

# Minimum Phase (2A)

---

Copyright (c) 2011 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

Please send corrections (or suggestions) to [youngwlim@hotmail.com](mailto:youngwlim@hotmail.com).

This document was produced by using OpenOffice and Octave.

# Properties of a Minimum Phase System

Lowest Time Delay

Group Delay

Energy Compaction

Invertible

Min Phase Filter

{ flat response  
correct phase response

Equalizer

{ flat response  
incorrect phase response

# Minimum Phase System

Stable Causal System

All its poles are in the left half of the s plane

Minimum Phase System



Maximum Phase System



Mixed Phase System



# Minimum Phase System Properties (1)

## Minimum Phase System

Amplitude Response is known

$$A(\omega) = |H(j\omega)| \quad 0 \leq \omega < \infty$$



Minimum Phase Response  
can be computed

$$\Phi_{min}(\omega) = \arg\{H(j\omega)\}$$

## Non-Minimum Phase System

Poles / Zeros in the right half s plane

Amplitude Response is known

$$A(\omega) = |H(j\omega)| \quad 0 \leq \omega < \infty$$

Phase Response  
always greater

$$\Phi(\omega) \geq \Phi_{min}(\omega)$$

# Minimum Phase System Properties (2)

## Minimum Phase System

Phase Response  $\Phi(\omega)$  can be unambiguously determined from the amplitude response  $A(\omega)$



## Non-Minimum Phase System

Not valid

## Verification of a Minimum Phase System

Check the progression of  $\Phi(\omega)$  and  $A(\omega)$  at high frequency

$$H(\omega) = \frac{N(s)}{D(s)}$$

Minimum Phase System



$$\Phi(\omega) = -90^\circ (n - m)$$

Minimum Phase System  
Non-Minimum

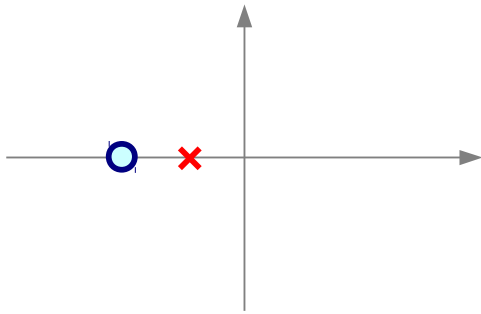


$$-20(n - m) \text{ dB / decade}$$

# Example

## Minimum Phase System

$$H(s) = \frac{1 + 2s}{1 + 4s}$$

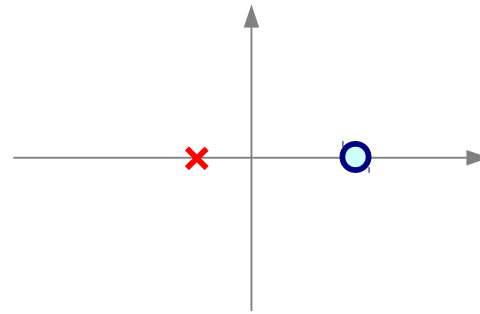


$$\begin{aligned} \frac{1 + j2\omega}{1 + j4\omega} &= \frac{1 + j2\omega}{1 + j4\omega} \cdot \frac{1 - j4\omega}{1 - j4\omega} \\ &= \frac{(1 + 8\omega^2) - j2\omega}{1 + 16\omega^2} \end{aligned}$$

$$\Phi(\omega) = -\tan^{-1}\left(\frac{2\omega}{1 + 8\omega^2}\right)$$

## Non-Minimum Phase System

$$H(s) = \frac{1 - 2s}{1 + 4s}$$



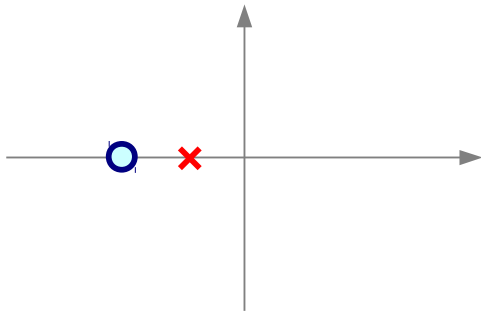
$$\begin{aligned} \frac{1 - j2\omega}{1 + j4\omega} &= \frac{1 - j2\omega}{1 + j4\omega} \cdot \frac{1 - j4\omega}{1 - j4\omega} \\ &= \frac{(1 - 8\omega^2) - j6\omega}{1 + 16\omega^2} \end{aligned}$$

$$\Phi(\omega) = -\tan^{-1}\left(\frac{6\omega}{1 + 8\omega^2}\right)$$

# Non-minimum Phase System

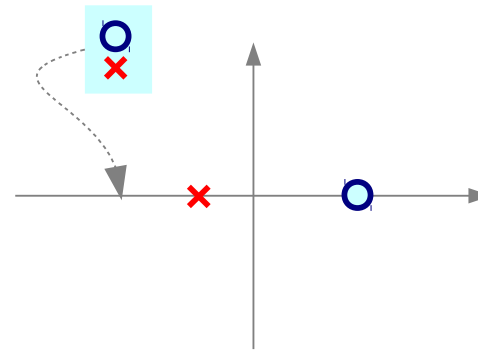
## Minimum Phase System

$$H(s) = \frac{1 + 2s}{1 + 4s}$$

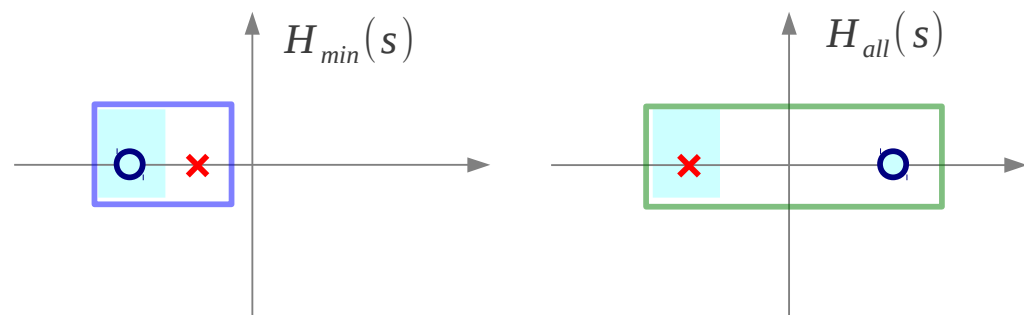


## Non-Minimum Phase System

$$H(s) = \frac{1 - 2s}{1 + 4s} = \frac{1 - 2s}{1 + 4s} \cdot \frac{1 + 2s}{1 + 2s}$$



$$\begin{aligned} H(s) &= \frac{1 - 2s}{1 + 4s} \cdot \frac{1 + 2s}{1 + 2s} \\ &= \frac{1 + 2s}{1 + 4s} \cdot \frac{1 - 2s}{1 + 2s} \\ &= H_{min}(s) \cdot H_{all}(s) \end{aligned}$$



A non-minimum phase system can always be decomposed into  $H_{min}(s) \cdot H_{all}(s)$



# All Pass Filter (1)

$$H_{all}(s) = \frac{1 - 2s}{1 + 2s}$$

Flat Magnitude

$$\begin{aligned} \left| \frac{1 - j2\omega}{1 + j2\omega} \right| &= \frac{|1 - j2\omega|}{|1 + j2\omega|} \\ &= \frac{\sqrt{1 + 4\omega^2}}{\sqrt{1 + 4\omega^2}} = 1 \end{aligned}$$

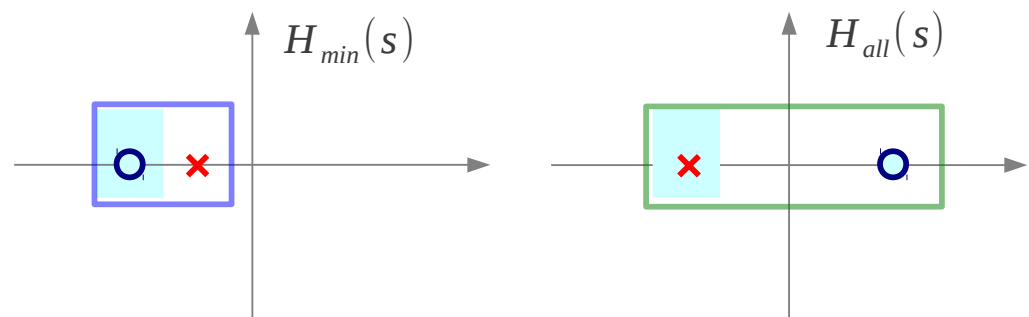
A Pure Phase Shifter

$$\begin{aligned} \frac{1 - j2\omega}{1 + j2\omega} &= \frac{1 - j2\omega}{1 + j2\omega} \cdot \frac{1 - j2\omega}{1 - j2\omega} \\ &= \frac{(1 - 4\omega^2) - j4\omega}{1 + 4\omega^2} \end{aligned}$$

$$|H_{all}(j\omega)| = \frac{1 + 4\omega^2}{1 + 4\omega^2} = 1$$

$$\arg\{H_{all}(j\omega)\} = -\tan^{-1}\left(\frac{4\omega}{1 - 4\omega^2}\right)$$

$$\begin{aligned} H(s) &= \frac{1 - 2s}{1 + 4s} \cdot \frac{1 + 2s}{1 + 2s} \\ &= \frac{1 + 2s}{1 + 4s} \cdot \frac{1 - 2s}{1 + 2s} \\ &= H_{min}(s) \cdot H_{all}(s) \end{aligned}$$



A non-minimum phase system can always be decomposed into  $H_{min}(s) \cdot H_{all}(s)$

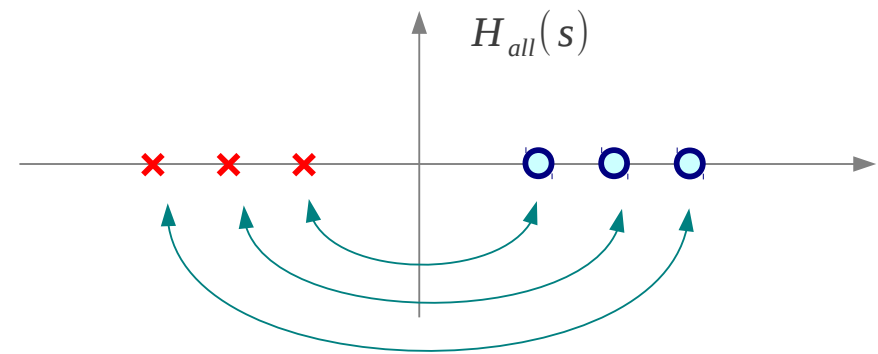
# All Pass Filter (2)

$$G_{all}(s) = \pm \frac{(s - \bar{s}_1)(s - \bar{s}_2) \cdots (s - \bar{s}_n)}{(s - s_1)(s - s_2) \cdots (s - s_n)}$$

zero  $\bar{s}_i$   
pole  $s_i$   complex conjugate

Flat Magnitude

A Pure Phase Shifter



# All Pass Filter (3)

$$\begin{aligned}H_{all}(s) &= \frac{s - 0.5}{s + 0.5} \\ &= \frac{s + 0.5 - 1}{s + 0.5}\end{aligned}$$

$$H(s) = 1 - \frac{2}{(s + 0.5)}$$



Inverse Laplace Transform

$$h(t) = \delta(t) - e^{-0.5t}$$

$$H_{all}(j\omega) = \frac{j\omega - 0.5}{j\omega + 0.5}$$

Flat Magnitude

$$|H_{all}(j\omega)| = \frac{\sqrt{\omega^2 + 0.25}}{\sqrt{\omega^2 + 0.25}} = 1$$

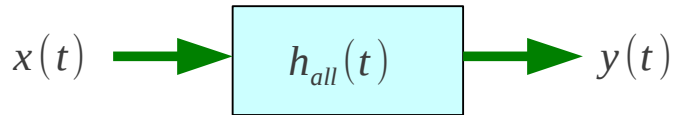
Phase Shifter

$$\arg\{H_{all}(j\omega)\} = -2 \tan^{-1}\left(\frac{\omega}{0.5}\right)$$

Group Delay

$$\begin{aligned}-\frac{d}{d\omega}(\arg\{H_{all}(j\omega)\}) \\ &= -\frac{d}{d\omega} \left( -2 \tan^{-1}\left(\frac{\omega}{0.5}\right) \right) \\ &= \frac{4}{(1 + \omega^2/0.25)} > 0\end{aligned}$$

# All Pass Filter (4)

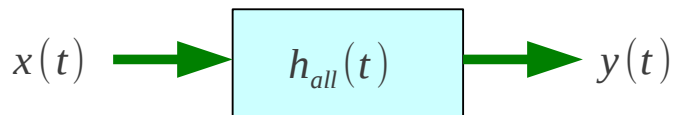


## Parseval's Theorem

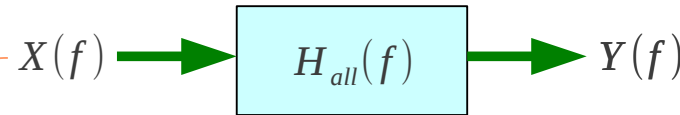
$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |y(t)|^2 dt$$

## Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2 dt \geq \int_{-\infty}^{t_0} |y(t)|^2 dt$$



The energy build-up in the input is more **rapid** than in the output



## Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 dt$$

$$\int_{-\infty}^{+\infty} |y(t)|^2 dt = \int_{-\infty}^{+\infty} |Y(f)|^2 dt$$

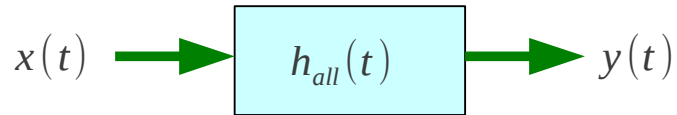
$$Y(f) = H_{all}(f)X(f) \Rightarrow \int_{-\infty}^{+\infty} |H_{all}(f)|^2 |X(f)|^2 dt$$

$$|H_{all}(f)| = 1 \Rightarrow \int_{-\infty}^{+\infty} |X(f)|^2 dt$$

Allpass Filter

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |y(t)|^2 dt$$

# All Pass Filter (5)

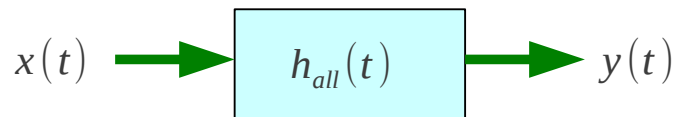


## Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |y(t)|^2 dt$$

## Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2 dt \geq \int_{-\infty}^{t_0} |y(t)|^2 dt$$



The energy build-up in the input is more **rapid** than in the output

*Truncated input*

$$x_1(t) = \begin{cases} x(t) & (t \leq t_0) \\ 0 & (t > t_0) \end{cases}$$

For  $t \leq t_0$   $\Rightarrow x_1(t) = x(t) \Rightarrow$

$$y_1(t) = \int_{-\infty}^{t_0} h(t-\tau)x_1(\tau)d\tau = \int_{-\infty}^t h(t-\tau)x(\tau)d\tau = y(t)$$

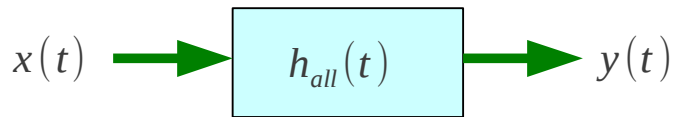
$$\int_{-\infty}^{+\infty} |x_1(t)|^2 dt = \int_{-\infty}^{+\infty} |y_1(t)|^2 dt$$

For  $t > t_0$   $\Rightarrow x_1(t) = 0 \Rightarrow$

$$\int_{-\infty}^{t_0} |x_1(t)|^2 dt = \int_{-\infty}^{+\infty} |y_1(t)|^2 dt = \int_{-\infty}^{t_0} |y_1(t)|^2 dt + \int_{t_0}^{+\infty} |y_1(t)|^2 dt$$

$$\int_{-\infty}^{t_0} |x(t)|^2 dt \geq \int_{-\infty}^{t_0} |y(t)|^2 dt \quad \text{For } t \leq t_0$$

# All Pass Filter (6)

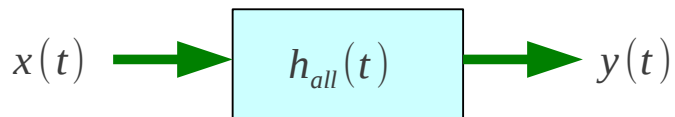


## Parseval's Theorem

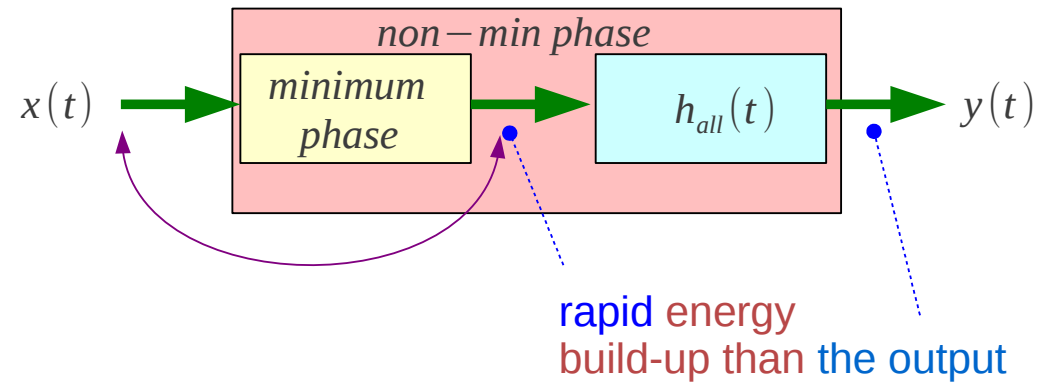
$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |y(t)|^2 dt$$

## Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2 dt \geq \int_{-\infty}^{t_0} |y(t)|^2 dt$$



The energy build-up in the input is more **rapid** than in the output



The signal energy until  $t_0$  of the minimum phase  $\geq$  any other causal signal with the same magnitude response

Thus minimum phase signals

➔ **maximally concentrated toward time 0** when compared against all causal signals having the same magnitude response

minimum phase signals

➔ **minimum delay signals**

# Properties of a Minimum Phase System

---

# Properties of a Minimum Phase System

---



# Properties of a Minimum Phase System

---

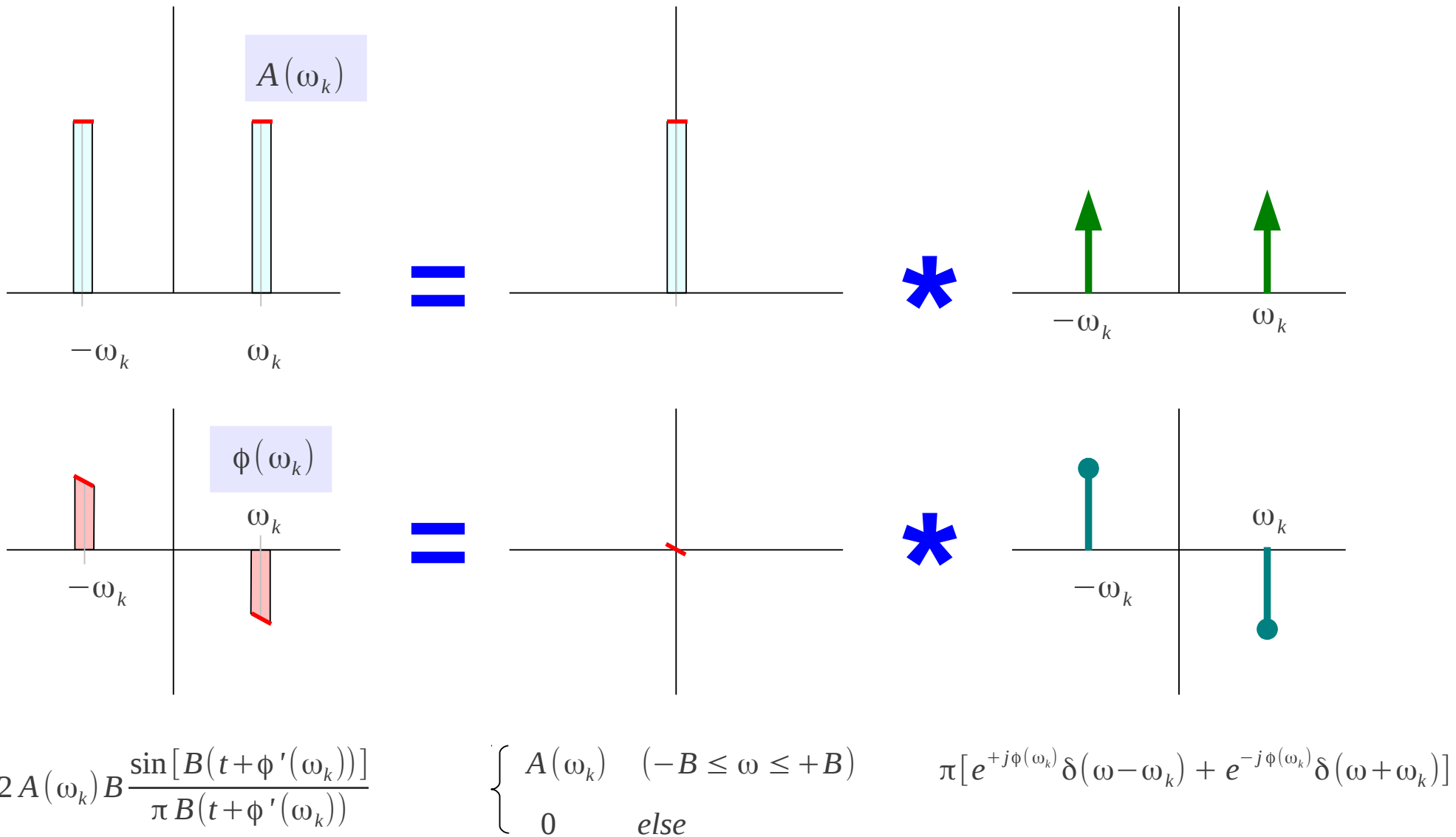
# Properties of a Minimum Phase System

---

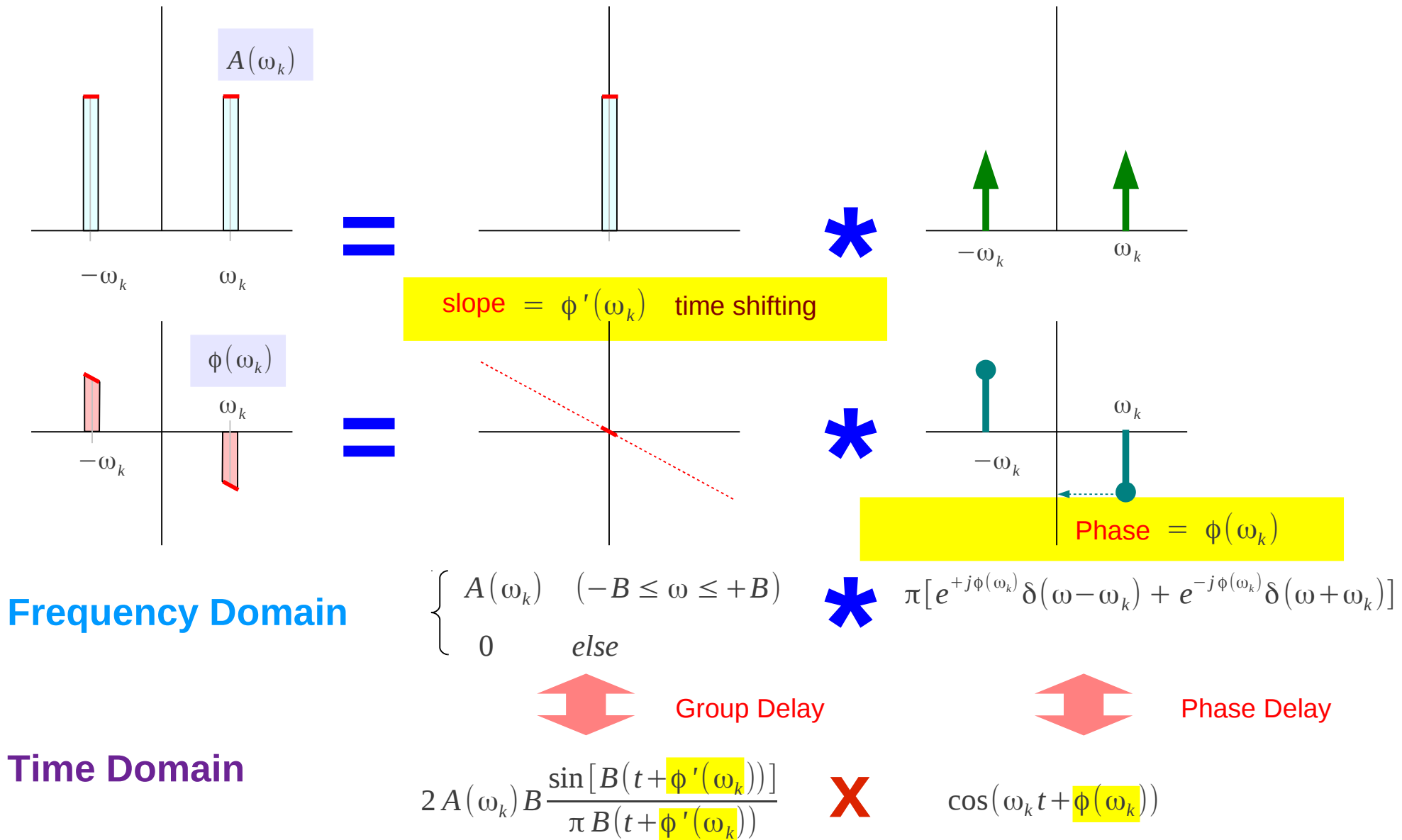
# Properties of a Minimum Phase System

---

# Simple LPF – Approximation (2)



# Simple LPF – Approximation (3)





## References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] <http://www.libinst.com/tpfd.htm>
- [4] K.Shin, J.K. Hammond, “Fundamentals of Signal Processing for Sound and Vibration Engineers”
- [5] [www.radiolab.com.au/DesignFile/DN004.pdf](http://www.radiolab.com.au/DesignFile/DN004.pdf)