

Complex Integration (2B)

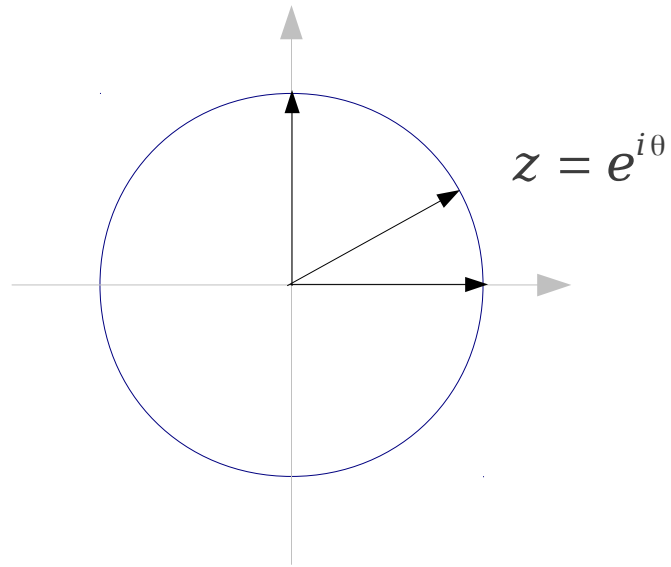
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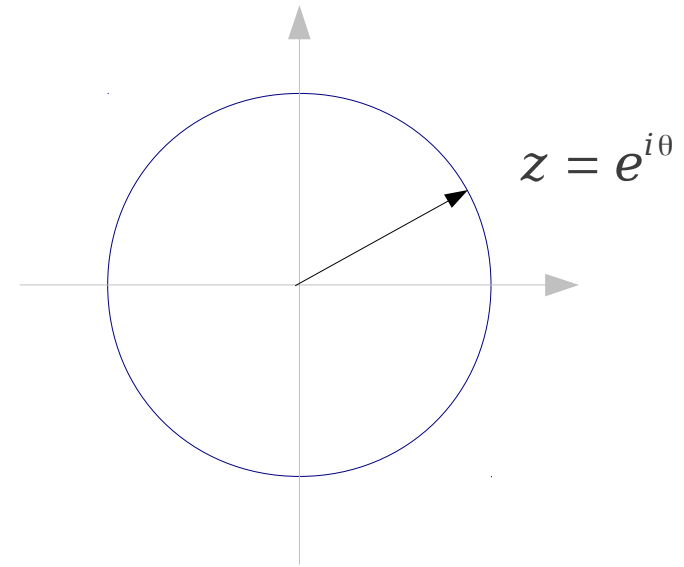
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Unit Circular Contour



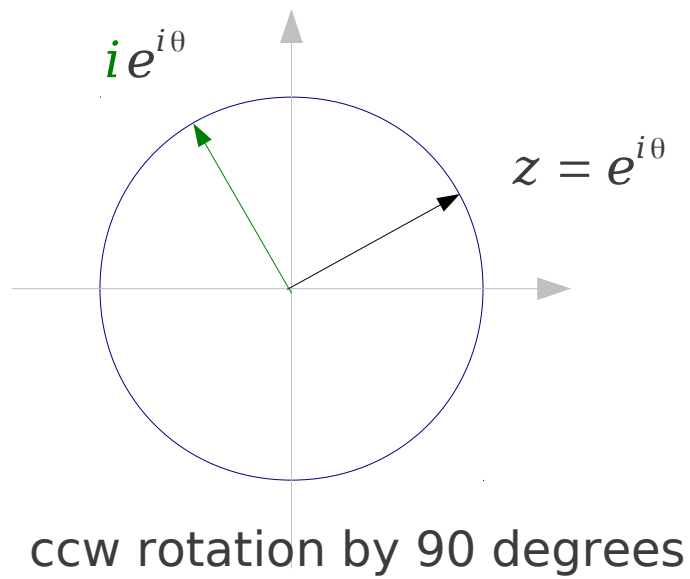
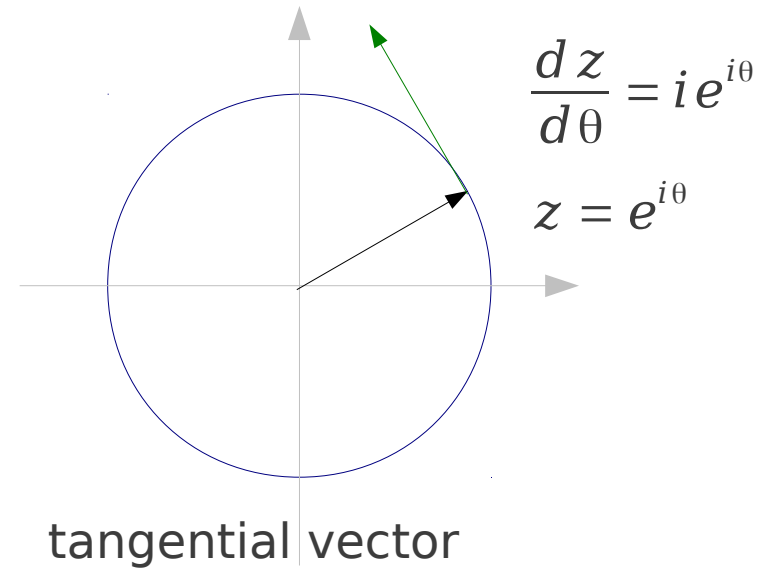
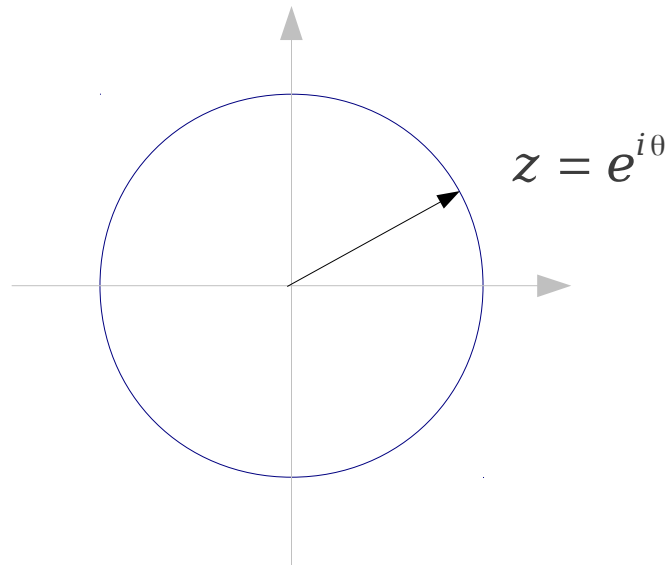
$$\frac{dz}{d\theta} = ie^{i\theta}$$



$$z(t) = x(t) + iy(t) \quad a \leq t \leq b$$

$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt$$

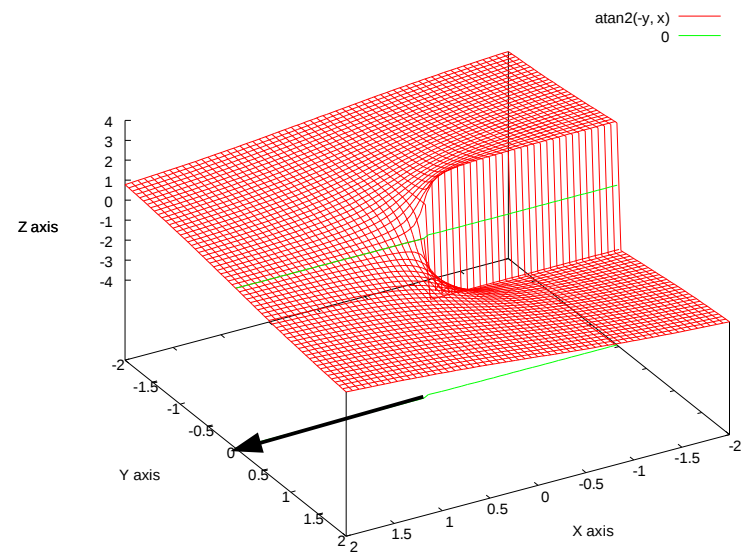
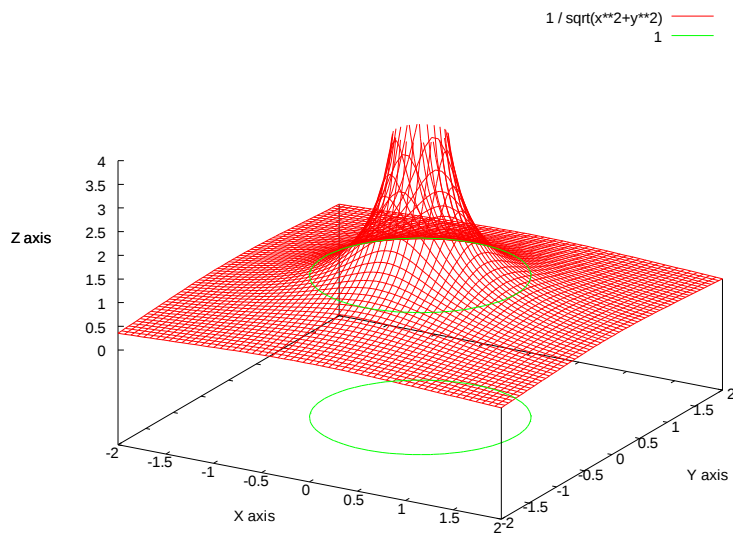
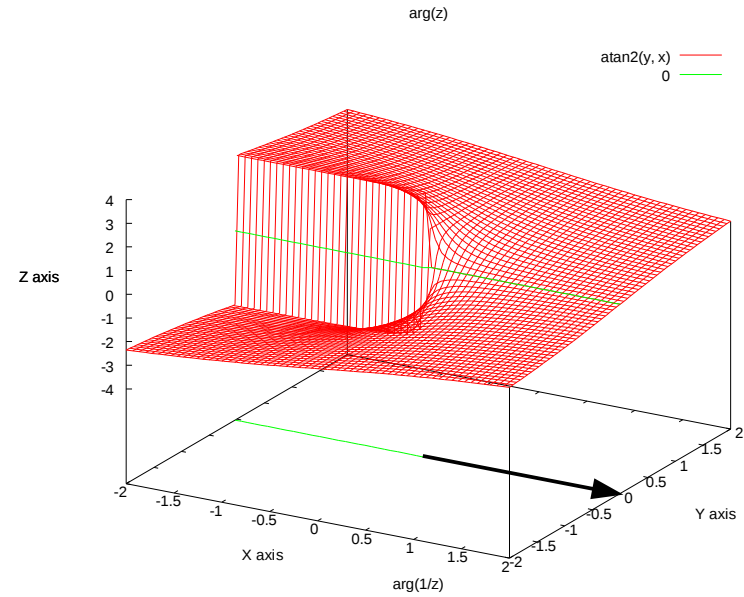
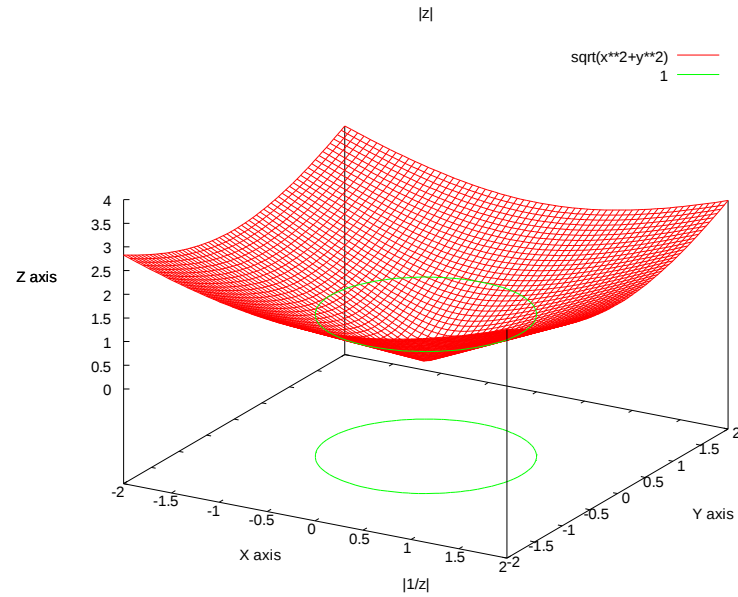
Unit Circular Contour



No radial axis change
leading phase by 90 degrees

$$dz = ie^{i\theta} d\theta$$

Functions z , $1/z$ on the unit circle (1)



plot code for $f(z)=z$

```
# Plot  $f(z) = z$ 
# Base on 3D gnuplot demo - contour plot
# Licensing: This code is distributed under the GNU LGPL
license.
# Modified: 2012.12.17
# Author: Young W. Lim

# set terminal pngcairo transparent enhanced font "arial,10"
fontscale 0.8 size 400, 250
# set output 'contours.1.png'
set view 60, 30, 0.85, 1.1
set samples 60, 60
set isosamples 61, 61
#set contour base
#set contour surface
set contour both
set cntrparam levels discrete 1, 4

set title "|z|"
set xlabel "X axis"
set ylabel "Y axis"
set zlabel "Z axis"
set zlabel offset character 1, 0, 0 font "" textcolor lt -1 norotate
.emf'
replot
set term wxt
```

```
set xrange [-2: 2]
set yrange [-2: 2]
set zrange [0: 4]
splot sqrt(x**2+y**2)
```

```
set term emf
set output 'splot_z.mag.emf'
replot
set term wxt
```

```
pause -1
```

```
set cntrparam levels discrete 0
set zrange [-4: 4]
set title "arg(z)"
```

```
splot atan2(y, x)
```

```
set term emf
set output 'splot_z.arg
```

plot code for $f(z)=1/z$

```
# Plot  $f(z) = z$ 
# Base on 3D gnuplot demo - contour plot
# Licensing: This code is distributed under the GNU LGPL
license.
# Modified: 2012.12.17
# Author: Young W. Lim

# set terminal pngcairo transparent enhanced font "arial,10"
fontscale 0.8 size 400, 250
# set output 'contours.1.png'
set view 60, 30, 0.85, 1.1
set samples 60, 60
set isosamples 61, 61
#set contour base
#set contour surface
set contour both
set cntrparam levels discrete 1, 4

set title "|1/z|"
set xlabel "X axis"
set ylabel "Y axis"
set zlabel "Z axis"
set zlabel offset character 1, 0, 0 font "" textcolor lt -1 norotate
```

```
set xrange [-2: 2]
set yrange [-2: 2]
set zrange [0: 4]
plot 1 / sqrt(x**2+y**2)

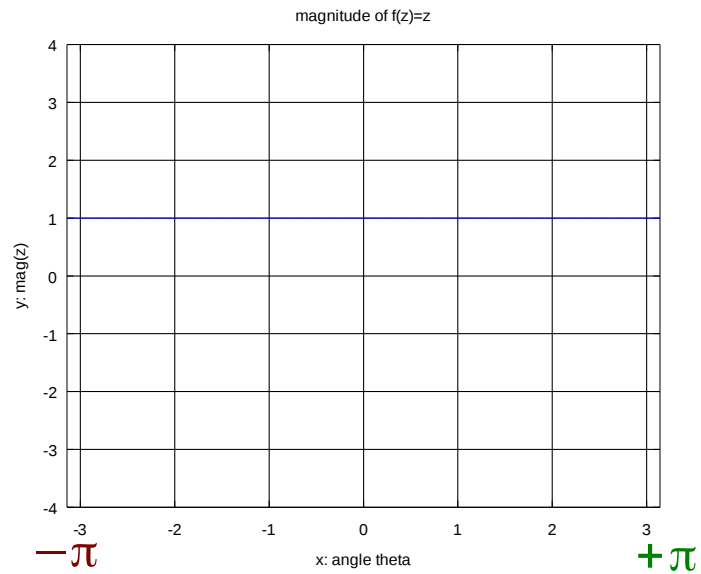
set term emf
set output 'plot_1_z.mag.emf'
replot
set term wxt

pause -1

set view 47, 150, 0.85, 1.1
set cntrparam levels discrete 0
set zrange [-4: 4]
set title "arg(1/z)"
plot atan2(-y, x)

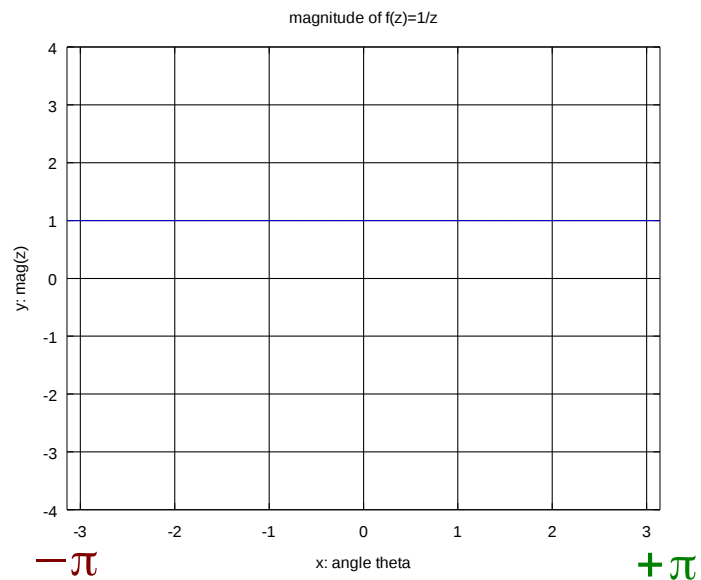
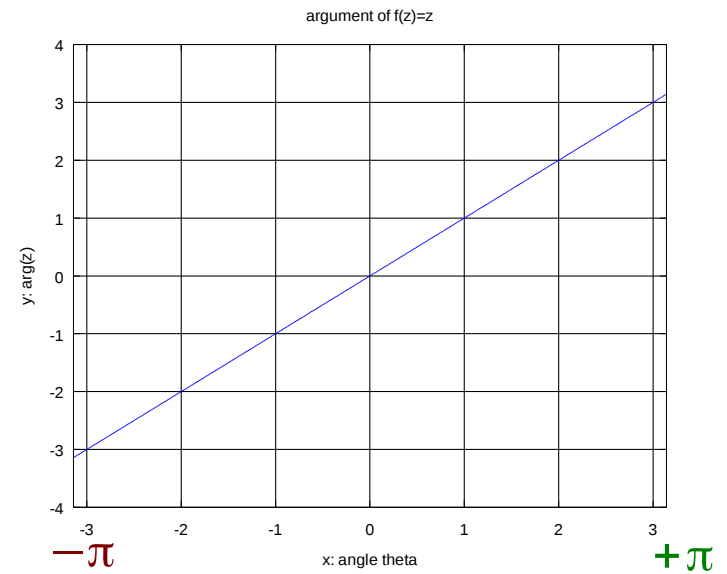
set term emf
set output 'plot_1_z.arg.emf'
replot
set term wxt
```

Plot around the unit circle (1)



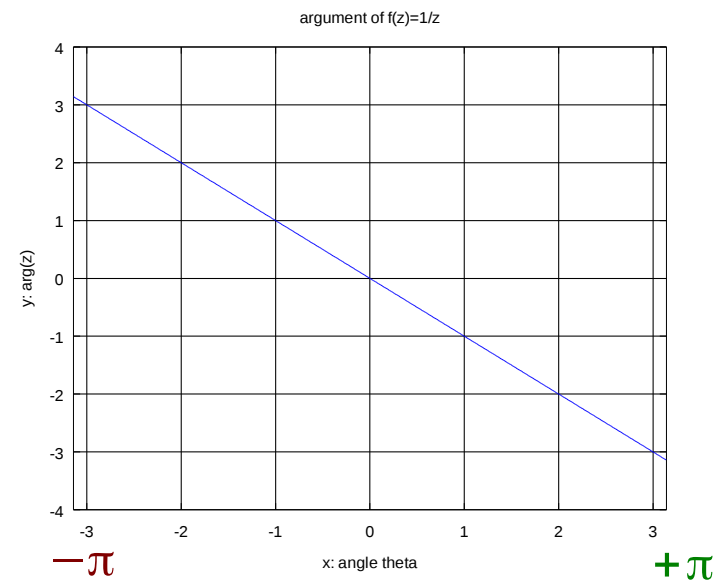
$+\pi$

$-\pi$

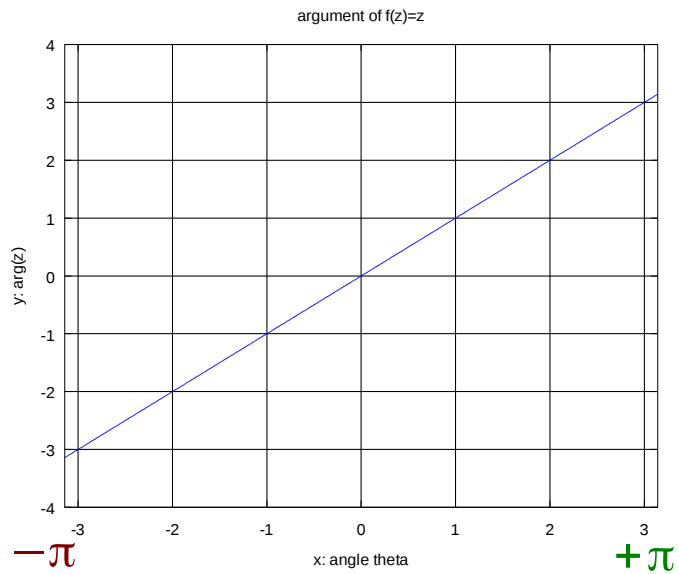


$+\pi$

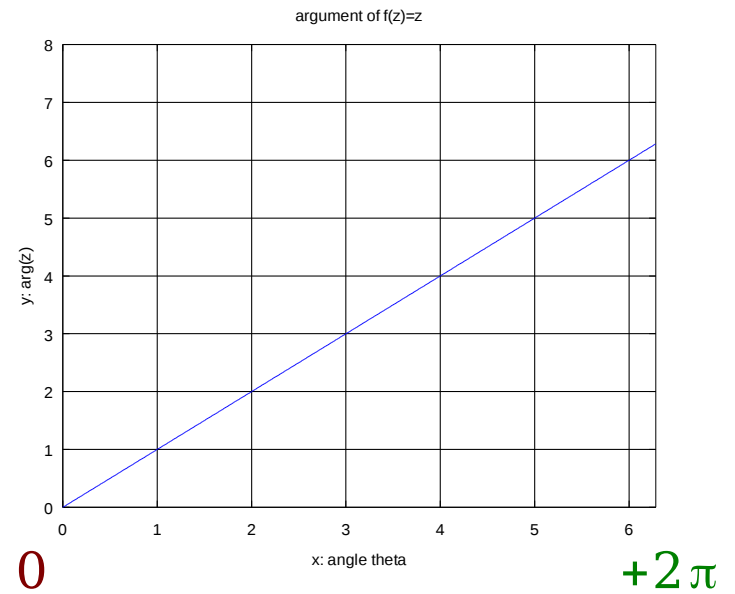
$-\pi$



Plot around the unit circle (2)

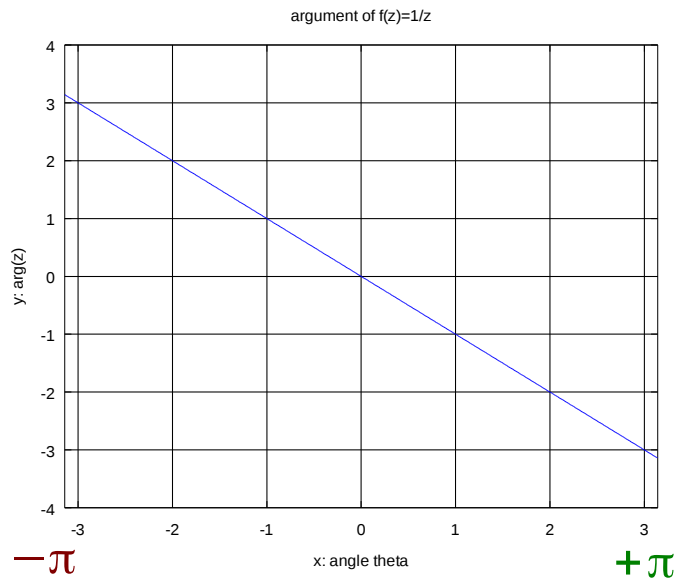


$+2\pi$

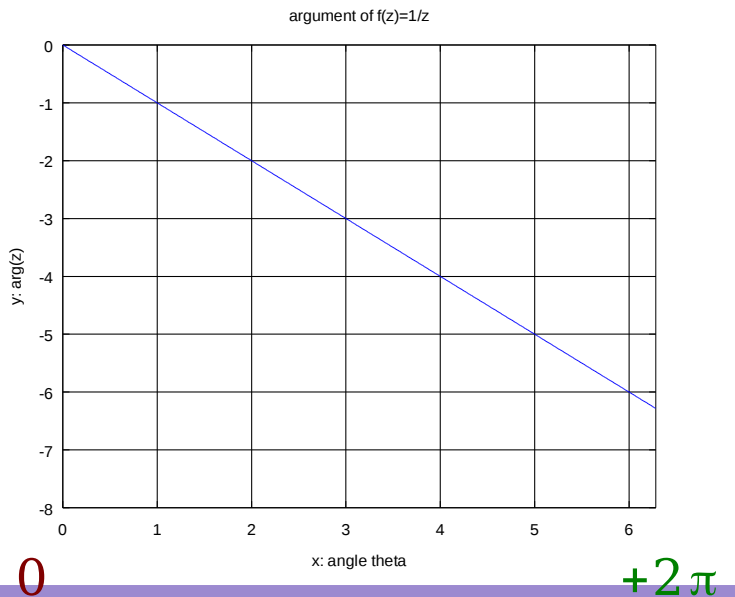


0

0



-2π



plot unit circle code

```
%-----  
% Plot  $f(z) = z$  on the unit circle  
% Licensing: This code is distributed under the GNU LGPL license.  
% Modified: 2012.12.17  
% Author: Young W. Lim  
%-----  
t = -pi : 0.01 : pi;  
z = e.^(j*t);  
  
plot(t, abs(z))  
title("magnitude of  $f(z)=z$ ");  
xlabel("x: angle theta");  
ylabel("y: mag(z)");  
grid on  
axis([-pi pi -4 +4]);  
print -demf uc_z.mag.emf  
pause  
plot(t, arg(z))  
title("argument of  $f(z)=z$ ");  
xlabel("x: angle theta");  
ylabel("y: arg(z)");  
grid on  
axis([-pi pi -4 +4]);  
print -demf uc_z.arg.emf  
  
t = -pi : 0.01 : pi;  
z = e.^(-j*t);  
  
plot(t, abs(z))  
title("magnitude of  $f(z)=1/z$ ");  
xlabel("x: angle theta");  
ylabel("y: mag(z)");  
grid on  
axis([-pi pi -4 +4]);  
print -demf uc_1_z.mag.emf  
pause  
plot(t, arg(z))  
title("argument of  $f(z)=1/z$ ");  
xlabel("x: angle theta");  
ylabel("y: arg(z)");  
grid on  
axis([-pi pi -4 +4]);  
print -demf uc_1_z.arg.emf
```

Contour Integration

$$\oint_C z \, dz$$

$$= \int_0^{2\pi} e^{i\theta} i e^{i\theta} d\theta$$

$$= \left[\frac{1}{2} e^{i2\theta} \right]_0^{2\pi} = 0$$

$$\oint_C z^2 \, dz$$

$$= \int_0^{2\pi} e^{i2\theta} i e^{i\theta} d\theta$$

$$= \left[\frac{1}{3} e^{i3\theta} \right]_0^{2\pi} = 0$$

$$\oint_C \frac{1}{z} \, dz$$

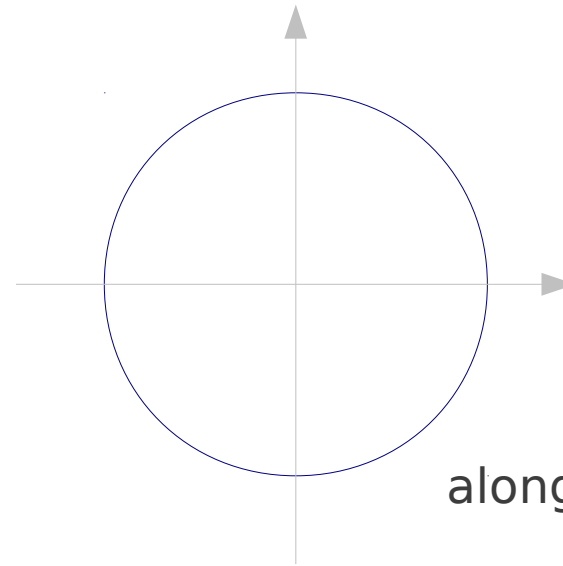
$$= \int_0^{2\pi} e^{-i\theta} i e^{i\theta} d\theta$$

$$= [i]_0^{2\pi} = 2\pi i$$

$$\oint_C z^2 \, dz$$

$$= \int_0^{2\pi} e^{-i2\theta} i e^{i\theta} d\theta$$

$$= [-e^{-i\theta}]_0^{2\pi} = 0$$



$$z = e^{i\theta}$$

$$dz = i e^{i\theta} d\theta$$

along C

$$\oint_C dz$$

$$= \int_0^{2\pi} i e^{i\theta} d\theta$$

$$= [e^{i\theta}]_0^{2\pi} = 0$$

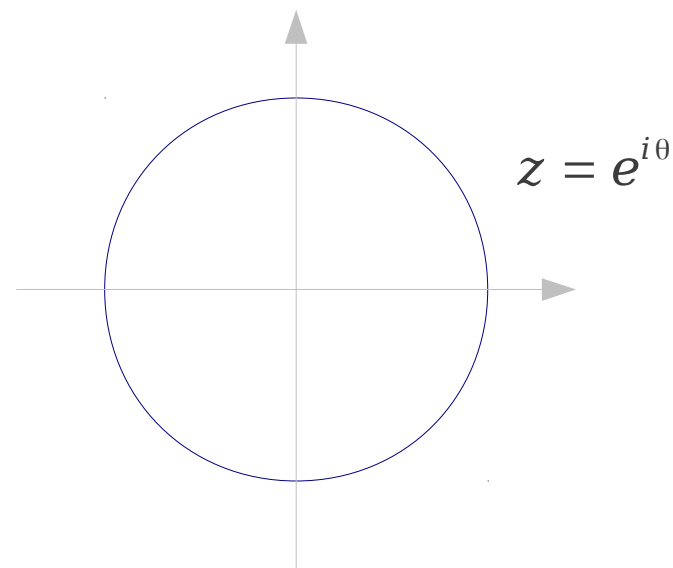
Cauchy's Integral Formula

$f(z)$: **analytic** on and inside simple close curve C

➔
$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$

the value of $f(z)$
at a point $z = a$ inside C

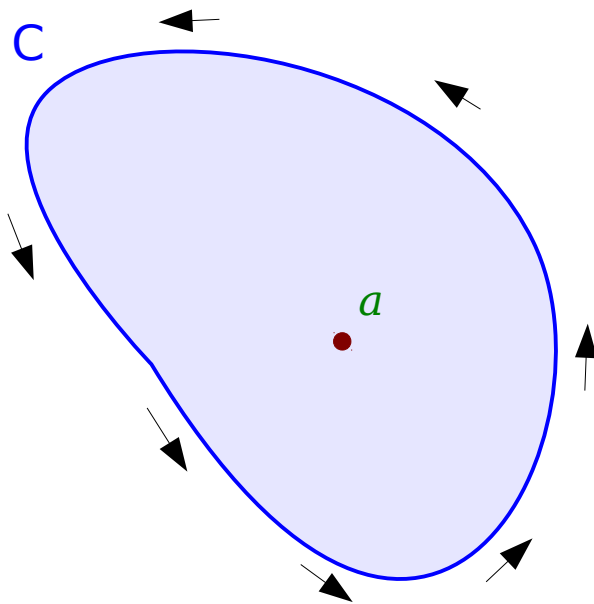
$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw$$



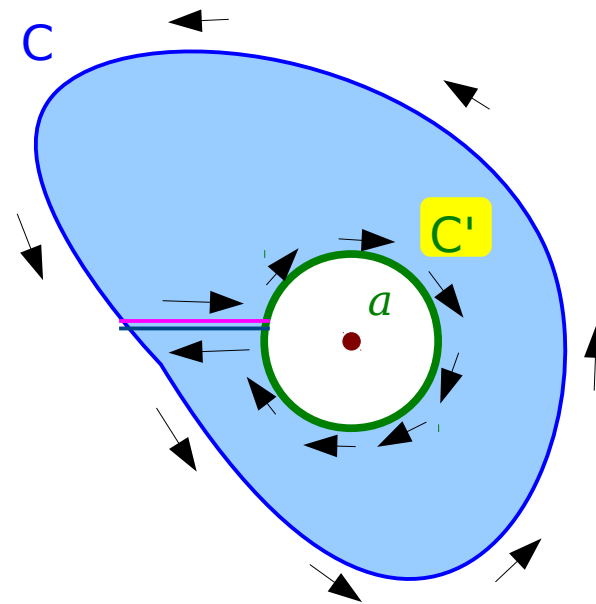
Cauchy's Integral Formula

$f(z)$: **analytic** on and inside simple close curve C

➔
$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$



$$\oint_C f(z) dz = 0$$



$$\oint_{\text{ccw } C} \frac{f(z)}{z-a} dz + \oint_{\text{cw } C'} \frac{f(z)}{z-a} dz = 0$$

$$\begin{aligned} & \oint_{\text{ccw } C} \frac{f(z) dz}{z-a} \\ &= \oint_{\text{ccw } C'} \frac{f(z) dz}{z-a} \end{aligned}$$

Cauchy's Integral Formula

$f(z)$: **analytic** on and inside simple close curve C

➔ $f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$

along C' $z - a = \rho e^{i\theta}$

$z = a + \rho e^{i\theta}$

$dz = i\rho e^{i\theta} d\theta$

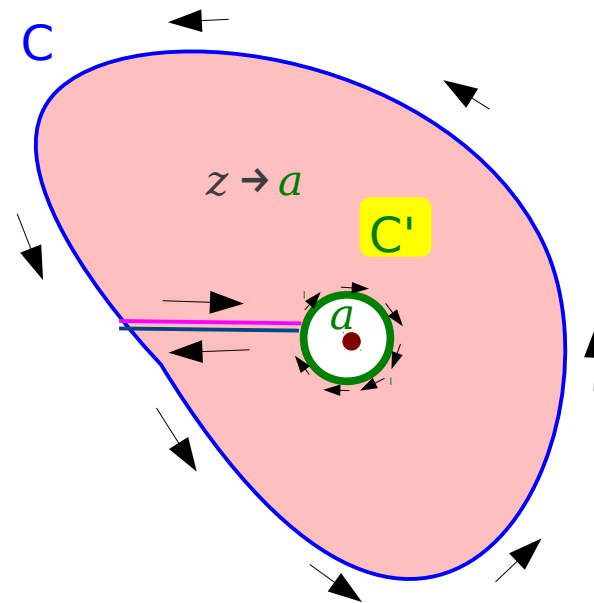
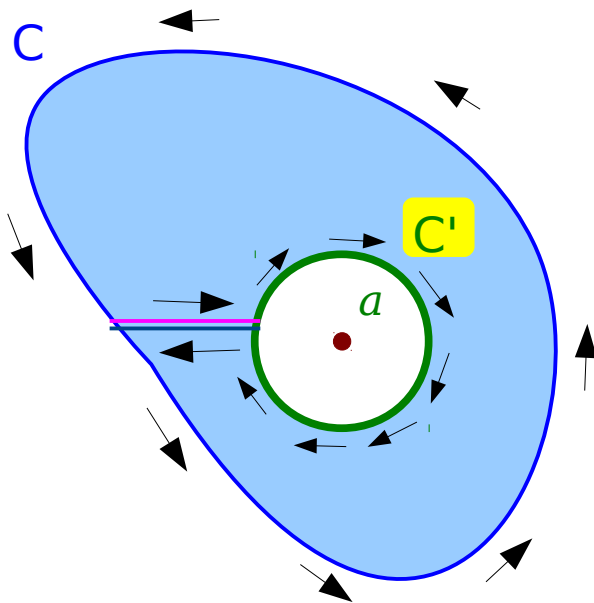
$\frac{dz}{z-a} = \frac{i\rho e^{i\theta} d\theta}{\rho e^{i\theta}}$

$\oint_{\text{ccw } C} \frac{f(z) dz}{z-a}$

$= \int_0^{2\pi} f(z) i d\theta$

$= 2\pi i f(a)$

as $z \rightarrow a \rightarrow \rho \rightarrow 0$



$\oint_{\text{ccw } C} \frac{f(z) dz}{z-a} = \oint_{\text{ccw } C'} \frac{f(z) dz}{z-a}$

$= 2\pi i f(a)$

References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”