Correlation (1A)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Probability Density Function

Probability Density Function $f(x;\ t) = \dfrac{\partial}{\partial x} F(x;\ t)$

$$P[\, a \le X(t) \le b \,] = \int_a^b f(x;\ t)\,dx$$

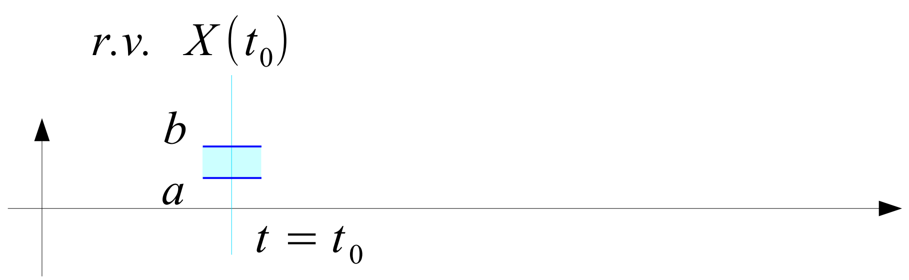
Cumulative Distribution Function $F(x;\ t) = \displaystyle\int_{-\infty}^{x} f(x;\ t)\,dx$

$$P[\, X(t) \le b \,] = \int_{-\infty}^{b} f(x;\ t)\,dx = F(b;\ t)$$

$X(t)$	*random variable X(t) at a given time t*
$f(x;\ t)$	*The probability density function of random variable X(t)*

$$P[\, x \le X(t) \le x + dt \,] = f(x;\ t)\, dx$$

$$P[\, X(t) = x \,] \Rightarrow f(x;\ t)$$



Probability Density Function

$X(t)$ *random variable X(t) at a given time t*

$f(x;\ t)$ *The probability density function of random variable X(t)*

$$P[\ x \le X(t) \le x + dt\ ] = f(x;\ t)\, dx$$

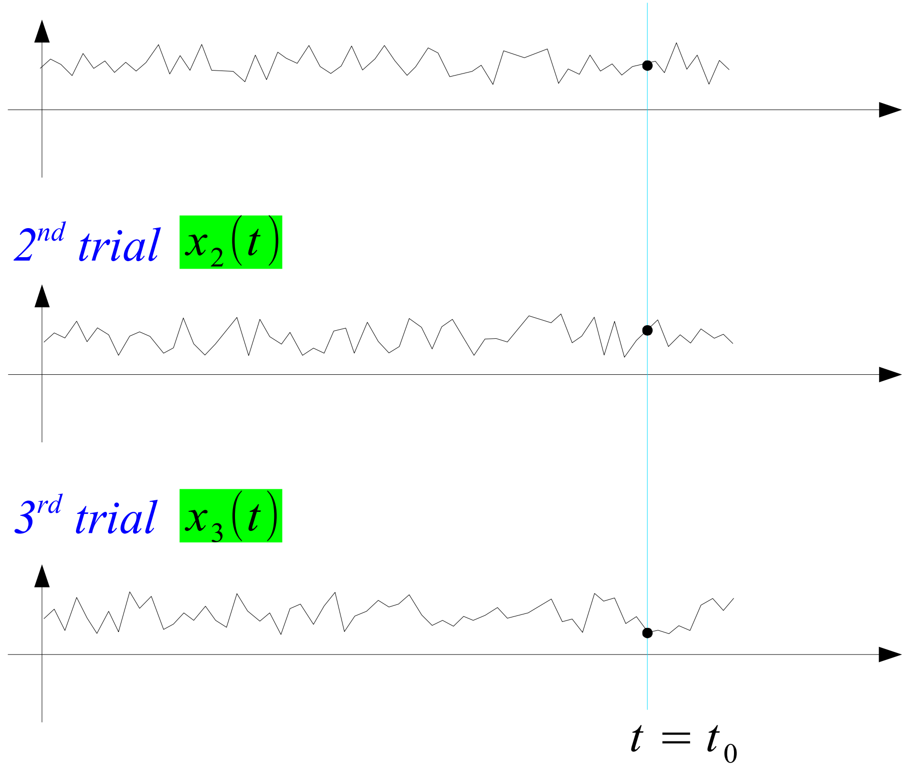
$$P[\ X(t) = x\ ] \Rightarrow f(x;\ t)$$

1st trial $x_1(t)$

$X(t_0)$ *random variable at a given time* t_0

2nd trial $x_2(t)$

3rd trial $x_3(t)$



$X(t_0) = x_1(t_0)$

r.v. $X(t_0)$ *has a value* $x_1(t_0)$ *at time* t_0 *in the **1st trial***

$X(t_0) = x_2(t_0)$

r.v. $X(t_0)$ *has a value* $x_2(t_0)$ *at time* t_0 *in the **2nd trial***

$X(t_0) = x_3(t_0)$

r.v. $X(t_0)$ *has a value* $x_3(t_0)$ *at time* t_0 *in the **3rd trial***

Joint Probability Density Function

Joint Probability Density Function $f(x_1, x_2;\ t_1, t_2) = \dfrac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2;\ t_1, t_2)$

$$P\left[\{a \le X(t_1) \le b\} \cap \{c \le X(t_2) \le d\}\right] = \int_a^b \int_c^d f(x_1, x_2;\ t_1, t_2)\, dx_1\, dx_2$$

Joint Cumulative Distribution Function $F(x_1, x_2;\ t_1, t_2) = \displaystyle\int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(x_1, x_2;\ t_1, t_2)\, dx_1\, dx_2$

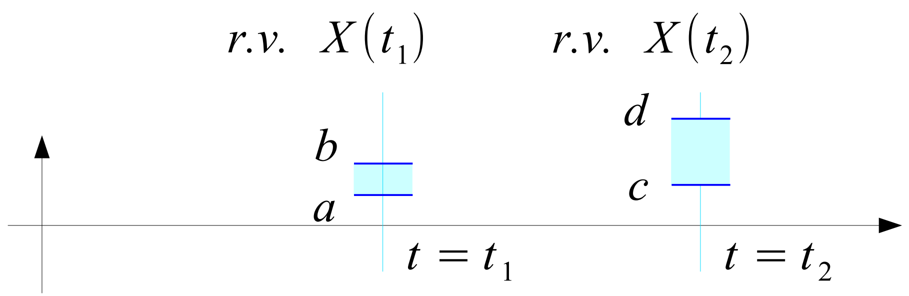
$$P\left[\{X(t_1) \le b\} \cap \{X(t_2) \le d\}\right] = \int_{-\infty}^{b} \int_{-\infty}^{d} f(x_1, x_2;\ t_1, t_2)\, dx_1\, dx_2 = F(b, d;\ t_1, t_2)$$

$X(t_1),\ X(t_2)$ *random variables at time t_1 and t_2*

$f(x_1, x_2;\ t_1, t_2)$ *joint probability density function*

$P\left[\{X(t_1) = x_1\} \cap \{X(t_2) = x_2\}\right]$

$\Rightarrow f(x_1, x_2;\ t_1, t_2)$



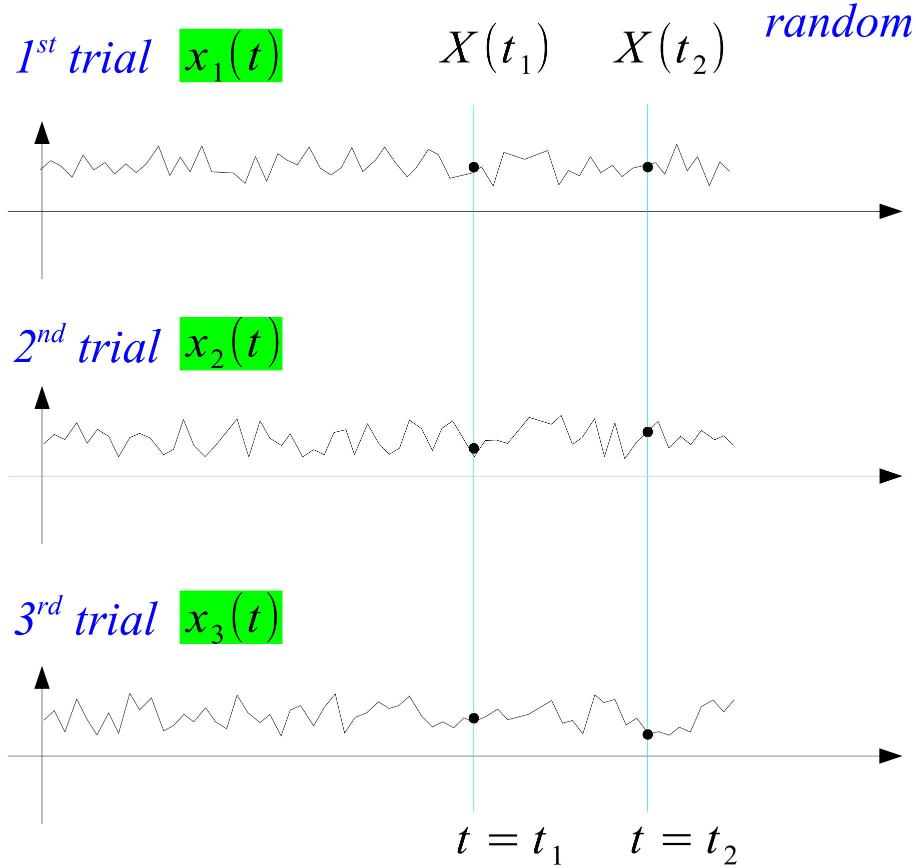
Probability Density Function

$X(t_1),\ X(t_2)$ *random variables at time t_1 and t_2*

$f(x_1, x_2;\ t_1, t_2)$ *joint probability density function*

$$P\left[\{X(t_1) = x_1\} \cap \{X(t_2) = x_2\}\right]$$
$$\Rightarrow f(x_1, x_2;\ t_1, t_2)$$

random variable at a given time t_0



$X(t_1) = x_1(t_1),\quad X(t_2) = x_1(t_2)$

r.v. $X(t_1)$ *has a value* $x_1(t_1)$
r.v. $X(t_2)$ *has a value* $x_1(t_2)$
at times $t_1,\ t_2$ *in the* ***1st trial***

$X(t_1) = x_2(t_1),\quad X(t_2) = x_2(t_2)$

r.v. $X(t_1)$ *has a value* $x_2(t_1)$
r.v. $X(t_2)$ *has a value* $x_2(t_2)$
at times $t_1,\ t_2$ *in the* ***2nd trial***

$X(t_1) = x_3(t_1),\quad X(t_2) = x_3(t_2)$

r.v. $X(t_1)$ *has a value* $x_3(t_1)$
r.v. $X(t_2)$ *has a value* $x_3(t_2)$
at times $t_1,\ t_2$ *in the* ***3rd trial***

Moments of a Random Process

The n-th Moment

$$E[\, X^n(t) \,] = \int_{-\infty}^{+\infty} x^n \, f(x\,;\, t) \, dx$$

The n-th Central Moment

$$E[\, (X(t) - \mu(t))^n \,] = \int_{-\infty}^{+\infty} (x - \mu(t))^n \, f(x\,;\, t) \, dx$$

1st Moment

$$E[\, X(t) \,] = \int_{-\infty}^{+\infty} x \, f(x\,;\, t) \, dx$$

$$= \mu(t)$$

1st Central Moment

$$E[\, (X(t) - \mu(t)) \,] = \int_{-\infty}^{+\infty} (x - \mu(t)) \, f(x\,;\, t) \, dx$$

2nd Moment

$$E[\, X^2(t) \,] = \int_{-\infty}^{+\infty} x^2 \, f(x\,;\, t) \, dx$$

2nd Central Moment

$$E[\, (X(t) - \mu(t))^2 \,] = \int_{-\infty}^{+\infty} (x - \mu(t))^2 \, f(x\,;\, t) \, dx$$

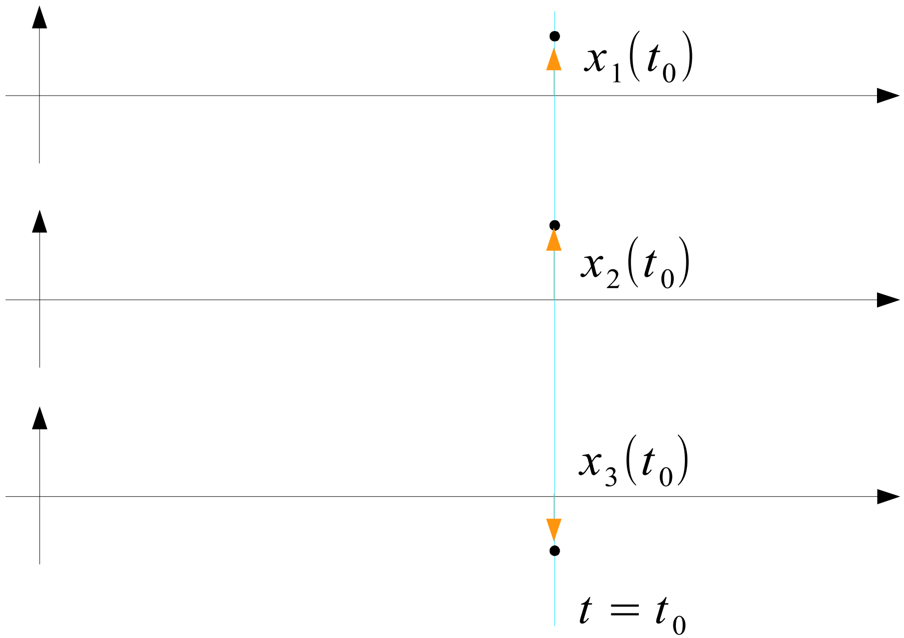
$$= \sigma^2(t)$$

Moments of a Random Process

$X(t)$ Random Variable at a given time t

$x_i(t)$ outcome of i^{th} realization at a given time t

Ensemble



$$E[\,X(t_0)\,] = \int_{-\infty}^{+\infty} x \; f(x;\, t_0)\, dx = \mu(t_0)$$

Ensemble Average

$$\bar{x}(t) = \frac{1}{N}\sum_{n=1}^{N} x_n(t)$$

$N \rightarrow \infty$

$$E[\,X(t)\,] = \int_{-\infty}^{+\infty} x \; f(x;\, t)\, dx$$

$$= \mu(t)$$

Stationarity

First-Order Stationary Process

$$f(x_1;\ t_1) = f(x_1;\ t_1+k) \qquad \forall k$$

$$f(x;\ t) \Rightarrow f(x)$$

Second-Order Stationary Process

$$f(x_1, x_2;\ t_1, t_2) = f(x_1, x_2;\ t_1+k, t_2+k) \qquad \forall k$$

$$f(x_1, x_2;\ t_1, t_2) = f(x_1, x_2;\ 0, t_2-t_1) \qquad k = -t_1$$

$$f(x_1, x_2;\ t_1, t_2) \Rightarrow f(x_1, x_2;\ t_2-t_1)$$

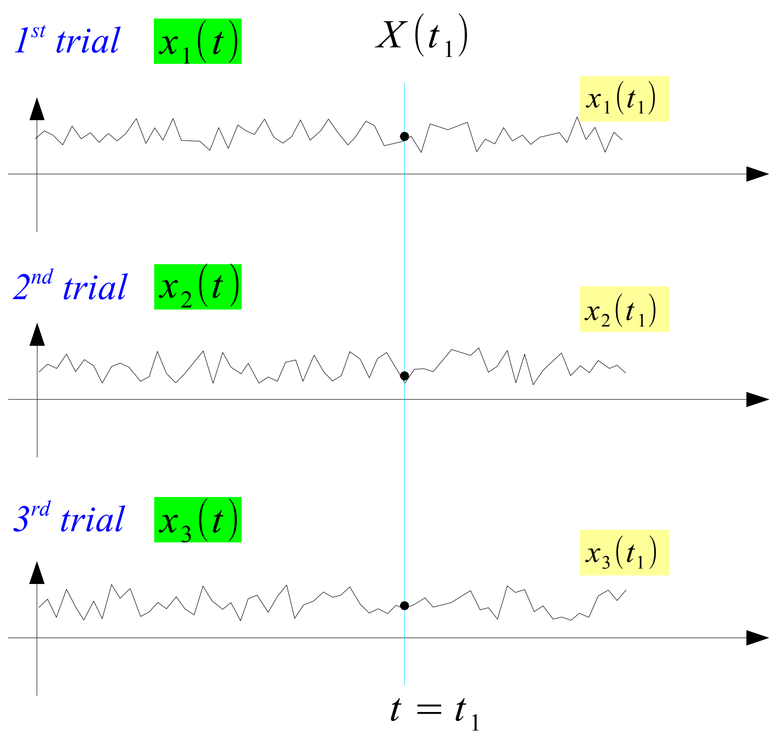
Nth-Order Stationary Process

$$f(x_1, x_2, \cdots, x_n;\ t_1, t_2, \cdots, t_n) = f(x_1, x_2, \cdots, x_n;\ t_1+k, t_2+k, \cdots, t_n+k) \qquad \forall k$$

Mean Function

Mean Function

$$\mu_x(t) = E[\, X(t) \,]$$



Ensemble Average

$$\overline{x(t)} = \frac{1}{N} \sum_{n=1}^{N} x_n(t)$$

$$N \rightarrow \infty$$

$$E[\, X(t) \,] = \int_{-\infty}^{+\infty} x \; f(x\,;\,t)\; dx$$

$$= \mu(t)$$

First Order Stationary Process

The n-th Moment

$$E[\,X^n(t)\,] = \int_{-\infty}^{+\infty} x^n \, f(x\,;\,t)\, dx$$

The n-th Central Moment

$$E[\,(X(t)-\mu(t))^n\,] = \int_{-\infty}^{+\infty} (x-\mu(t))^n \, f(x\,;\,t)\, dx$$

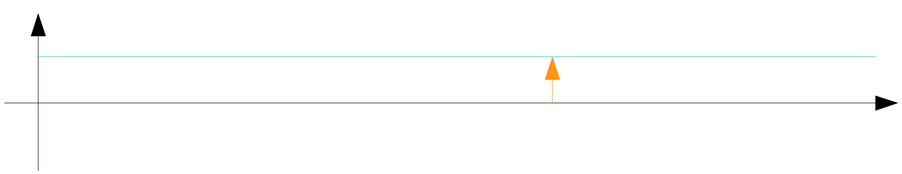
$$f(x,t) \Rightarrow f(x)$$

*if **f** does not change with time*

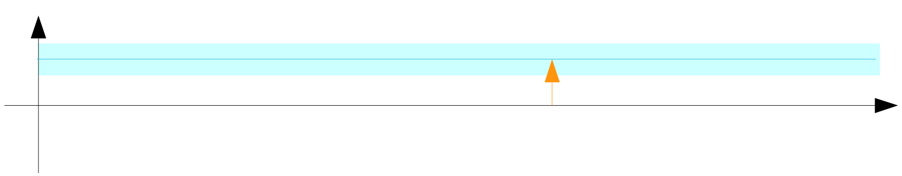
$$E[\,X^n(t)\,] \Rightarrow E[\,X^n\,]$$

$$E[\,(X(t)-\mu(t))^n\,] \Rightarrow E[\,(X-\mu)^n\,]$$

First Order Stationary Process



$$E[\,X(t)\,] \Rightarrow E[\,X\,] = \mu$$

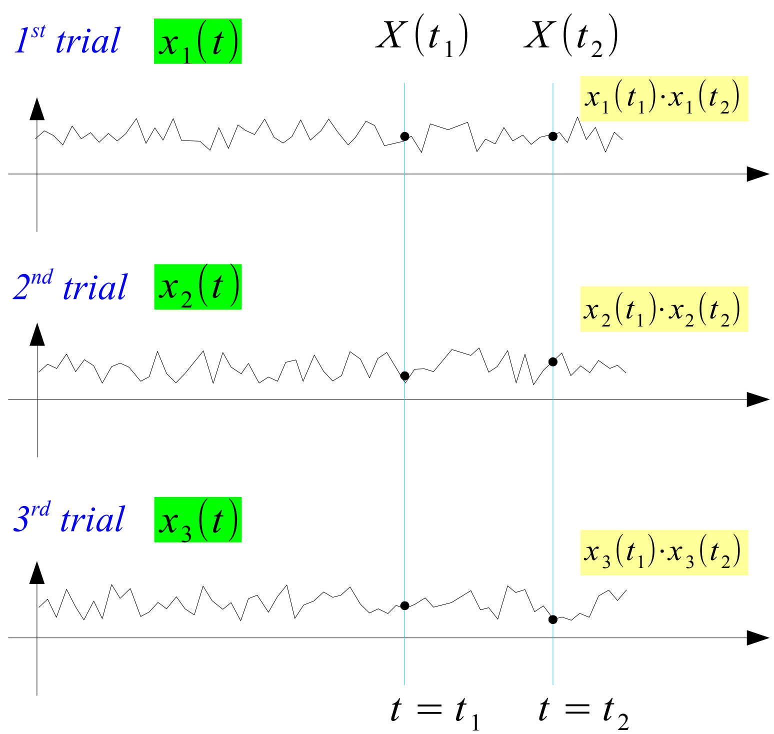


$$E[\,(X(t)-\mu(t))^2\,] \Rightarrow E[\,(X-\mu)^2\,] = \sigma^2$$

AutoCorrelation Functions

Auto-correlation Function

$$R_{xx}(t_1, t_2) = E[\,X(t_1)\,X(t_2)\,]$$



Ensemble Average

$$\overline{x(t_1)\,x(t_2)} = \frac{1}{N}\sum_{n=1}^{N} x_n(t_1)\cdot x_n(t_2)$$

$$N \to \infty$$

$$E[\,X(t_1)\,X(t_2)\,] = R_{xx}(t_1, t_2)$$

$$= \int_{-\infty}^{+\infty} x_1\, x_2\, f(x_1, x_2;\ t_1, t_2)\, dx_1\, dx_2$$

AutoCorrelation Function Examples

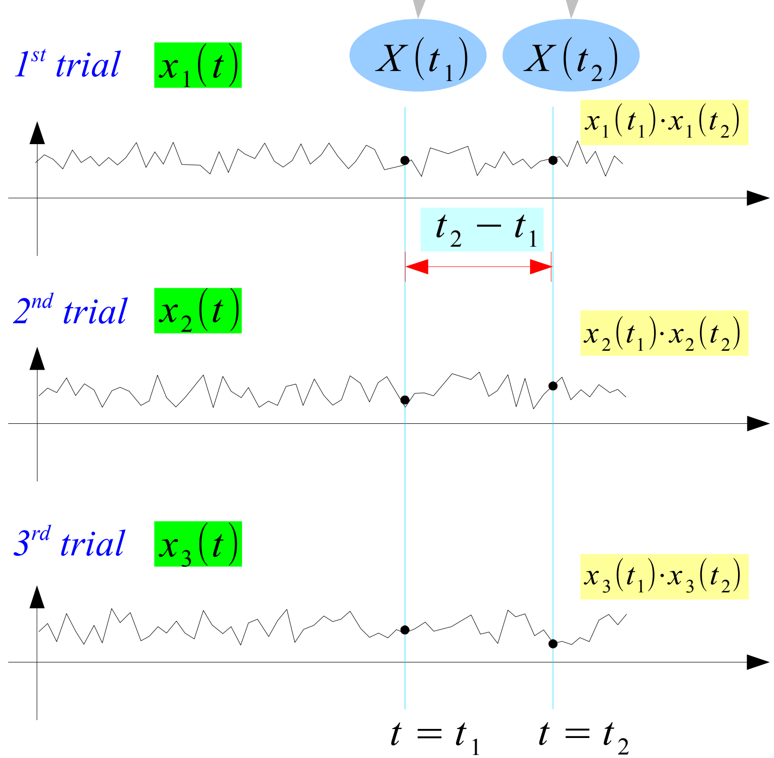
auto-correlation function

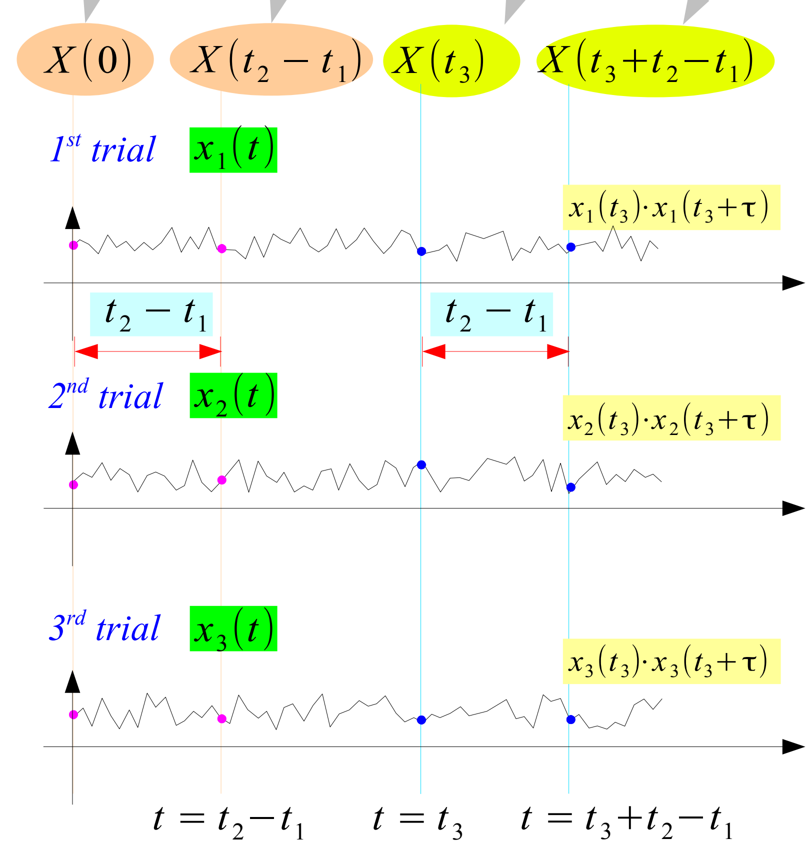
$\tau = t_2 - t_1$

$$R_{xx}(t_1, t_2) = E[\, X(t_1)\, X(t_2)\,]$$

$$R_{xx}(0, t_2 - t_1) = E[\, X(0)\, X(t_2 - t_1)\,]$$

$$R_{xx}(t_3, t_3 + t_2 - t_1) = E[\, X(t_3)\, X(t_3 + t_2 - t_1)\,]$$





Second Order Stationary Process

auto-correlation function

$$R_{xx}(t_1, t_2) = E[\,X(t_1)\,X(t_2)\,]$$

2nd Order Stationary Process

$$f(x_1, x_2;\ t_1, t_2) \Rightarrow f(x_1, x_2;\ t_2 - t_1)$$

$$R_{xx}(t_1, t_2) = R_{xx}(0, t_2 - t_1) = R_{xx}(t_3, t_3 + t_2 - t_1) =$$

$$E[\,X(t_1)\,X(t_2)\,] = E[\,X(0)\,X(t_2 - t_1)\,] =$$

$$E[\,X(t_3)\,X(t_3 + t_2 - t_1)\,]$$

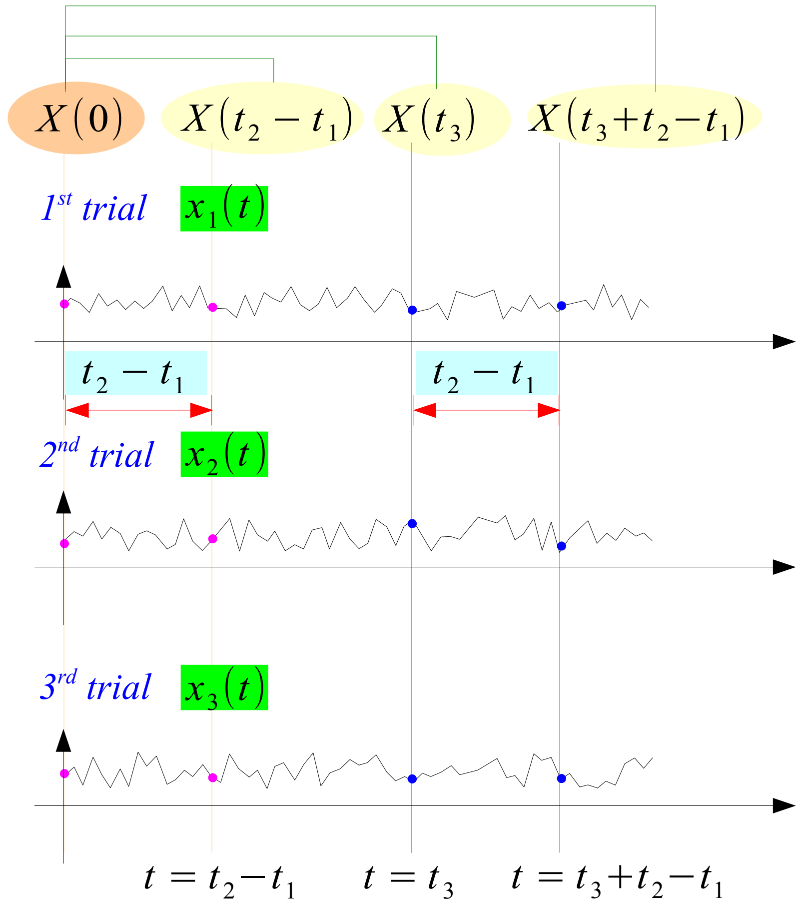
$$\Rightarrow R_{xx}(t_1 - t_2) = R_{xx}(\tau) \qquad \tau = t_2 - t_1$$

$$R_{xx}(\tau) = E[\,X(t)\,X(t + \tau)\,]$$

$$R_{xx}(t_3 + t_2 - t_1) = E[\,X(0)\,X(t_3 + t_2 - t_1)\,]$$

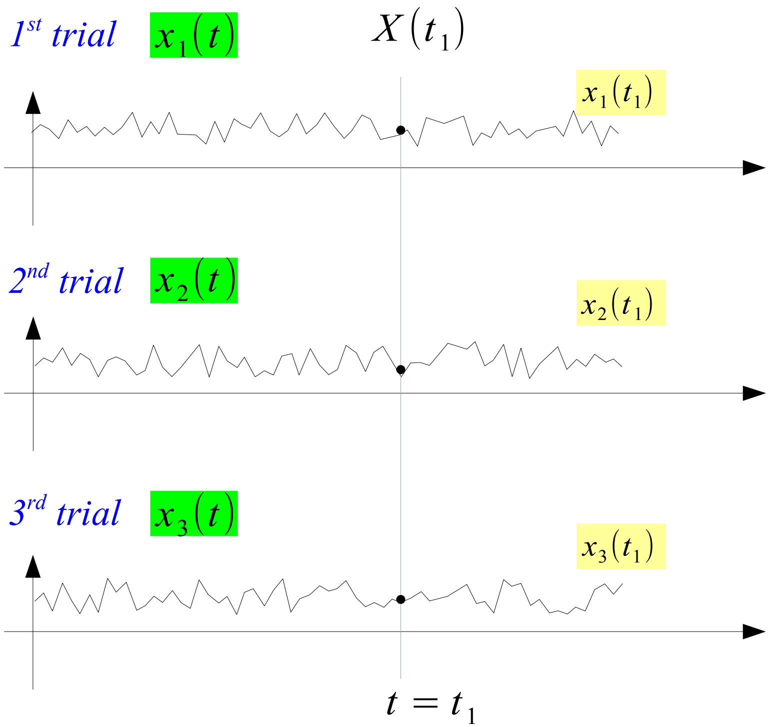
$$R_{xx}(t_3) = E[\,X(0)\,X(t_3)\,]$$

$$R_{xx}(t_2 - t_1) = E[\,X(0)\,X(t_2 - t_1)\,]$$



Time Average & Mean

mean function



Time Average

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)\, dt = \mu_x$$

Ensemble Average

$$\overline{x(t)} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} x_n(t)$$

$$E[\, X(t) \,] = \int_{-\infty}^{+\infty} x\, f(x\,;\, t)\, dx = \mu(t)$$

Stationary Process

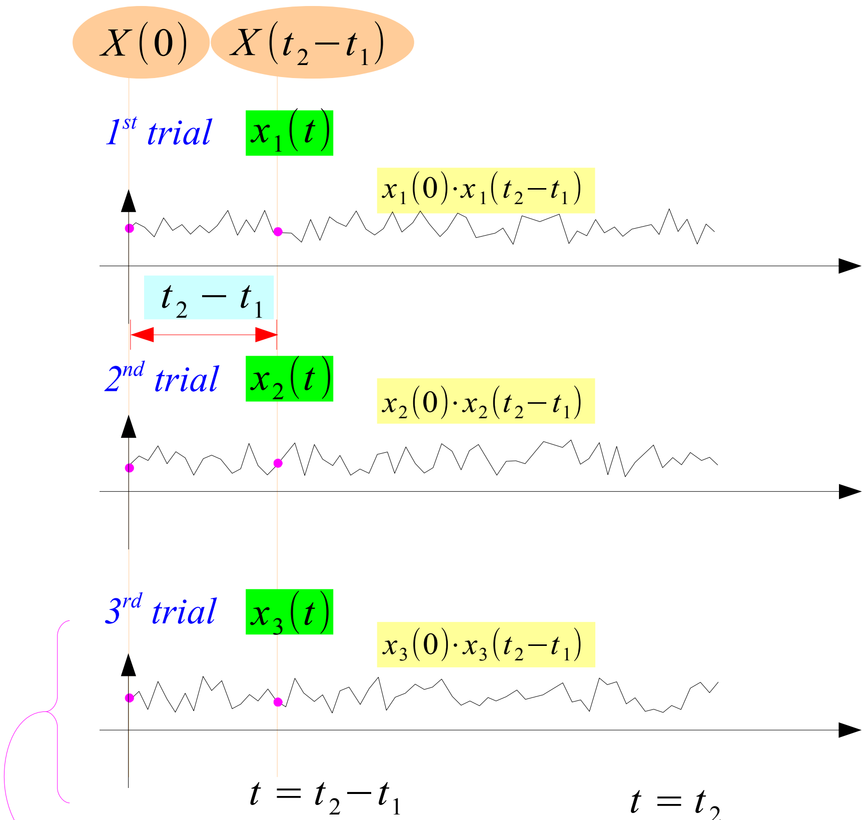
$$\mu(t) = const$$

Ergodic Process

$$\mu(t) = \mu_x$$

Time Average & AutoCorrelation

autocorrelation function



Time Average

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)\, x(t + t_2 - t_1)\, dt$$

Ensemble Average

$$\overline{x(t_1)\, x(t_2)} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} x_n(t_1) \cdot x_n(t_2)$$

$$E[\, X(t_1)\, X(t_2)\,] = R_{xx}(t_1,\, t_2)$$

Stationary Process

$$E[\, X(t_1)\, X(t_2)\,] = R_{xx}(t_2 - t_1)$$

Ergodic Process

$$R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)\, x(t + \tau)\, dt$$

Time Average & Ensemble Average

autocorrelation function

Ensemble Average

Time Average

Time Average & Ensemble Average

autocorrelation function

Ensemble Average

Moving Average Filter

Matched Filter

Correlation Functions (1)

auto-covariance function

$$C_{xx}(t_1, t_2) = E\left[\underline{(X(t_1) - \mu_x(t_1))} \; \underline{(X(t_2) - \mu_x(t_2))} \right]$$

$$C_{xx}(t_2 - t_1) = E\left[(X(t_1) - \mu_x(t_1)) \; (X(t_2) - \mu_x(t_2)) \right]$$

$$C_{xx}(\tau) = E\left[(X(t) - \mu_x) \; (X(t+\tau) - \mu_x) \right]$$

auto-correlation function

$$R_{xx}(t_1, t_2) = E\left[\underline{X(t_1)} \; \underline{X(t_2)} \right]$$

$$R_{xx}(\tau) = E\left[X(t) \; X(t + \tau) \right]$$

Correlation Functions (2)

cross-covariance function

$$C_{xy}(t_1, t_2) = E\left[\underline{(X(t_1) - \mu_x(t_1))} \; \underline{(Y(t_2) - \mu_y(t_2))} \right]$$

$$C_{xy}(\tau) = E\left[(X(t) - \mu_x) \; (Y(t+\tau) - \mu_y) \right]$$

cross-correlation function

$$R_{xy}(t_1, t_2) = E\left[\underline{X(t_1)} \; \underline{Y(t_2)} \right]$$

$$R_{xy}(\tau) = E\left[X(t) \, Y(t + \tau) \right]$$

Ergodicity

$$\mu_x = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)\, dt$$

$$\hat{\mu}_x = \frac{1}{T} \int_0^T x(t)\, dt = \bar{x}$$

$$\bar{\psi}_x^2 = \frac{1}{T} \int_0^T x^2(t)\, dt = \overline{x^2}$$

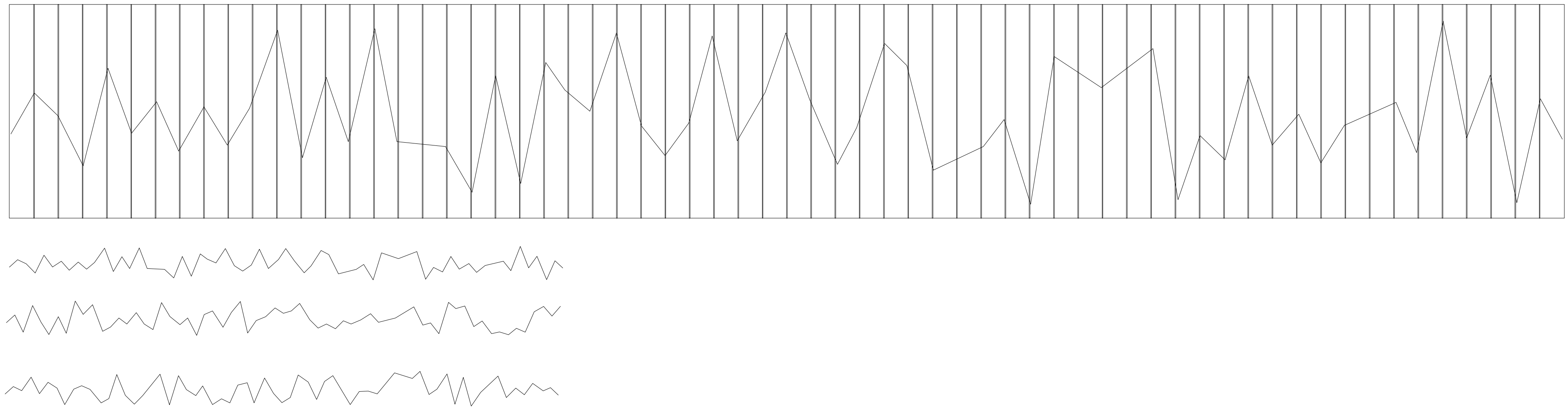
$$C_{xy}(\tau) = E\left[(X(t) - \mu_x)\ (Y(t+\tau) - \mu_y) \right]$$

$$C_{xy}(\tau) = \lim_{T \to 0} \frac{1}{T} \int_0^T (x(t) - \mu_x)\ (y(t+\tau) - \mu_y)\, dt$$

$$\hat{C}_{xy}(\tau) = \frac{1}{T-\tau} \int_0^{T-\tau} (x(t) - \bar{x})\ (y(t+\tau) - \bar{y})\, dt \qquad 0 \le \tau < T$$

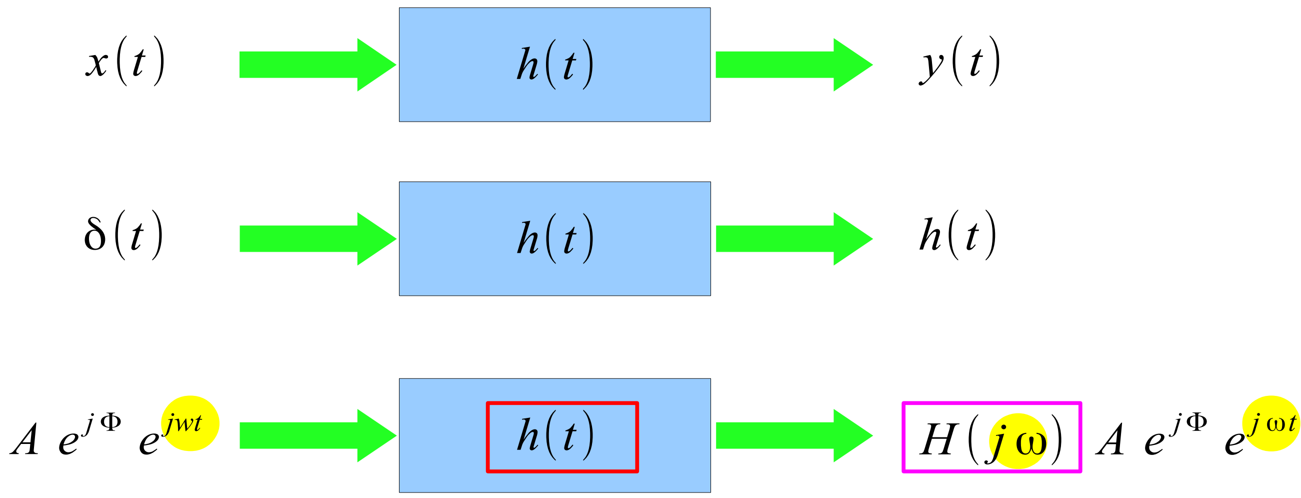
$$\hat{C}_{xy}(\tau) = \frac{1}{T-|\tau|} \int_0^{T-|\tau|} (x(t) - \bar{x})\ (y(t+\tau) - \bar{y})\, dt \qquad -T < \tau \le 0$$

Time Average



Frequency Response

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$



single frequency component : ω

single frequency component : ω

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] K.H. Shin, J.K. Hammond, Fundamentals of Signal Processing for Sound and Vibration Engineers, Wiley, 2008