# Line Integrals (4A)

- Line Integral
- Path Independence

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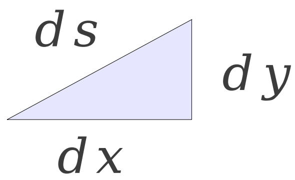
# Line Integral In the Plain

$$x = f(t) \quad \Rightarrow \quad \frac{dx}{dt} = f'(t) \quad \Rightarrow \quad dx = f'(t)\, dt$$

$$y = g(t) \quad \Rightarrow \quad \frac{dy}{dt} = g'(t) \quad \Rightarrow \quad dy = g'(t)\, dt$$

$$ds = \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Curve C $\quad a \le t \le b$




$$\int_C G(x, y)\, dx = \int_a^b G(f(t), g(t))\, f'(t)\, dt$$

$$\int_C G(x, y)\, dy = \int_a^b G(f(t), g(t))\, g'(t)\, dt$$

$$\int_C G(x, y)\, ds = \int_a^b G(f(t), g(t)) \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

# Line Integral In Space

$$x = f(t) \quad \Rightarrow \quad \frac{dx}{dt} = f'(t) \quad \Rightarrow \quad dx = f'(t)\,dt$$

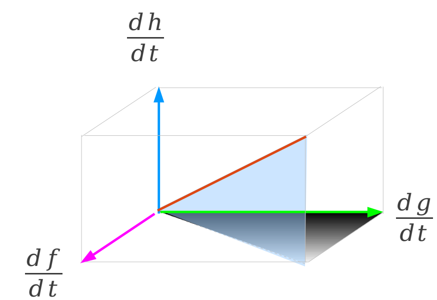
$$y = g(t) \quad \Rightarrow \quad \frac{dy}{dt} = g'(t) \quad \Rightarrow \quad dy = g'(t)\,dt$$

$$z = h(t) \quad \Rightarrow \quad \frac{dz}{dt} = h'(t) \quad \Rightarrow \quad dz = h'(t)\,dt$$

Curve C $\quad a \le t \le b$

$$ds = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$$

$$\int_C G(x, y, z)\,dz = \int_a^b G(f(t), g(t), h(t))\,h'(t)\,dt$$




$$\int_C G(x, y, z)\,ds = \int_a^b G(f(t), g(t), h(t))\sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$$

# Line Integral using $\boldsymbol{r}(t)$

Arc Length Parameter

s increases in the direction of increasing t

$$s(t) = \int_{t_0}^{t} |\boldsymbol{v}(\tau)|\, d\tau = \int_{t_0}^{t} \|\boldsymbol{r}'(\tau)\|\, d\tau = \int_{t_0}^{t} \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2 + [h'(\tau)]^2}\, d\tau$$

$$ds = |\boldsymbol{v}(t)|\, dt \qquad ds = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

$$\int_C G(x, y, z)\, ds = \int_a^b G(\boldsymbol{r}(t))\, |\boldsymbol{r}'(t)|\, dt$$

$$= \int_a^b G(f(t), g(t), h(t))\, |\boldsymbol{v}(t)|\, dt$$

$$= \int_a^b G(f(t), g(t), h(t)) \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

# Line Integral Notation

In many applications

$$\int_C G(x, y)\, ds = \int_C P(x, y)\, dx + \int_C Q(x, y)\, dy$$

$$= \int_C P(x, y)\, dx + Q(x, y)\, dy$$

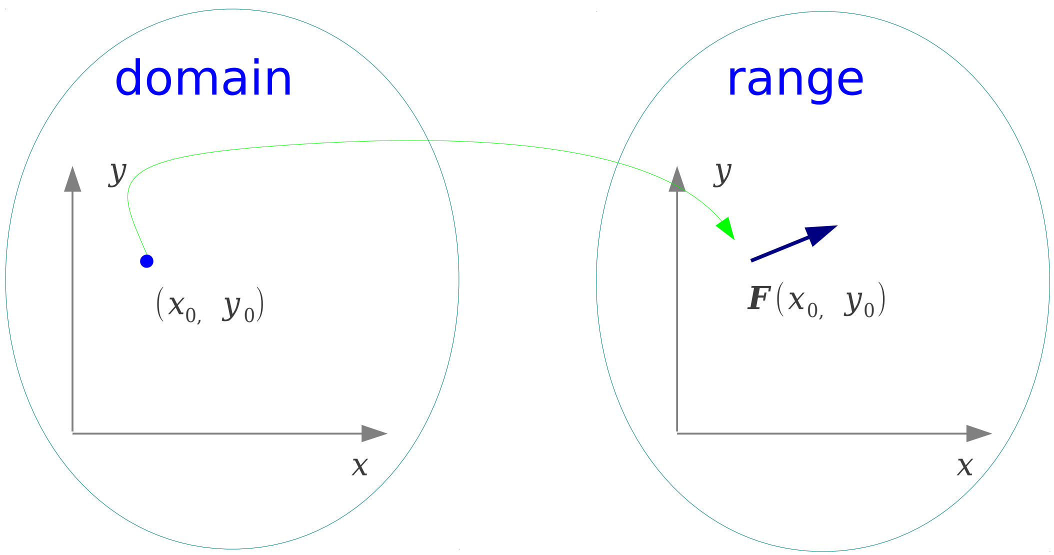
$$= \int_C P\, dx + Q\, dy$$

$$\int_C G(x, y, z)\, ds = \int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$$

# Line Integral over a 2-D Vector Field (1)

a given point in a 2-d space ➡ A vector

$(x_{0}, \ y_{0})$ $\langle P(x_{0}, y_{0}), \ Q(x_{0}, y_{0}) \rangle$




2 functions

$(x_{0}, \ y_{0}) \longrightarrow P(x_{0}, y_{0})$

$(x_{0}, \ y_{0}) \longrightarrow Q(x_{0}, y_{0})$

$(x_{0}, \ y_{0})$

only points that are
on the curve

$\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} \longrightarrow \boldsymbol{F}(x_{0}, \ y_{0}) = P(x_{0}, y_{0})\boldsymbol{i} + Q(x_{0}, y_{0})\boldsymbol{j}$

$x = f(t) \quad y = g(t) \quad a \leq t \leq b$

# Line Integral over a 2-D Vector Field (2)

$$\boldsymbol{r}(t) = f(t)\,\boldsymbol{i} + g(t)\,\boldsymbol{j}$$

$$\frac{d\boldsymbol{r}}{dt} = f'(t)\,\boldsymbol{i} + g'(t)\,\boldsymbol{j} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j}$$

$$\frac{d\boldsymbol{r}}{dt}dt = \left(\frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j}\right)dt = dx\,\boldsymbol{i} + dy\,\boldsymbol{j}$$

$$d\boldsymbol{r} = dx\,\boldsymbol{i} + dy\,\boldsymbol{j}$$

$$\boldsymbol{F}(x, y) = P(x, y)\,\boldsymbol{i} + Q(x, y)\,\boldsymbol{j}$$

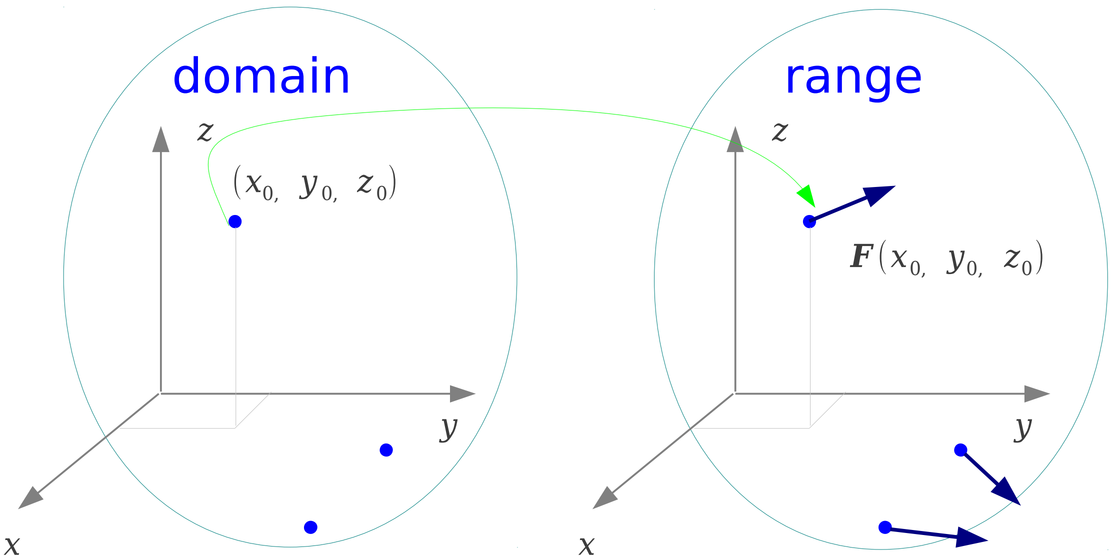
$$\boldsymbol{F} \cdot d\boldsymbol{r} = P(x, y)\,dx + Q(x, y)\,dy$$

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C P(x, y)\,dx + Q(x, y)\,dy$$

# Line Integral over a 3-D Vector Field (1)

A given point in a 3-d space ➡ A vector

$(x_{0,}\ \ y_{0,}\ \ z_0)$ $\quad \langle\, P(x_0, y_0, z_0),\ Q(x_0, y_0, z_0),\ R(x_0, y_0, z_0)\,\rangle$




3 functions

$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow P(x_{0,} y_{0,} Z_0)$

$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow Q(x_{0,} y_{0,} z_0)$

$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow R(x_{0,} y_{0,} z_0)$

$(x_{0,}\ \ y_{0,}\ \ z_0)$

only points that are on the curve $\longrightarrow$ $\boldsymbol{F}(x_{0,}\ \ y_{0,}\ \ z_0) = P(x_{0,} y_{0,} z_0)\boldsymbol{i} + Q(x_{0,} y_{0,} z_0)\boldsymbol{j} + R(x_{0,} y_{0,} z_0)\boldsymbol{k}$

$\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{j}$

$x = f(t) \quad y = g(t) \quad z = h(t) \quad a \le t \le b$

# Line Integral over a 3-D Vector Field (2)

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$$

$$\frac{d\mathbf{r}}{dt} = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

$$\frac{d\mathbf{r}}{dt}dt = \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right)dt = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$$

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$$

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

$$\mathbf{F}\cdot d\mathbf{r} = P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz$$

$$\int_c \mathbf{F}\cdot d\mathbf{r} = \int_C P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dy$$

# Work (1)

$$W = \boldsymbol{F} \cdot \boldsymbol{d}$$

A force field $\boldsymbol{F}(x, y) = P(x, y)\boldsymbol{i} + Q(x, y)\boldsymbol{j}$

A smooth curve $C : x = f(t),\ \ y = g(t),\ \ a \le t \le b$

Work done by **F** along C $W = \int_c \boldsymbol{F}(x, y) \cdot d\boldsymbol{r}$

$$= \int_C P(x, y)\, dx + Q(x, y)\, dy$$

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} \qquad d\boldsymbol{r} = \frac{d\boldsymbol{r}}{ds} ds \qquad d\boldsymbol{r} = \boldsymbol{T}\, ds$$

$$W = \int_c \boldsymbol{F} \cdot d\boldsymbol{r} = \int_c \boldsymbol{F} \cdot \boldsymbol{T}\, ds$$

# Work (2)

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} \qquad d\mathbf{r} = \frac{d\mathbf{r}}{ds}ds \qquad d\mathbf{r} = \mathbf{T}\,ds$$

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

$$= \int_{t_1}^{t_0} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt$$

$$= \int_{t_0}^{t_1} \left(P\frac{df}{dt} + Q\frac{dg}{dt} + R\frac{dh}{dt}\right)dt$$

$$= \int_{t_0}^{t_1} \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right)dt$$

$$= \int_{t_0}^{t_1} P\,dx + Q\,dy + R\,dz$$

$$\mathbf{F}(x,y,z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

$$= P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t)$$

# Circulation

A Simple Closed Curve C → Circulation

$$circulation = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_C \boldsymbol{F} \cdot \boldsymbol{T}\, d\boldsymbol{s}$$

# Conservative Vector Field

A vector function $\boldsymbol{F}$ in 2-d or 3-d space is conservative

⇐ $\boldsymbol{F}$ can be written as the gradient of a scalar function $\Phi$ $\quad \boldsymbol{F} = \nabla \Phi$

$$\boldsymbol{r}(t) = x(t)\,\boldsymbol{i} + y(t)\,\boldsymbol{j} \qquad a \le t \le b$$

$$\boldsymbol{F}(x, y) = P(x, y)\,\boldsymbol{i} + Q(x, y)\,\boldsymbol{j}$$ conservative

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \nabla \Phi \cdot d\boldsymbol{r} = \Phi(B) - \Phi(A)$$

$$A = (x(a), y(a))$$
$$B = (x(b), y(b))$$

Path Independence

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

# Connected

**Connected**

Every pair of points A and B in the region can be joined
by a piecewise smooth curve that lies entirely in the region

**Simply Connected**

Connected and every simple closed curve lying entirely within the region
can be shrunk, or contracted, to a point without leaving the region

➡ The interior of the curve lies also entirely in the region

➡ No holes in the region

**Disconnected**

Cannot be joined by a piecewise smooth curve
that lies entirely in the region

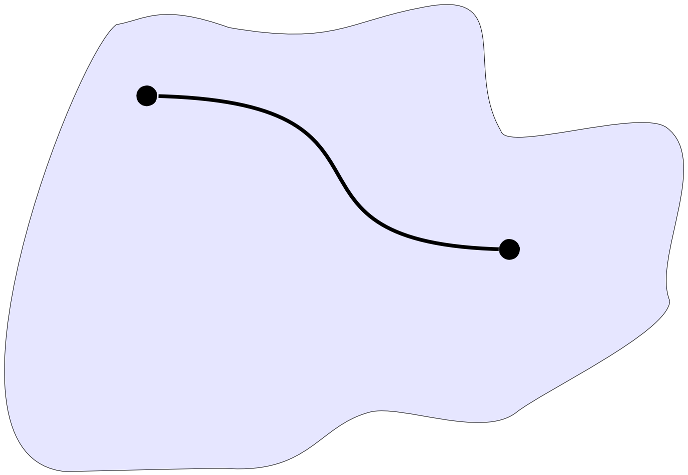
**Multiply Connected** Many holes within the region

**Open Connected** Contains no boundary points

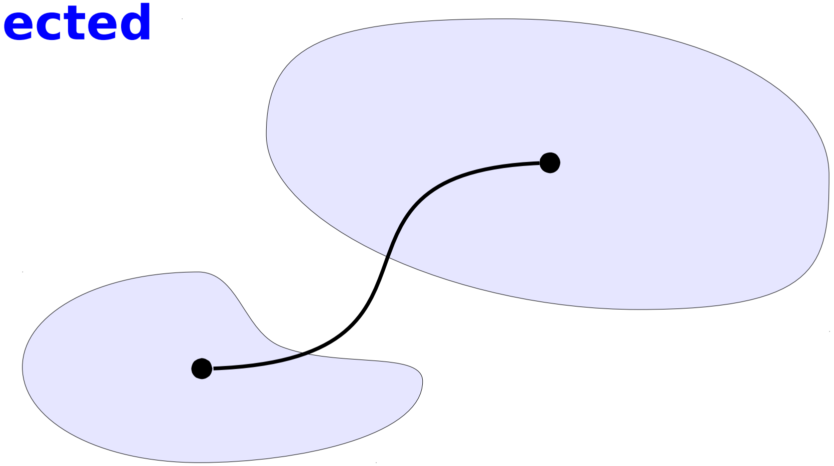
# Connected

**Connected**

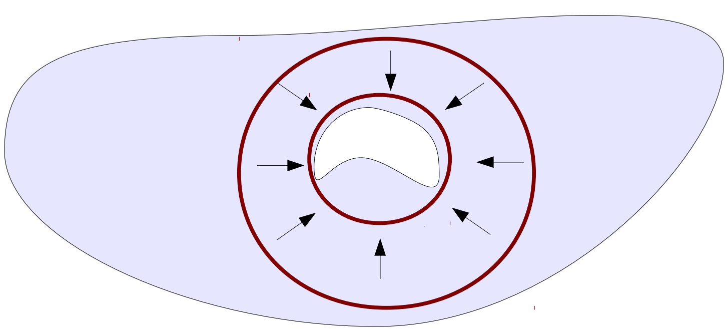
**Simply Connected**



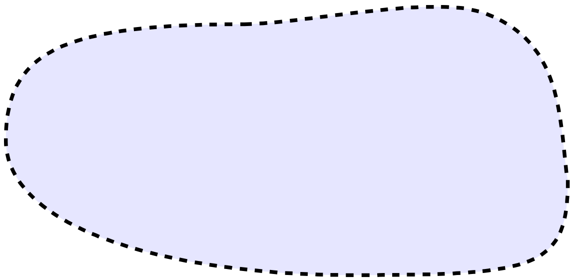
**Disconnected**



**Multiply Connected**



**Open Connected**



# Equivalence

In an open connected region

Path Independence $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$

⟺ Conservative $\boldsymbol{F} = \nabla \Phi$

⟺ Closed path C $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$

# Equivalence in 2-D

In an open connected region

Path Independence $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ $\Leftrightarrow$

Conservative $\boldsymbol{F} = \nabla \Phi$ $\Leftrightarrow$

Closed path C $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ $\Leftrightarrow$ $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

$$\boldsymbol{F} = P\,\boldsymbol{i} + Q\,\boldsymbol{j} \qquad \boldsymbol{F} = \nabla \Phi = \frac{\partial \Phi}{\partial x}\boldsymbol{i} + \frac{\partial \Phi}{\partial y}\boldsymbol{j}$$

# Equivalence in 3-D

In an open connected region

Path Independence $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ ⟺

Conservative $\boldsymbol{F} = \nabla \Phi$ ⟺

Closed path C $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ ⟺

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

$$curl\,\boldsymbol{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right)\boldsymbol{k}$$

$$\boldsymbol{F} = P\,\boldsymbol{i} + Q\,\boldsymbol{j} + R\,\boldsymbol{k} \qquad \boldsymbol{F} = \nabla\Phi = \frac{\partial \Phi}{\partial x}\boldsymbol{i} + \frac{\partial \Phi}{\partial y}\boldsymbol{j} + \frac{\partial \Phi}{\partial z}\boldsymbol{k}$$

# 2-Divergence

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y + \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x \qquad = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y$$

Flux density

$$= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

Divergence of **F**

Flux Density

## References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, "Mathematical Methods in the Physical Sciences"
[4] D.G. Zill, "Advanced Engineering Mathematics"