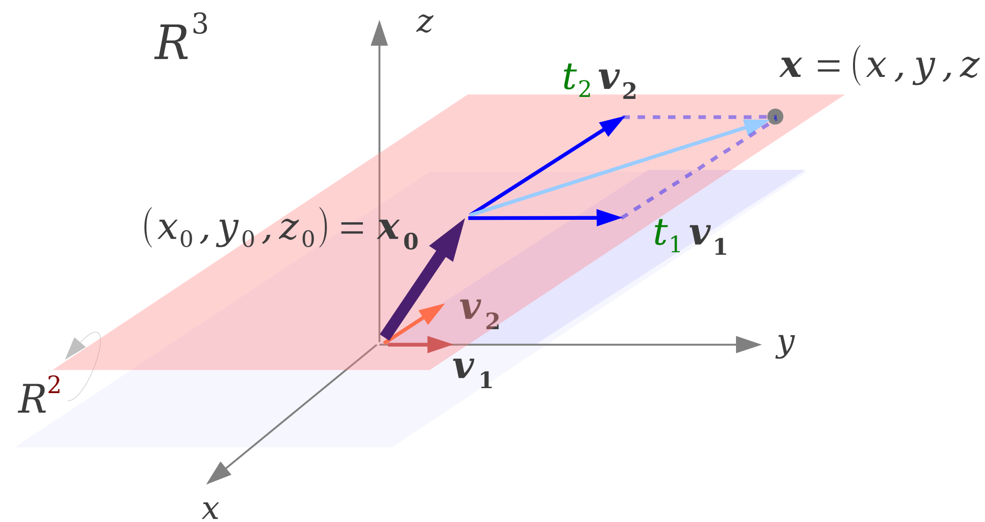
$$x = t\,a$$
$$y = t\,b$$

$$\mathbf{x} = t_1 \mathbf{v_1} + t_2 \mathbf{v_2}$$

$$x = t_1 a_1 + t_2 a_2$$
$$y = t_1 b_1 + t_2 b_2$$
$$z = t_1 c_1 + t_2 c_2$$

Parametric Equation (2)





$$\mathbf{x} = \mathbf{x_0} + t\,\mathbf{v}$$

$$x = x_0 + t\,a$$
$$y = y_0 + t\,b$$

$$\mathbf{x} = \mathbf{x_0} + t_1\mathbf{v_1} + t_2\mathbf{v_2}$$

$$x = x_0 + t_1 a_1 + t_2 a_2$$
$$y = y_0 + t_1 b_1 + t_2 b_2$$
$$z = z_0 + t_1 c_1 + t_2 c_2$$

Parametric Equation (3)

Line Equation in R^n

$\mathbf{x_0}$, $\mathbf{v}$ vectors in R^n

$$\mathbf{x} = \mathbf{x_0} + t\,\mathbf{v}$$

$\mathbf{x_0}$, $\mathbf{x_1}$ vectors in R^n

$$\mathbf{v} = (\mathbf{x_1} - \mathbf{x_0})$$

$$\mathbf{x} = \mathbf{x_0} + t(\mathbf{x_1} - \mathbf{x_0})$$

$$\mathbf{x} = (1 - t)\mathbf{x_0} + t\,\mathbf{x_1}$$

Plane Equation in R^n

$\mathbf{x_0}$, $\mathbf{v_1}$, $\mathbf{v_2}$ vectors in R^n

$$\mathbf{x} = \mathbf{x_0} + t_1\mathbf{v_1} + t_2\mathbf{v_2}$$

Linear System (1)

Linear Equations

Corresponding Homogeneous Equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

$$\boldsymbol{a} = (a_1 , a_2 , \cdots , a_n)$$

$$\boldsymbol{x} = (x_1 , x_2 , \cdots , x_n)$$

normal vector

$$\boldsymbol{a} \cdot \boldsymbol{x} = b$$

$$\boldsymbol{a} \cdot \boldsymbol{x} = 0$$

each **solution** vector $\boldsymbol{x}$ of a **homogeneous** equation
orthogonal to the coefficient vector $\boldsymbol{a}$

Homogeneous Linear System

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= 0 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= 0 \end{aligned}$$

$$\begin{aligned} \boldsymbol{r_1} \cdot \boldsymbol{x} &= 0 \\ \boldsymbol{r_2} \cdot \boldsymbol{x} &= 0 \\ &\cdots \\ \boldsymbol{r_m} \cdot \boldsymbol{x} &= 0 \end{aligned}$$

Linear System (2)

Homogeneous Linear System

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
\qquad
\begin{aligned}
\mathbf{r_1}\cdot\mathbf{x} &= 0 \\
\mathbf{r_2}\cdot\mathbf{x} &= 0 \\
\cdots & \\
\mathbf{r_m}\cdot\mathbf{x} &= 0
\end{aligned}
$$

each **solution** vector $\mathbf{x}$ of a **homogeneous** equation
orthogonal to the row vector $\mathbf{r_i}$ of the coefficient matrix

Homogeneous Linear System $\mathbf{A}\cdot\mathbf{x} = 0$ $\quad \mathbf{A} : m\times n$

solution set consists of all vectors in R^n
that are **orthogonal** to every row vector of $\mathbf{A}$

Linear System (3)

Non-Homogeneous Linear System $\quad \mathbf{A}\cdot\mathbf{x} = b \quad \mathbf{A} : m\times n$

Homogeneous Linear System $\quad \mathbf{A}\cdot\mathbf{x} = 0$

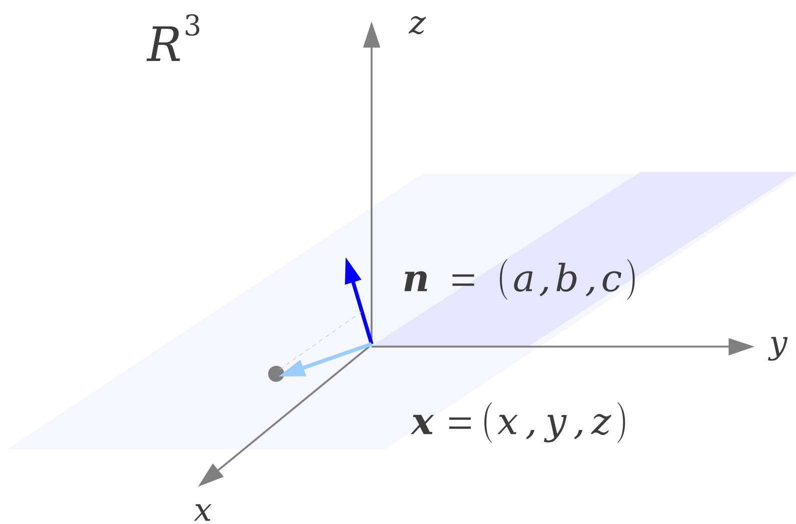
a particular solution

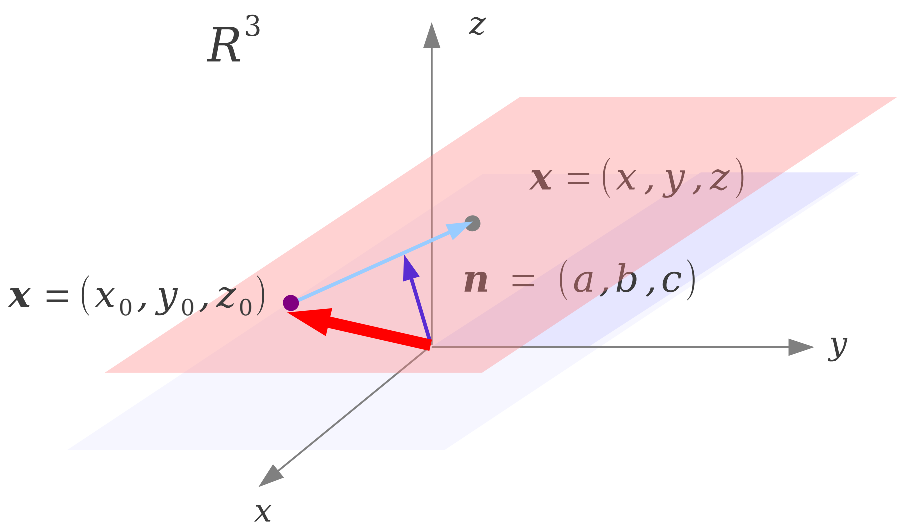
$$\mathbf{A}\cdot\mathbf{x} = b$$

solution set consists of all vectors in R^n that are **orthogonal** to every row vector of $\mathbf{A}$

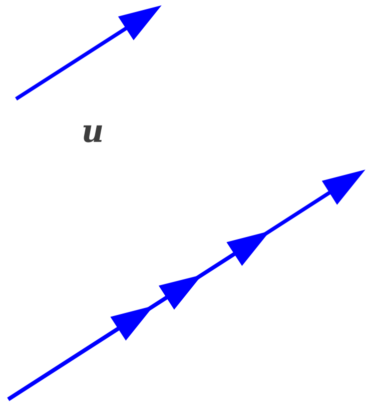
+

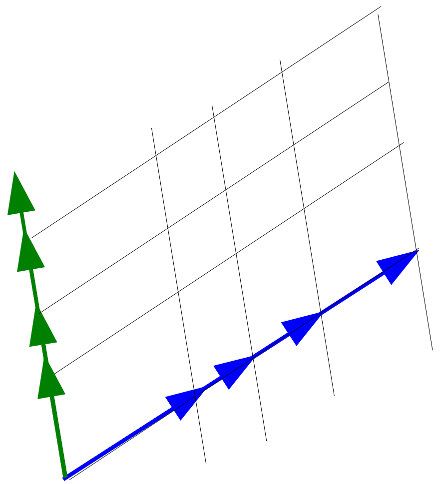
a particular solution $\mathbf{x_0}$ $\quad \mathbf{A}\cdot\mathbf{x_0} = b$



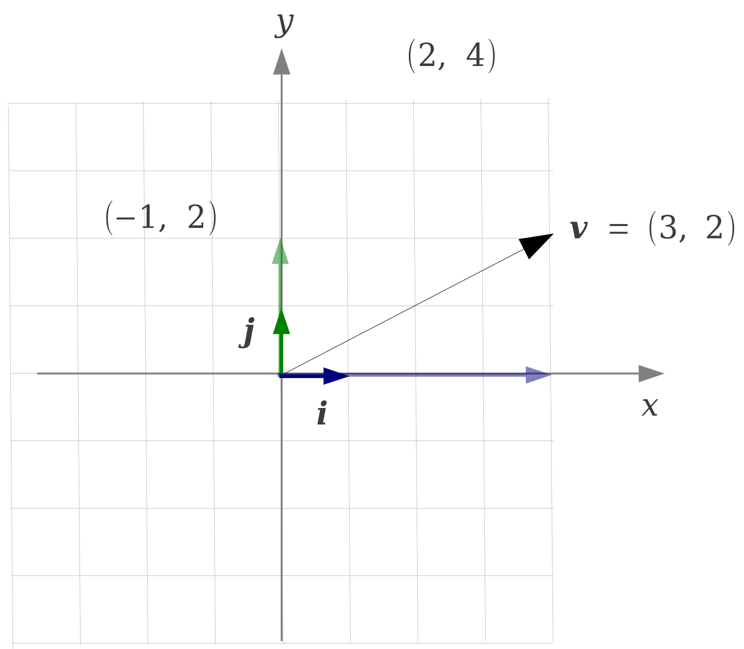


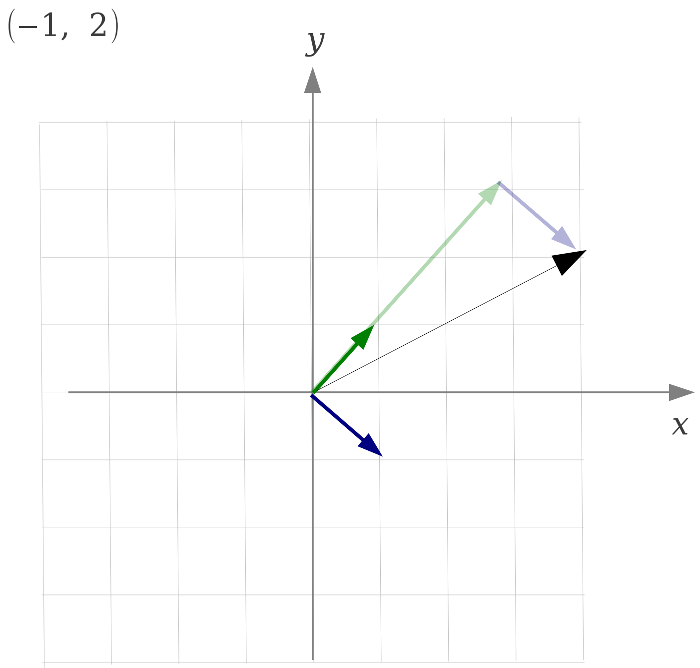
Vector Addition





Basis





$\boldsymbol{v} = 3\boldsymbol{i} + 2\boldsymbol{j}$

$a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} = 0$ ⇒ $a = b = 0$

$a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}1\\-1\end{bmatrix} = 0$ ⇒ $a = b = 0$

basis

basis

Cross Product (1)

Determinant of order 3

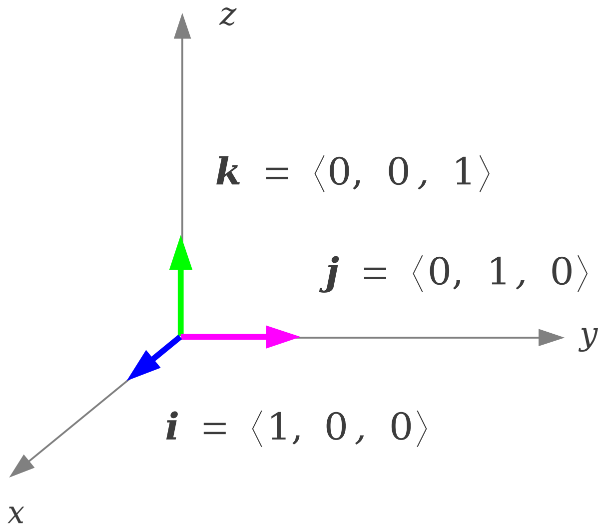
$$\boldsymbol{a} = a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k} = \langle a_1, a_2, a_3 \rangle$$

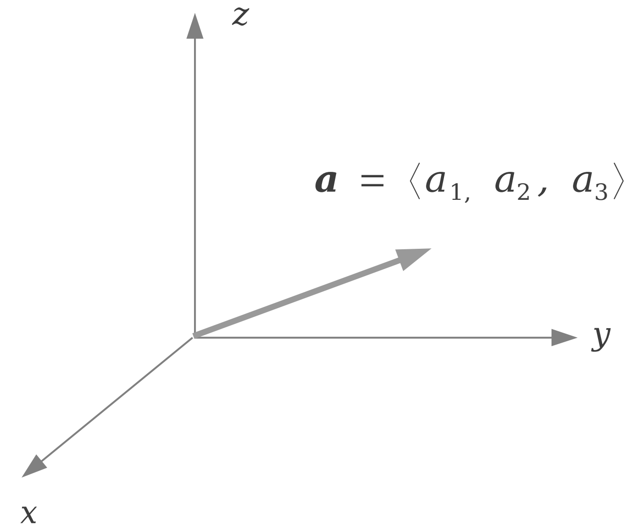
$$\boldsymbol{b} = b_1 \boldsymbol{i} + b_2 \boldsymbol{j} + b_3 \boldsymbol{k} = \langle b_1, b_2, b_3 \rangle$$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \boldsymbol{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2 b_3 - a_3 b_2)\boldsymbol{i} - (a_1 b_3 - a_3 b_1)\boldsymbol{i} + (a_1 b_2 - a_2 b_1)\boldsymbol{i}$$

Cross Product (2)





$$\boldsymbol{i} \times \boldsymbol{j} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \boldsymbol{k}$$

normal to $\boldsymbol{i}$ & $\boldsymbol{j}$

$$\boldsymbol{j} \times \boldsymbol{i} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\boldsymbol{k}$$

$$\boldsymbol{j} \times \boldsymbol{k} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \boldsymbol{i}$$

normal to $\boldsymbol{j}$ & $\boldsymbol{k}$

$$\boldsymbol{k} \times \boldsymbol{j} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\boldsymbol{i}$$

$$\boldsymbol{k} \times \boldsymbol{i} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \boldsymbol{j}$$

normal to $\boldsymbol{k}$ & $\boldsymbol{i}$

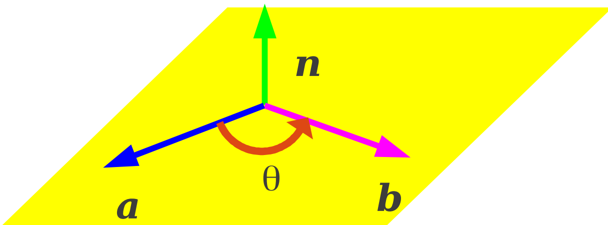
$$\boldsymbol{i} \times \boldsymbol{k} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\boldsymbol{j}$$

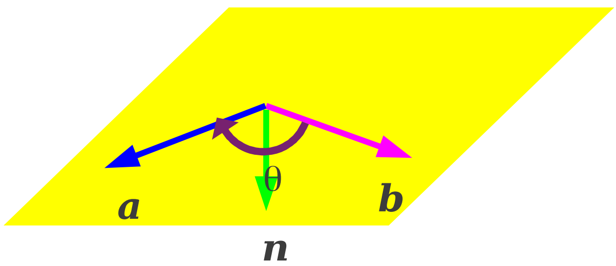
Right Hand Rule

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

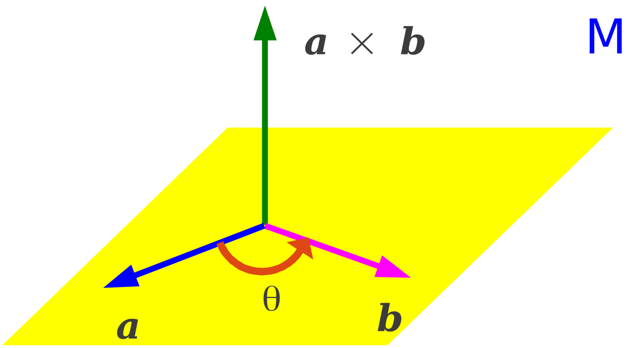
$$\boldsymbol{b} \times \boldsymbol{a} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

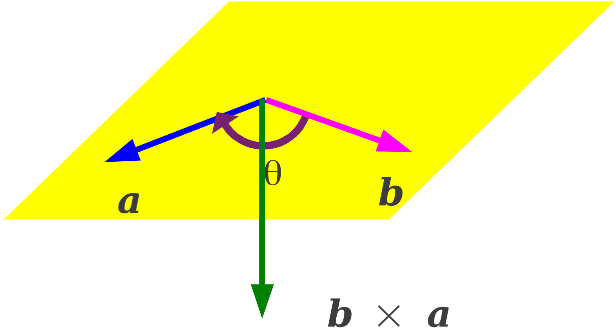
Normal direction $\boldsymbol{n}$





Magnitude = $\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \, \sin\theta$





Line Equations (1)

Vector Equation

$$\boldsymbol{r} = \boldsymbol{r}_2 + t\,\boldsymbol{a}$$

Parameter

Direction Vector

$$\boldsymbol{r}_2 = \langle x_{2,}\ y_{2,}\ z_2 \rangle$$
$$\boldsymbol{a} = \langle a_{1,}\ a_{2,}\ a_3 \rangle$$

Parametric Equation

$$x = x_2 + t\,a_1$$
$$y = y_2 + t\,a_2$$
$$z = z_2 + t\,a_3$$

Component

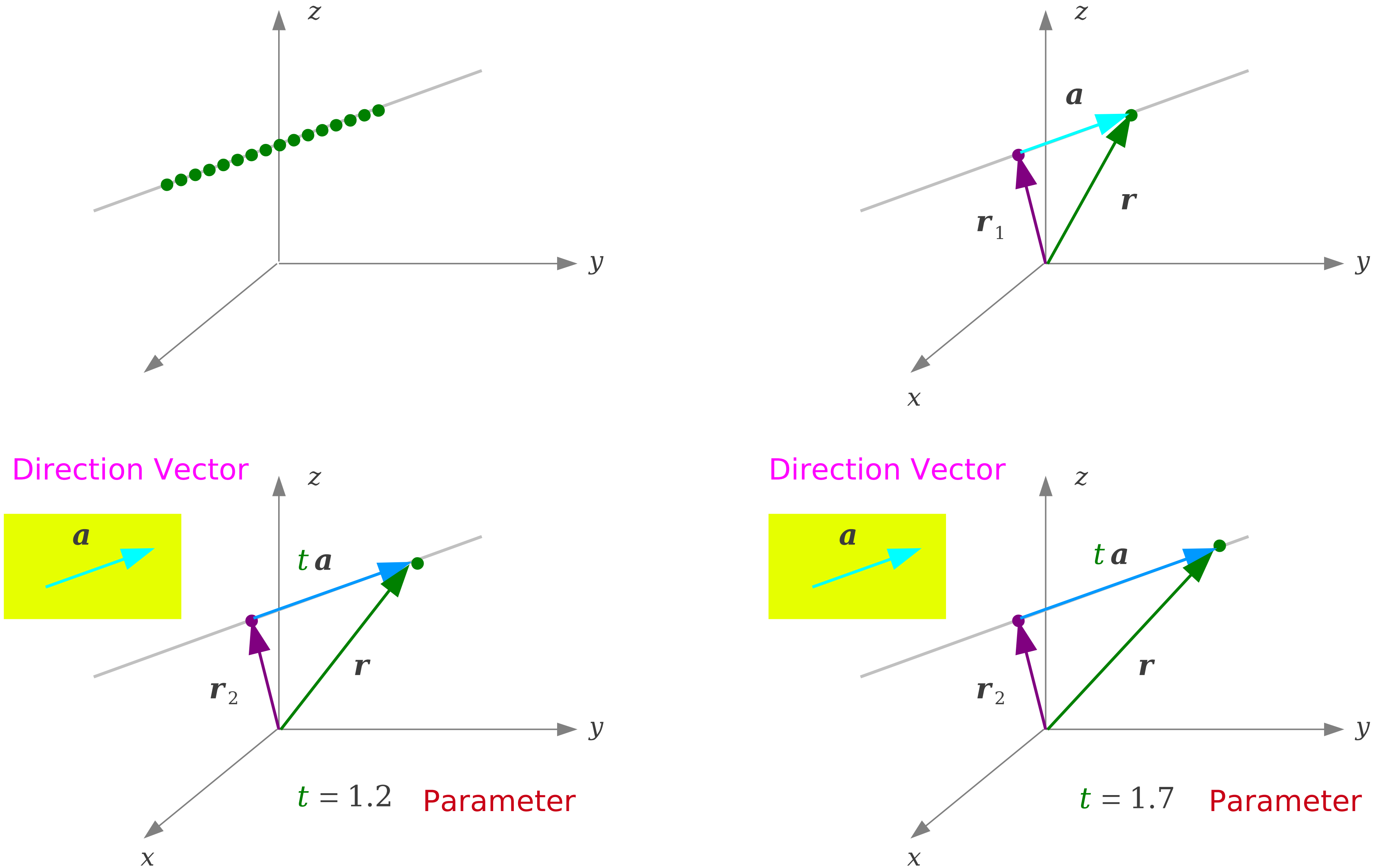
$$t\,a_1 = x - x_2$$
$$t\,a_2 = y - y_2$$
$$t\,a_3 = z - z_2$$

Symmetric Equation

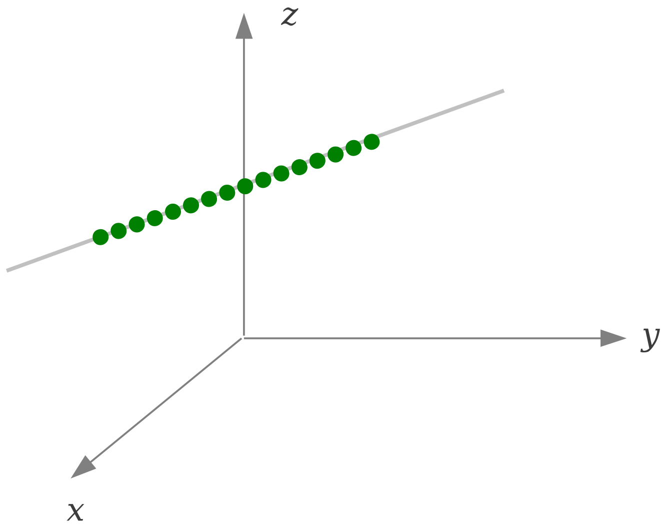
$$t = \frac{x - x_2}{a_1} = \frac{y - y_2}{a_2} = \frac{z - z_2}{a_3}$$

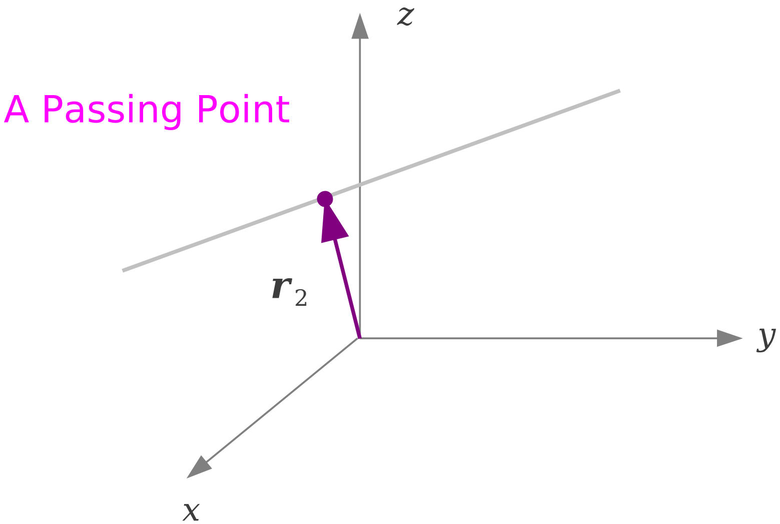
Elimination of parameter

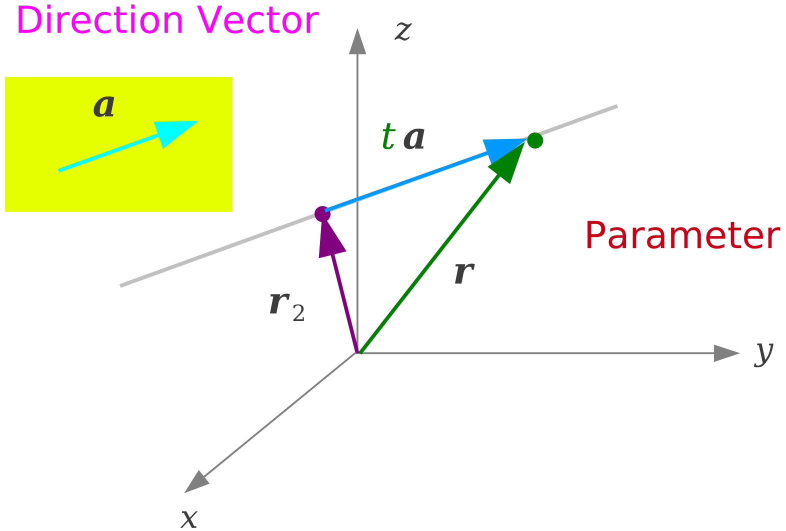
Line Equations (2)



Line Equations (3)







$$\boldsymbol{r} = \boldsymbol{r}_2 + t\,\boldsymbol{a}$$

$$\boldsymbol{r}_2 = \langle x_2, y_2, z_2 \rangle$$

$$\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$$

Plane Equations (1)

Vector equation

$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_1) = 0$$

Normal Vector

$$\boldsymbol{r} = \langle x, y, z \rangle$$

$$\boldsymbol{r}_1 = \langle x_1, y_1, z_1 \rangle$$

$$\boldsymbol{n} = \langle a, b, c \rangle$$

$$\boldsymbol{r} - \boldsymbol{r}_1 = \langle x - x_1, \; y - y_1, \; z - z_1 \rangle$$

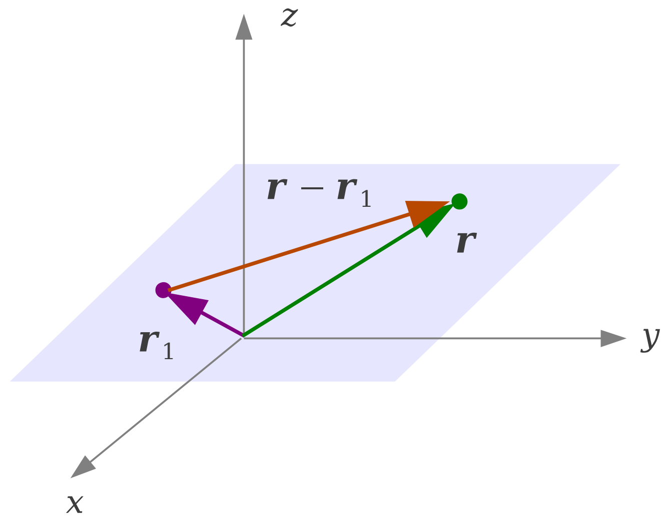
$$\boldsymbol{n} = \langle \quad a, \quad b, \quad c \rangle$$

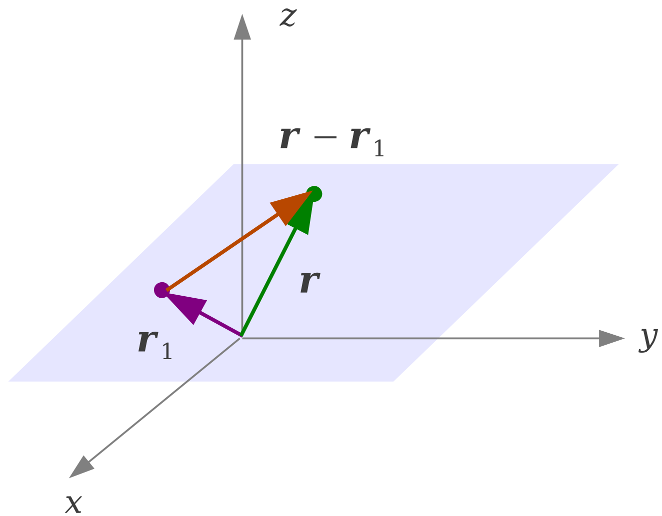
Cartesian equation

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

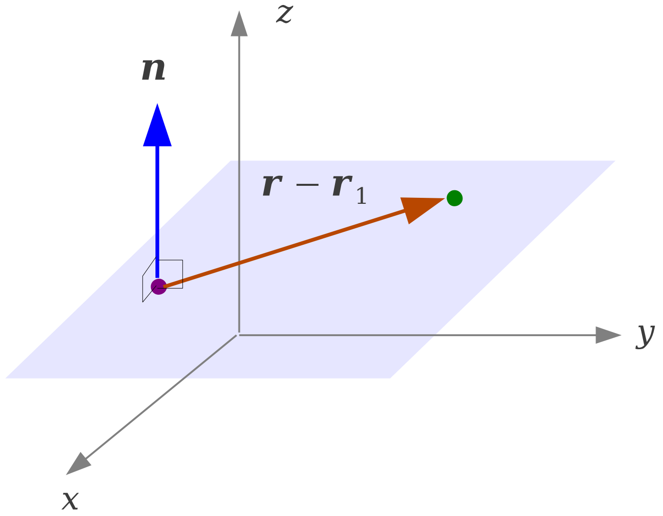
Plane Equations (2)

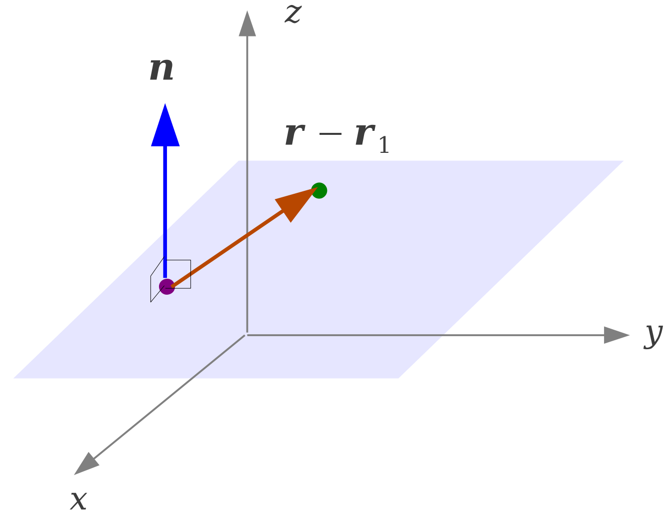
No Parameter

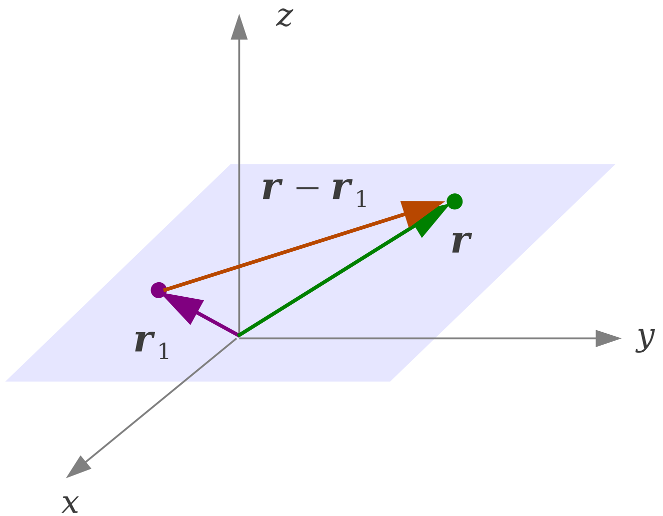




Plane Equations (3)







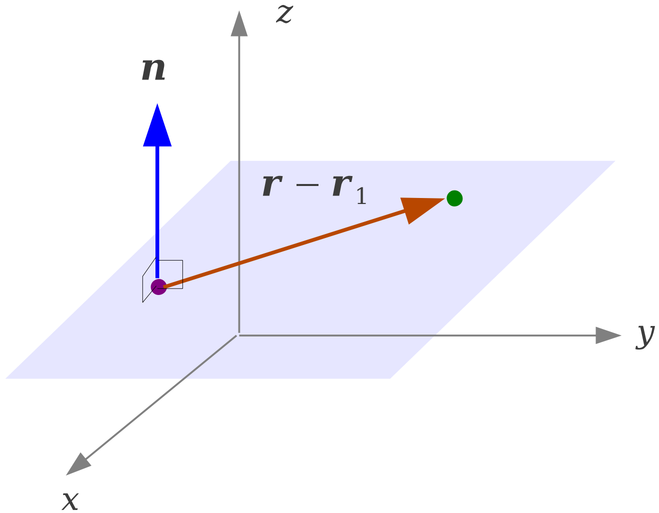
$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

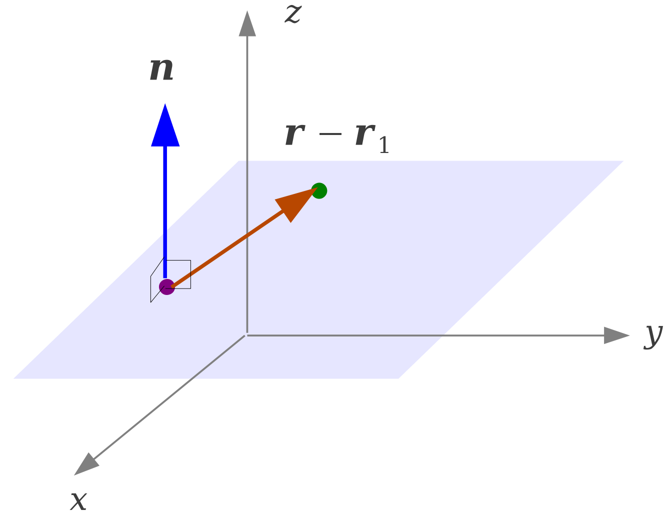
$$\mathbf{r} = \langle x, y, z \rangle$$

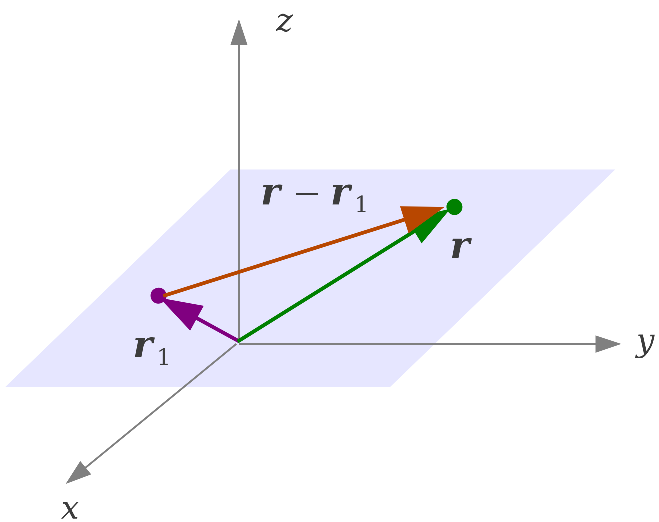
$$\mathbf{r}_1 = \langle x_1, y_1, z_1 \rangle$$

$$\mathbf{n} = \langle a, b, c \rangle$$

Plane Equations (3)







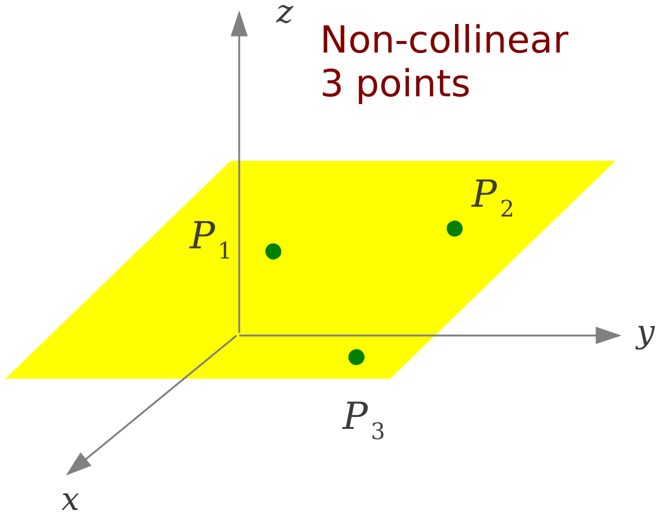
$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_1) = 0$$

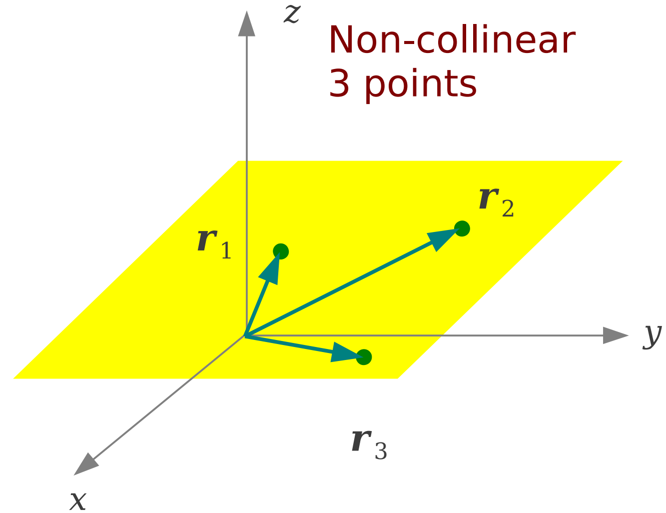
$$\boldsymbol{r} = \langle x, y, z \rangle$$

$$\boldsymbol{r}_1 = \langle x_1, y_1, z_1 \rangle$$

$$\boldsymbol{n} = \langle a, b, c \rangle$$

Normal Vector & 3 Points



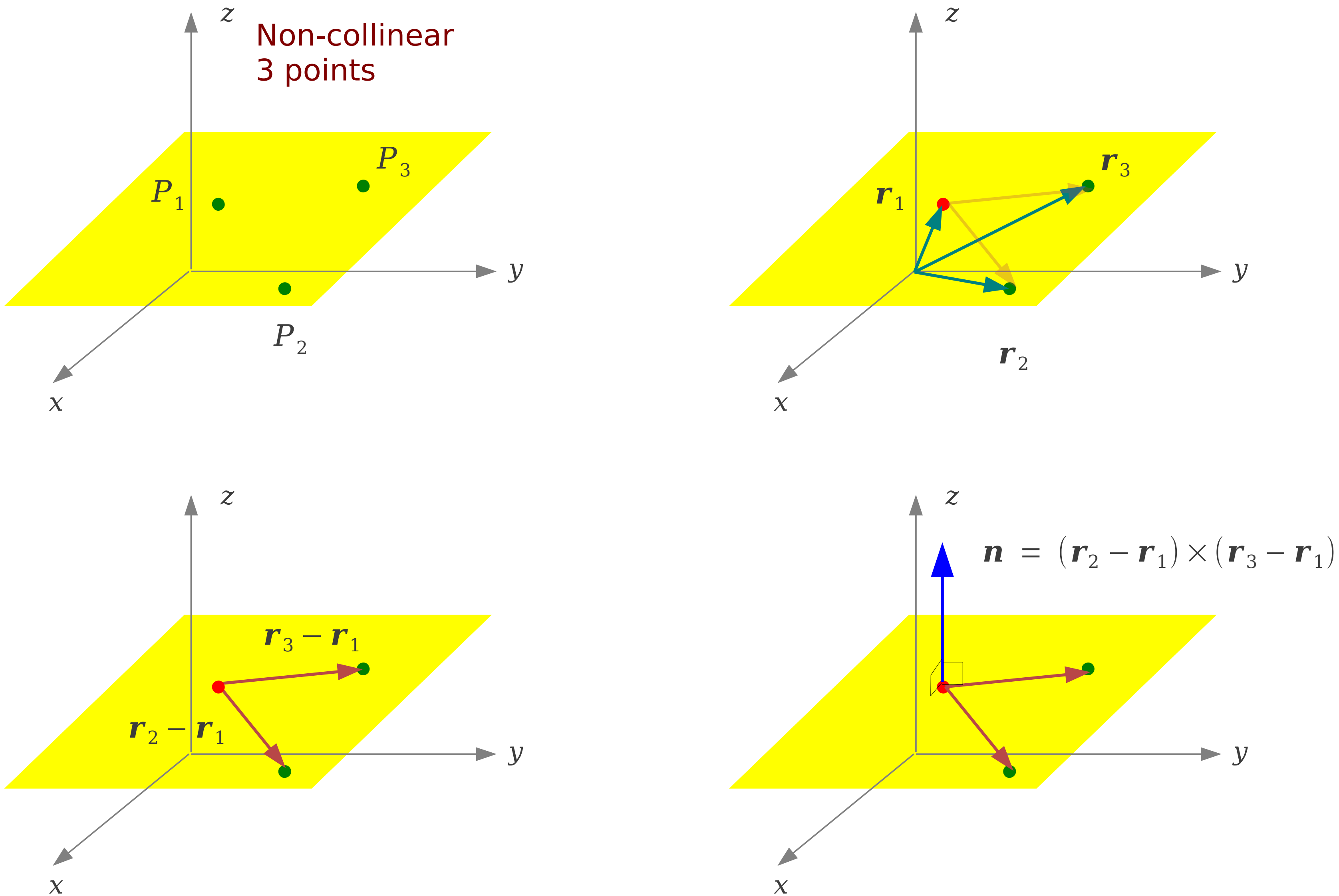


Graph of a plane

Line intersection of two planes

Point of intersection of a line and plane

Normal Vector & 3 Points



Vector Triple Product

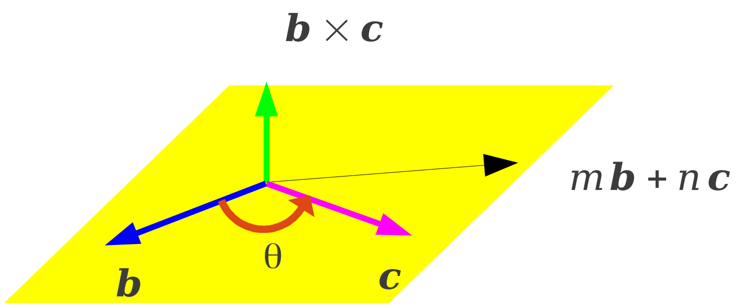
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$$

⇨ Perpendicular to $\boldsymbol{b} \times \boldsymbol{c}$

⇨ Any vector perpendicular to $\boldsymbol{b} \times \boldsymbol{c}$ lies in the plane perpendicular to $\boldsymbol{b} \times \boldsymbol{c}$

⇨ lies in the plane of $\boldsymbol{b}$ and $\boldsymbol{c}$

Perpendicular to the plane of



$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \Rightarrow m\boldsymbol{b} + n\boldsymbol{c}$$

$$(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$$

Scalar Triple Product

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$$

$$\boldsymbol{a} = a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k}$$

$$\boldsymbol{b} = b_1 \boldsymbol{i} + b_2 \boldsymbol{j} + b_3 \boldsymbol{k}$$

$$\boldsymbol{c} = c_1 \boldsymbol{i} + c_2 \boldsymbol{j} + c_3 \boldsymbol{k}$$

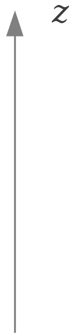
$$\boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \boldsymbol{i} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k}) \cdot \left(\boldsymbol{i} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right)$$

$$= \left(a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

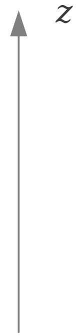
Normal Vector & 3 Points

Non-collinear
3 points

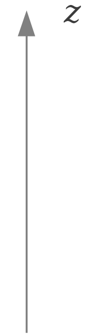


$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Normal Vector & 3 Points



Non-collinear
3 points



References

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