Matrices and Matrix Operations (3A)

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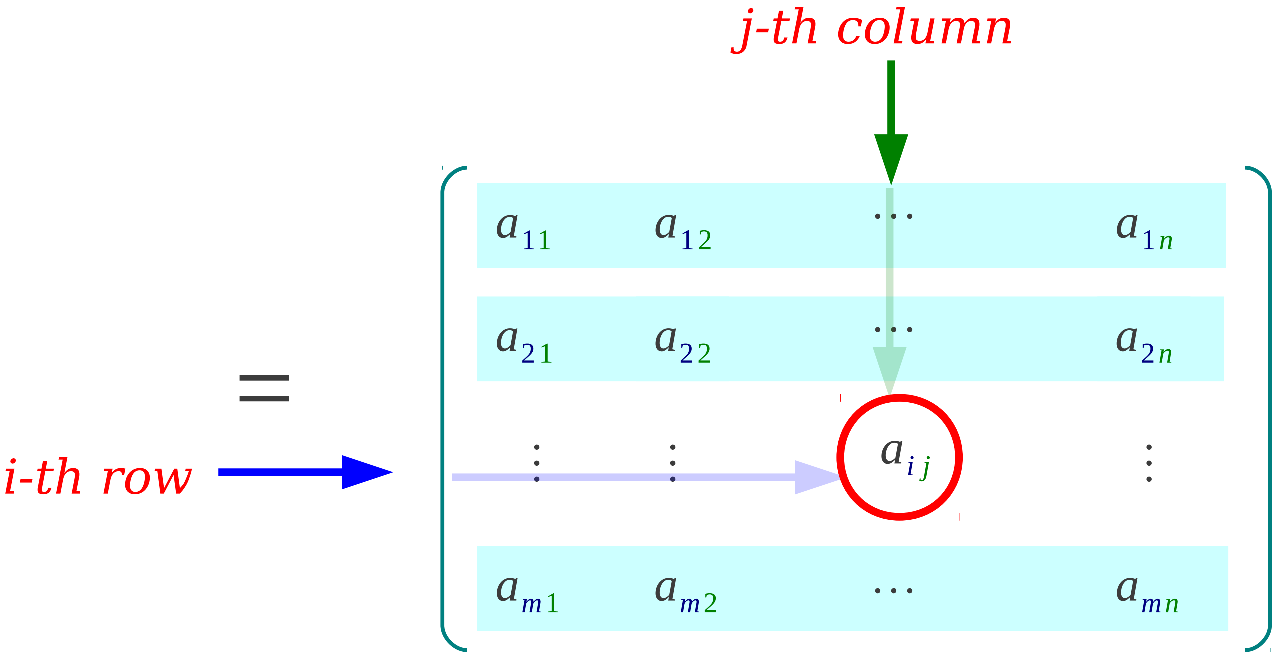
A General m x n Matrix

$$A = [a_{ij}]_{m \times n} = [a_{ij}]$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

A Element of a Matrix

$$A = [a_{ij}]_{m \times n} = [a_{ij}]$$



$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \boxed{a_{ij}} & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$(A)_{m \times n} = a_{ij}$$

Matrix Multiplication (1)

1st row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

2nd row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

m-th row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\boldsymbol{A} = [a_{ij}]_{m \times n}$$

$$\boldsymbol{x} = [x_{ij}]_{n \times 1}$$

$$\boldsymbol{b} = [b_{ij}]_{m \times 1}$$

$$\boldsymbol{A} \quad \boldsymbol{x} = \boldsymbol{b}$$

$$m \times n \quad n \times 1 \qquad m \times 1$$

Matrix Multiplication (2)

1st column

1st row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

$$\boldsymbol{A} = [a_{ij}]_{m\times n}$$

$$\boldsymbol{B} = [b_{ij}]_{n\times 2}$$

$$\boldsymbol{C} = [c_{ij}]_{m\times 2}$$

1st column

2nd row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

$$\boldsymbol{A} \quad \boldsymbol{B} = \boldsymbol{C}$$

$$m\times n \quad n\times 2 \qquad m\times 2$$

Matrix Multiplication (3)

2nd column

1st row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

$$\boldsymbol{A} = [a_{ij}]_{m\times n}$$

$$\boldsymbol{B} = [b_{ij}]_{n\times 2}$$

$$\boldsymbol{C} = [c_{ij}]_{m\times 2}$$

2nd column

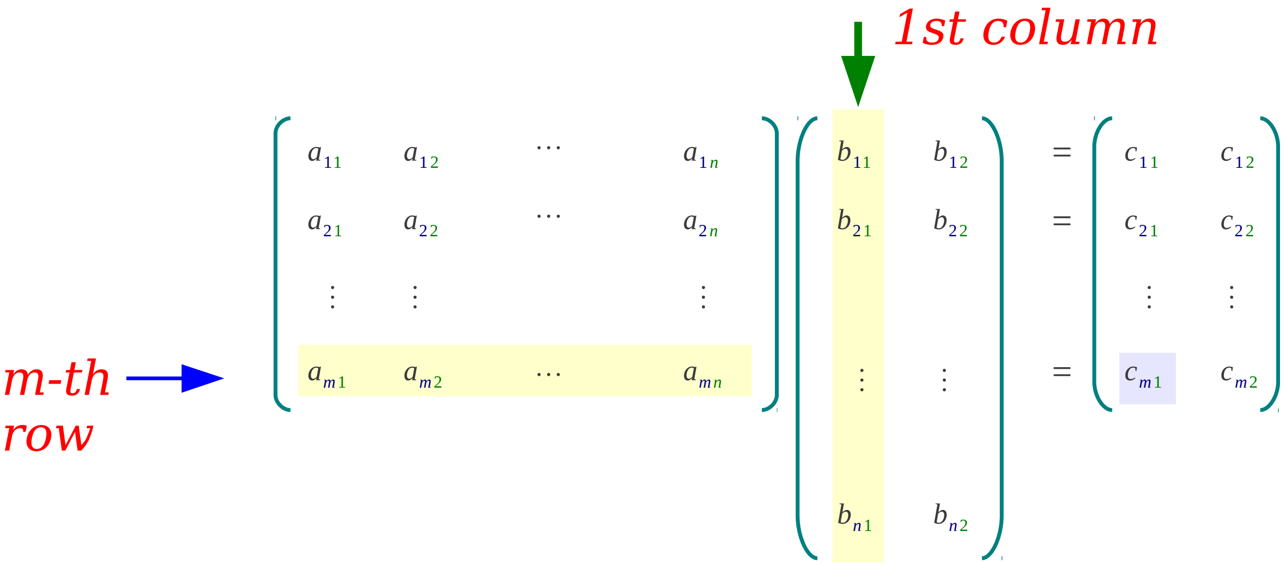
2nd row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

$$\boldsymbol{A} \quad \boldsymbol{B} = \boldsymbol{C}$$

$$\boxed{m}\times n \quad n\times\boxed{2} \qquad \boxed{m\times 2}$$

Matrix Multiplication (4)



1st column

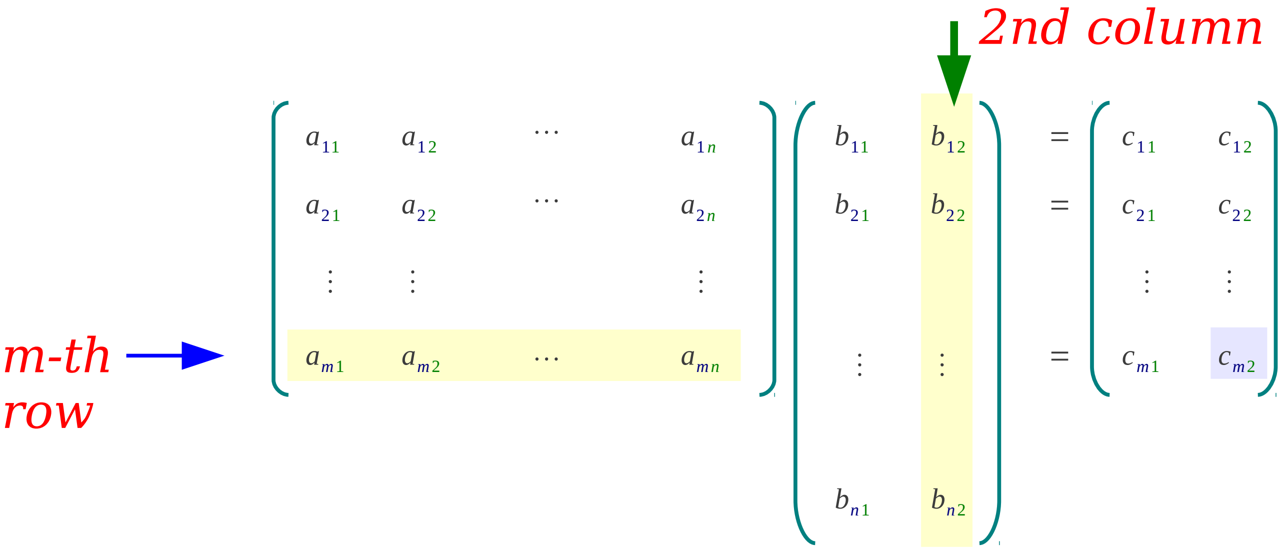
m-th row

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

$$\mathbf{A} = [a_{ij}]_{m\times n}$$

$$\mathbf{B} = [b_{ij}]_{n\times 2}$$

$$\mathbf{C} = [c_{ij}]_{m\times 2}$$



2nd column

m-th row

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

$$\mathbf{A} \quad \mathbf{B} = \mathbf{C}$$

$$\boxed{m} \times n \quad n \times \boxed{2} \qquad \boxed{m \times 2}$$

Matrix Multiplication (4)

1st column

m-th row

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

$$\mathbf{A} = [a_{ij}]_{m\times n}$$

$$\mathbf{B} = [b_{ij}]_{n\times 2}$$

$$\mathbf{C} = [c_{ij}]_{m\times 2}$$

2nd column

m-th row

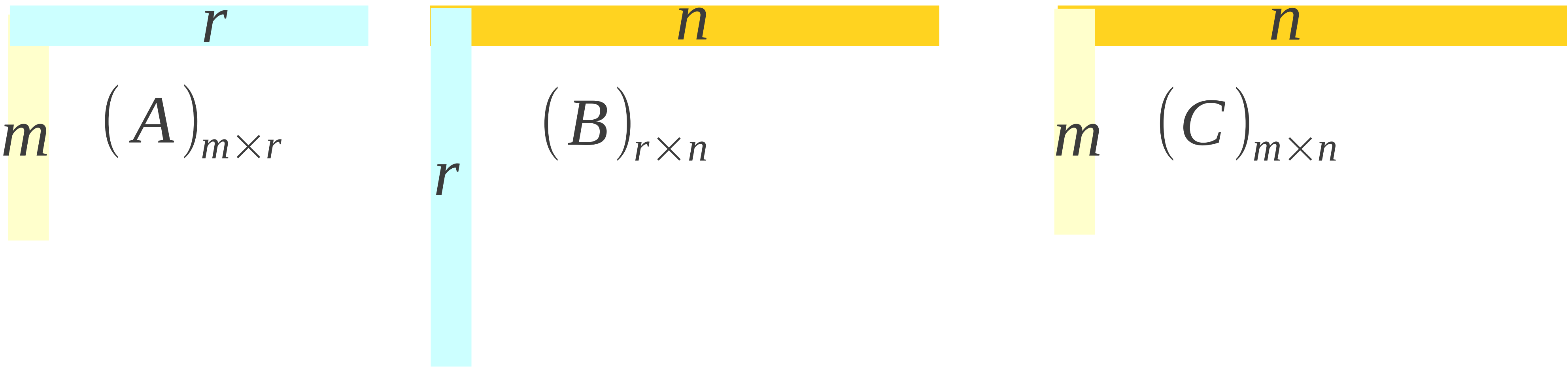
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

$$\mathbf{A} \quad \mathbf{B} = \mathbf{C}$$

$$m \times n \quad n \times 2 \qquad m \times 2$$

Multiplication of Matrices (1)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \boxed{c_{ii}} & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rn} \end{pmatrix}$$



$(A)_{m\times r}$ $(B)_{r\times n}$ $(C)_{m\times n}$

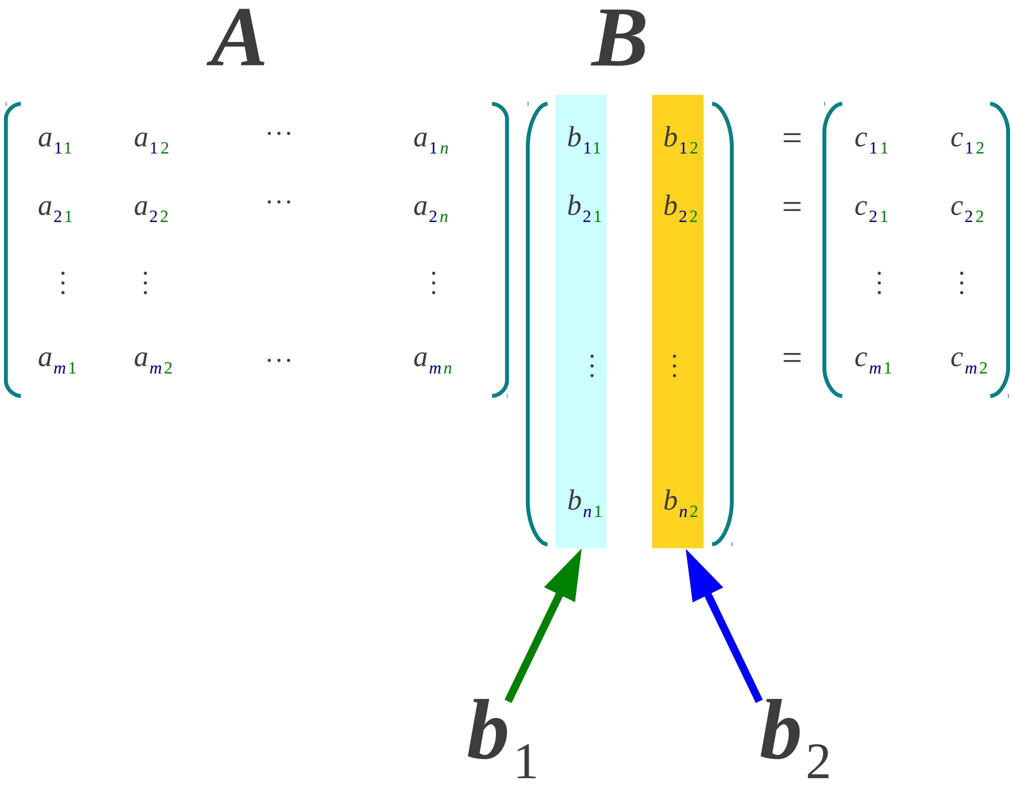
Multiplication of Matrices (2)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{r2} & \cdots & b_{rn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \boxed{c_{ii}} & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rn} \end{pmatrix}$$

$$(AB)_{ij} = c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} \quad \cdots \quad + a_{ir}b_{rj}$$

$$= \sum_{k=1}^{r} a_{ik}b_{kj}$$

Partitioned Matrix



$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

$$\mathbf{A} = [a_{ij}]_{m\times n}$$

$$\mathbf{B} = [b_{ij}]_{n\times 2}$$

$$\mathbf{C} = [c_{ij}]_{m\times 2}$$

$$\underset{n\times 2}{\mathbf{B}} = [\underset{n\times 1}{\mathbf{b}_1} \quad \underset{n\times 1}{\mathbf{b}_2}]$$

$$\underset{m\times 2}{\mathbf{A}\mathbf{B}} = [\underset{m\times 1}{\mathbf{A}\mathbf{b}_1} \quad \underset{m\times 1}{\mathbf{A}\mathbf{b}_2}]$$

$$m\times n \quad n\times 2 \qquad m\times n \quad n\times 1 \qquad m\times n \quad n\times 1$$

Partitioned Matrix

$$\begin{array}{cccc} & \boldsymbol{A} & \boldsymbol{B} & \\ \begin{array}{c} \boldsymbol{a}_1 \rightarrow \\ \boldsymbol{a}_2 \rightarrow \\ \\ \boldsymbol{a}_m \rightarrow \end{array} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} & \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} & = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix} \end{array}$$

$$\boldsymbol{A} = [a_{ij}]_{m \times n}$$

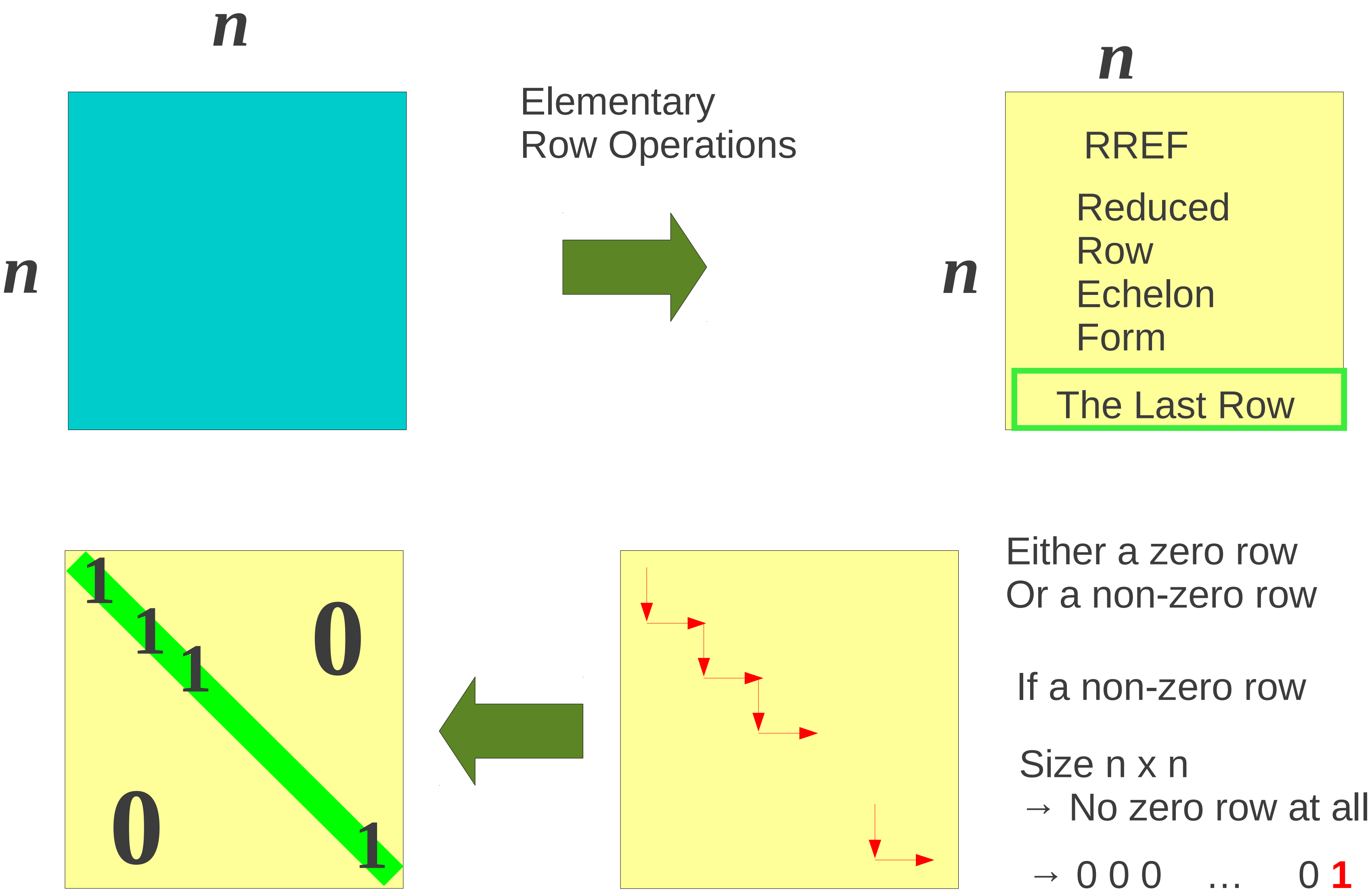
$$\boldsymbol{B} = [b_{ij}]_{n \times 2}$$

$$\boldsymbol{C} = [c_{ij}]_{m \times 2}$$

$$\underset{m \times n}{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_m \end{bmatrix} \begin{array}{l} 1 \times n \\ 1 \times n \\ \\ 1 \times n \end{array}$$

$$\underset{\substack{m \times 2 \\ \\ m \times n \quad n \times 2}}{\boldsymbol{A}\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{a}_1 \boldsymbol{B} \\ \boldsymbol{a}_2 \boldsymbol{B} \\ \vdots \\ \boldsymbol{a}_m \boldsymbol{B} \end{bmatrix} \begin{array}{ll} 1 \times 2 & \quad 1 \times n \;\; n \times 2 \\ 1 \times 2 & \quad 1 \times n \;\; n \times 2 \\ & \\ 1 \times 2 & \quad 1 \times n \;\; n \times 2 \end{array}$$

RREF of a Square Matrix



Elementary
Row Operations

RREF

Reduced
Row
Echelon
Form

The Last Row

Either a zero row
Or a non-zero row

If a non-zero row

Size n x n
→ No zero row at all

→ 0 0 0 … 0 1

Non-singular Matrix

An Invertible,
Non-singular
Matrix

$$AB = BA = I$$

All square matrix

Unique

$$AC = CA = I \Rightarrow B = C$$

Notation

$$AA^{-1} = A^{-1}A = I$$

Inverse of a 2x2 Matrix

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Invertible when $ad - bc \neq 0$

$$\boldsymbol{A}^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

Properties of Negative Powers

$$Assume \quad \mathbf{A} \quad is \quad invertible$$

$$(\mathbf{A})^{-1} \quad is \quad invertible$$

$$((\mathbf{A})^{-1})^{-1} \; = \; A$$

$$\mathbf{A}^{n} \quad is \quad invertible$$

$$(\mathbf{A}^{n})^{-1} = \mathbf{A}^{-n} = (\mathbf{A}^{-1})^{n}$$

$$k\,\mathbf{A} \quad is \quad invertible$$

$$(k\,\mathbf{A}^{n})^{-1} = k^{-1}\mathbf{A}^{-1}$$

References

[1] http://en.wikipedia.org/
[2] Anton & Busby, “Contemporary Linear Algebra”
[3] Anton & Rorres, “Elementary Linear Algebra”