The Rate of Convergence to Perfect Competition of a Simple Matching and Bargaining Mechanism

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Abstract

We study the steady state of a market with inflowing cohorts of buyers and sellers who are randomly matched pairwise and bargain under private information. Two bargaining protocols are considered: take-it-or-leave-it offering and the double auction. There are frictions due to costly search and time discounting, parameterized by a single number $\tau > 0$ proportional to the waiting time until the next meeting. We study the efficiency of these mechanisms as the frictions are removed, i.e. $\tau \to 0$. We find that all equilibria of the take-it-or-leave-it offering mechanism converge to the Walrasian limit, at the fastest possible rate $O(\tau)$ among all bargaining mechanisms. For the double auction mechanism, we find that there are equilibria that converge at the linear rate, those that converge at a slower rate or even not converge at all.

Keywords: Matching and Bargaining, Search, Double Auctions, Foundations for Perfect Competition, Rate of Convergence

JEL Classification Numbers: C73, C78, D83.

1 Introduction

A number of papers in the literature on dynamic matching and bargaining have shown that, as frictions vanish, equilibria converge to perfect competition.[1] But it is also important to know how rapidly the equilibria converge. To our knowledge, this question has not been addressed in the literature.

In contrast, the rate of convergence to efficiency has been the focus of the literature on static double auctions. In particular, Rustichini, Satterthwaite, and Williams (1994)

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[1] Papers that address convergence include Rubinstein and Wolinsky (1985), Gale (1986), Gale (1987), Rubinstein and Wolinsky (1990), Mortensen and Wright (2002), and, with private information, Butters (1979), Wolinsky (1988), De Fraja and Sakovics (2001), Serrano (2002), Lauermann (2006), Satterthwaite and Shneyerov (2007), Atakan (2007), Shneyerov and Wong (2007).

show robust convergence of double-auction equilibria in the symmetric class at the fast rate $O(1/n)$ for the bid/offer strategies and the superfast rate $O\left(1/n^2\right)$ for the ex-ante traders' welfare, where n is the number of traders in the market.[2] Moreover, the double auction converges at the rate that is fastest among all incentive-compatible and individually rational mechanisms (Satterthwaite and Williams (2002); Tatur (2005)). Cripps and Swinkels (2005) substantially enrich the model by allowing correlation among bidders' valuations, and show convergence at the rate $O\left(1/n^{2-\varepsilon}\right)$, where $\varepsilon > 0$ is arbitrary small. Reny and Perry (2006) allow interdependent values and show that it is almost efficient and almost fully aggregates information as $n \to \infty$.

In this paper, we fill this gap by proving a rate of convergence result for a decentralized model of trade. We study the steady state of a market with continuously inflowing cohorts of buyers and sellers who are randomly matched pairwise and bargain under private information. The model is therefore in continuous time and is a replica of Mortensen and Wright (2002), but with private information. (In a companion paper Shneyerov and Wong (2007), this model is looked at from a different perspective; in particular, we prove existence and show that private information may be welfare enhancing.)

Our baseline model considers the take-it-or-leave-it offer protocol in which seller makes an offer with probability α (and the buyer makes an offer with a complementary probability).[3] There are frictions due to costly search and time discounting. We parameterize these frictions by a single number $\tau > 0$ proportional to the waiting time until the next meeting.[4] The inverse of τ can also be interpreted as a measure of (local) market size that analogous to the number of traders n in the centralized double auction literature.

We show that price range collapses to the Walrasian price and inefficiency vanishes at the *linear* rate as the frictions are removed, i.e. $\tau \to 0$. Moreover, we derive explicit bounds on the difference between traders' strategies (and hence welfare) and their competitive limits, for all τ. We show that these bounds become tight as the discount rate becomes small relative to the search costs.

Thus all nontrivial equilibria converge rapidly to Walrasian benchmark. We show that this is also true under full information as in Mortensen and Wright (2002). Using the notion of worst-case asymptotic optimality similar to Satterthwaite and Williams (2002), we show that the take-it-or-leave-it offer mechanism is worst-case asymptotic optimal: in terms of the welfare, the rate of convergence cannot be faster under any other bargaining mechanism (and even under full information as in Mortensen and Wright (2002)).[5]

As a robustness check, we also study the rate of convergence of the double auction mechanism. We find that equilibria can be either convergent at the linear rate, convergent at a slower than linear rate or even divergent. This result can be compared to the findings

[2]See also Gresik and Satterthwaite (1989), Satterthwaite and Williams (1989), Satterthwaite (1989), Williams (1991).

[3]Under full information, this model corresponds to the Nash bargaining solution. It is used extensively in labor market search models For a recent survey, see Rogerson, Shimer, and Wright (2005).

[4]Our parameter τ therefore corresponds to the period between matches as in the discrete time models, e.g. Satterthwaite and Shneyerov (2007).

[5]The notion of the "worst case" here is actually weaker because we only look at the most unfavorable equilibrium; all our results are for fixed distributions of traders' types. The definition of the worst case in Satterthwaite and Williams (2002) involves also searching for most unfavorable distributions of traders' types.

in Serrano (2002). In a dynamic setting, Serrano (2002) studies a mechanism that in some respects resembles a double auction (the set of bids is restricted to be a finite grid) and finds that "as discounting is removed, equilibria with Walrasian and non-Walrasian features persist".[6] Serrano points out, however, that "after removing the finite sets of traders' types and of allowed prices, the present model confirms Gale's one-price result and has a strong Walrasian flavor". We, on the other hand, find that in our model, non-convergent equilibria exist even if the bargaining protocol is the "unrestricted" double auction.

The structure of the paper is as follows. Section 2 introduces the baseline model with take-it-or-leave-it bargaining mechanism. Section 3 presents the elementary properties of equilibria of the baseline model. Section 4 derives the convergence rate to perfect competition as frictions are removed. Section 5 gives our worst-case asymptotic optimality result for the take-it-or-leave-it bargaining mechanism. Section 6 discussed necessary changes to the model for the double auction, presents and proves all the results for it. Section 7 concludes. The results for the full information case are contained in the Appendix.

2 The Baseline Model

The players of our baseline model are potential buyers and potential sellers of a homogeneous, indivisible good. Each buyer has a unit demand for the good, while each seller is able to produce one unit of the good. Potential buyers are heterogeneous in their valuations (or types) v over the good. Potential sellers are also heterogeneous in their costs (or types) c of producing the good. For simplicity, we assume $v, c \in [0, 1]$. Time is continuous and infinite horizon. The details of the model are described as follows:

- Entry: Potential buyers and potential sellers are continuously born at rate b and s respectively. The type of a new-born buyer is drawn i.i.d. from the c.d.f. $F(v)$ and the type of a new-born seller is drawn i.i.d. from the c.d.f. $G(c)$. Each trader's type will not change once it is drawn. Entry (or participation, or being active) is voluntary. Each potential trader decides whether to enter the market once they are born. Those who does not enter will get zero payoff. Those who enter must incur the participation cost continuously at the rate κ_B for buyers and κ_S for sellers, until they leave the market.

- Matching: Active buyers and active sellers are randomly and continuously matched pairwise with the rate of matching given by a matching function $M(B, S)/\tau$, where B and S are the masses of active buyers and active sellers currently in the market, and τ is a parameter proportional to the waiting time of a trader until the next meeting.

- Bargaining: Once a pair of buyer and seller is matched, they bargain over the trading decision and the term of trade without observing the type of their partner. The bargaining protocol is take-it-or-leave-it offering: either the seller or the buyer, called proposer, makes a take-it-or-leave-it offer and the other party, called responder, either accept or reject the offer. With probability $\alpha \in (0, 1)$ (independent across pairs), the seller makes the offer; with probability $1 - \alpha$, the buyer makes the offer.

[6] This simplified bargaining mechanism was introduced in Wolinsky (1990) and also used in Blouin and Serrano (2001).

- If a type v buyer and a type c seller successfully trade at a price p, then they leave the market with (current value) payoff $v - p$, and $p - c$ respectively. If the matched pair fails to trade, both traders can either stay in the market waiting for another match (and incur the participation costs) as if they were never matched, or simply exit and never come back. The instantaneous discount rate is $r \geq 0$.

We make the following assumptions on the primitives of our model.

Assumption (distributions of inflow types) The cumulative distributions $F(v)$ and $G(c)$ of inflow types have densities $f(v)$ and $g(c)$ on $(0, 1)$, bounded away from 0 and ∞: $0 < \underline{f} \leq f(v) \leq \bar{f} < \infty$, $0 < \underline{g} \leq g(c) \leq \bar{g} < \infty$.

Assumption (matching function) The matching function M is continuous on $\mathbb{R}^2_{++}$, nondecreasing in each argument, constant returns to scale (i.e. homogeneous of degree one), and satisfies $\lim_{B \to 0} M(B, S) = \lim_{S \to 0} M(B, S) = 0$.

The matching technology is parametrized by τ, the parameter proportional to a trader's waiting time until next meeting. To see why this interpretation is correct, notice that, given steady-state active trader masses B and S, trading opportunities for a buyer arrive at the Poisson rate $M(B, S)/(\tau B)$, and therefore the waiting time until the next meeting is exponentially distributed with mean $\tau \cdot B/M(B, S)$.[7] Similarly, the waiting time for the seller is exponentially distributed with mean $\tau \cdot S/M(B, S)$.

The inverse of τ can also be interpreted as a measure of market size that analogous to the number of traders n in the centralized double auction literature. To see why, recall that in a centralize market, traders are competing intratemporally with all other traders in the same side. On the contrary, in the dynamic matching environment here, traders, whenever they bargain with their partners, are not directly competing with all other traders in the same side because of the matching frictions. But they do intertemporally compete with others in the sense that their partners have the option to search another to trade with. Since $1/\tau$ is proportional to arrival rates, it measures the *local market size* that corresponds this intertemporal competition.

This parameter, τ, is crucial for us since we are interested in the limit of our bilateral matching and bargaining game as $\tau \to 0$, i.e. as matching becomes frictionless.[8]

It turns out to be more convenient to work with a normalized matching function. Let

$$\zeta \equiv \frac{B}{S}$$

be the steady-state ratio of buyers to sellers (or market tightness), and define

$$m(\zeta) \equiv M(\zeta, 1).$$

Since the matching technology is assumed to be constant returns to scale, it is easy to see that $m(\zeta)$ is also equal to $M(B, S)/S$, the probability that a seller is matched over a short

[7] See Karlin and Taylor (1975), p. 124.

[8] All of our results hold equally well if we fix τ and let r, κ_B and κ_S tend to 0 proportionally, instead of letting $\tau \to 0$.

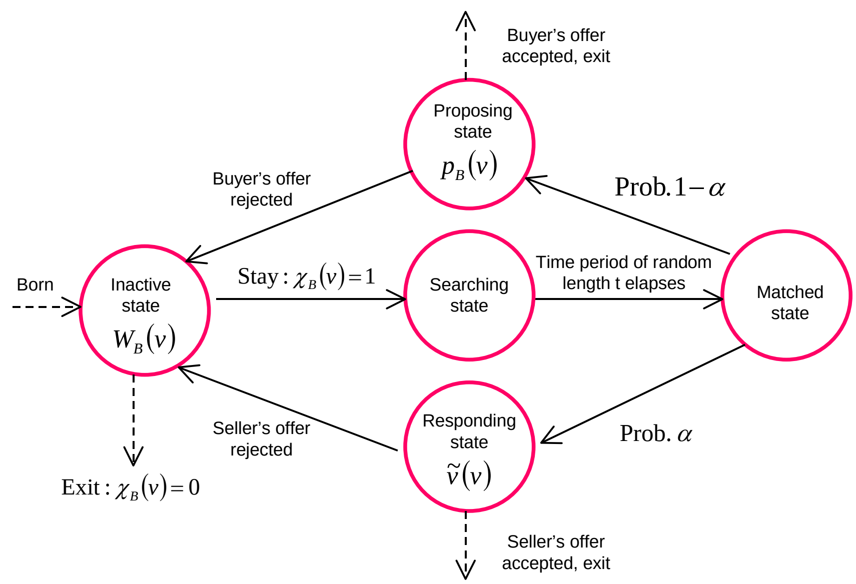

Figure 1: Markov chain of a buyer

time period of length dt divided by dt. Similarly, $m(\zeta)/\zeta$ is equal to $M(B,S)/B$, the probability that a buyer is matched over a over a short time period of length dt divided by dt. Note that $m(\zeta)$ and $m(\zeta)/\zeta$ are nondecreasing and nonincreasing respectively in ζ, and m is continuous on $\mathbb{R}_{++}$. In this notation, the Poisson arrival rates for buyers and sellers become

$$\ell_B(\zeta,\tau) \equiv \frac{m(\zeta)}{\tau\zeta}, \quad \ell_S(\zeta,\tau) \equiv \frac{m(\zeta)}{\tau}.$$

Notice that an uninteresting no-trade equilibrium always exists in which all potential traders do not enter. In the following, we will study steady-state market equilibria in which positive trade occurs. Let us simply call them *nontrivial steady-state equilibria*.

We now proceed to the definition of a nontrivial steady-state equilibrium of our baseline model. It is useful to represent each trader's world as a continuous-time Markov chain, as shown in Figure 1 for buyers. A trader is born into the "inactive" state, and has to decide immediately whether to enter to the market and search a partner, or simply exit. Let $\chi_B : [0,1] \to \{0,1\}$ and $\chi_S : [0,1] \to \{0,1\}$ be the buyers' and sellers' entry-decision functions in the inactive state. For example, $\chi_B(v) = 1$ means type v buyer enters; $\chi_S(c) = 0$ means type c seller does not enter. Let $A_B \subset [0,1]$ and $A_S \subset [0,1]$ be the sets of active buyers' and sellers' types, i.e.

$$\begin{aligned} A_B &\equiv \{v \in [0,1] : \chi_B(v) = 1\}, \\ A_S &\equiv \{c \in [0,1] : \chi_S(c) = 1\}. \end{aligned}$$

Once in the "searching" state, the trader waits until a new trading opportunity arrives. This happens after a time period of random length t has elapsed. (Recall that this length t is exponentially distributed with mean $1/\ell_B$ for buyers and $1/\ell_S$ for sellers.) The arrival of a trading opportunity moves a trader from the searching state to the "matched" state. Once in the matched state, the trader immediately proceeds either to the proposing state (with probability α for sellers and $1-\alpha$ for buyers), or to the responding state (with the complimentary probabilities). Let $p_B(v)$ and $p_S(c)$ be the proposing strategies used by buyers and sellers respectively.[9] Similarly, let $\tilde{v}(v)$ and $\tilde{c}(c)$ be the acceptance levels, characterizing the responding strategies of buyers and sellers respectively. Precisely, in a proposing state, type v buyers will propose the trading price $p_B(v)$, while in a responding state, they will accept a proposed price p if and only if $\tilde{v}(v) \geq p$. Analogous meanings apply to $p_S(c)$ and $\tilde{c}(c)$.

In the event when trading is successful, the matched pair leaves the market forever with their realized gains from trade. If trading is unsuccessful, each trader is immediately back in the inactive state of her Markov chain and the cycle repeats.

Let $\Phi(v), \Gamma(c)$ be the (endogenous) steady-state cumulative distributions of types of buyers and sellers who are active. The equilibria of our model can be defined as a collection[10]

$$E \equiv \{\chi_B, \chi_S, p_B, p_S, \tilde{v}, \tilde{c}, B, S, \Phi, \Gamma\}$$

such that:

(i) given the relevant beliefs made from E, every potential and active buyers (resp. sellers) find the entry policy given by χ_B (resp. χ_S), the proposing policy $p_B(\cdot)$ (resp. $p_S(\cdot)$) and the responding policy characterized by $\tilde{v}(\cdot)$ (resp. $\tilde{c}(\cdot)$) to be their optimal strategies sequentially;

(ii) E generates B, S, Φ, Γ in steady state.

The mathematical conditions for our equilibrium are as follows. Let us consider the sequential optimality of the responding strategies first. Let $W_B(v)$ be the (steady-state) equilibrium continuation payoff of a type v buyer in her inactive state, and let $W_S(c)$ be the equilibrium continuation payoff of a type c seller in her inactive state. Pick a type v buyer.[11] If she is in her responding state with an offer p at hand, her continuation payoff is $\max\{v-p, W_B(v)\}$. The first element $v-p$ is the continuation payoff if she accepts the offer p, while the second element $W_B(v)$ is the continuation payoff if she rejects and hence immediately get back to the inactive state. Similar logic applies to sellers' situation. Therefore, sequential optimality in the responding states requires the acceptance levels to be equal to what we shall call *dynamic types*[12]

$$\tilde{v}(v) \equiv v - W_B(v), \tag{1}$$

$$\tilde{c}(c) \equiv c + W_S(c). \tag{2}$$

Our dynamic type functions $\tilde{v}(v)$ and $\tilde{c}(c)$ allow us to characterize the proposing strategies in a simple manner. To this end, it is useful to consider the distributions of traders'

[9] Implicitly, every traders are assumed to use symmetric pure strategies. However, it is essentially without loss of generality. See Shneyerov and Wong (2007) for details.

[10] This definition is similar to the one in Satterthwaite and Shneyerov (2007).

[11] This type v buyer could be either active or not. If she is not active, we are considering an off-equilibrium path.

[12] The notion of dynamic types is due to Satterthwaite and Shneyerov (2007).

dynamic types, denoted as

$$\tilde{\Phi}(x) \equiv \int_{\tilde{v}(v)\leq x} d\Phi(v), \tag{3}$$

$$\tilde{\Gamma}(x) \equiv \int_{\tilde{c}(c)\leq x} d\Gamma(c). \tag{4}$$

Consider the situation where a type v buyer is in a proposing state and suppose sellers use their equilibrium responding policy characterized by $\tilde{c}\,(c)$ and sellers' distribution is at the equilibrium value Γ. If the buyer propose λ (one can think λ as a one-shot deviation) and this offer is accepted, her continuation payoff will be $v-\lambda$; and if her offer is rejected, she will be back to the inactive state immediately and her continuation payoff would be $W_B(v)$. Therefore, her continuation payoff in a proposing state, conditional on proposing λ, is

$$\int_{\tilde{c}(c)\leq\lambda}(v-\lambda)d\Gamma(c)+\int_{\tilde{c}(c)>\lambda}W_B(v)d\Gamma(c),$$

which can be rewritten as

$$\tilde{\Gamma}(\lambda)[\tilde{v}(v)-\lambda]+W_B(v).$$

Only the first term, which is the "capital gain part", depends on λ. Similar logic applies to sellers' situation. It is clear that sequential optimality in the proposing states is satisfied if and only if

$$p_B(v) \in \arg\max_{\lambda}\tilde{\Gamma}(\lambda)[\tilde{v}(v)-\lambda], \tag{5}$$

$$p_S(c) \in \arg\max_{\lambda}[1-\tilde{\Phi}(\lambda)][\lambda-\tilde{c}(c)]. \tag{6}$$

It follows that the equilibrium proposing strategies are determined as best-responses in the static monopoly problems where the distributions of responders' types are replaced by the distributions of the responders' dynamic types and the proposers' types are replaced by the proposers' dynamic types. As we have seen, this principle applies to the responding strategies as well. In general, the bargainers behave as if they are in a one-shot game with their types replaced by their dynamic types. Intuitively, trading with current partner lead a trader to give up the opportunity of searching and trading with another partner. Our dynamic type notions are simply adjusted with the traders' opportunity cost of further searching. This observation plays a very important role in both intuition and proofs of our results.

Turn to the matched state. Suppose that all traders always use their prescribed equilibrium strategies, $\{\chi_B,\chi_S,p_B,p_S,\tilde{v},\tilde{c}\}$ and that the stationary distributions of active seller and buyer types are at their equilibrium values Γ and Φ. Then a type v buyer's expected bargaining surplus from the meeting is equal to

$$\Pi_B(v)\equiv(1-\alpha)\int_{\tilde{c}(c)\leq p_B(v)}[v-p_B(v)]d\Gamma(c)+\alpha\int_{p_S(c)\leq\tilde{v}(v)}[v-p_S(c)]d\Gamma(c). \tag{7}$$

Further denote

$$q_B(v)\equiv(1-\alpha)\int_{\tilde{c}(c)\leq p_B(v)}d\Gamma(c)+\alpha\int_{p_S(c)\leq\tilde{v}(v)}d\Gamma(c), \tag{8}$$

the buyer's probability of a successful trade in a given meeting. With probability $1-q_B(v)$, the bargaining turn unsuccessful. The buyer's Markov chain then moves to the inactive state, giving a continuation payoff $W_B(v)$.

Now suppose a type v buyer chooses to enter, she has to wait and stay in the searching state until the next meeting. Since the buyer's waiting time before her next meeting is exponentially distributed with mean $1/\ell_B$,[13] the discounted value of one dollar to be received at the time of next meeting is equal to

$$R_B(\zeta,\tau) \equiv \int_{t=0}^{\infty} e^{-rt} d(1-e^{-\ell_B(\zeta,\tau)\cdot t}) = \frac{\ell_B(\zeta,\tau)}{r+\ell_B(\zeta,\tau)}. \tag{9}$$

Similarly, the accumulated discounted participation cost over the period until next meeting is equal to

$$K_B(\zeta,\tau) \equiv \int_{t=0}^{\infty} \left(\int_0^t \kappa_B e^{-rx} dx \right) d(1-e^{-\ell_B(\zeta,\tau)\cdot t}) = \frac{\kappa_B}{r+\ell_B(\zeta,\tau)}. \tag{10}$$

Then the searching state continuation payoff, provided that the type v buyer enters, is

$$R_B(\zeta,\tau)[\Pi_B(v) + (1-q_B(v))W_B(v)] - K_B(\zeta,\tau).$$

Since the entry decision is made in the inactive state and the trader gets 0 if she exits, the inactive state continuation payoff, $W_B(v)$, must satisfy the following recursive equation:

$$W_B(v) = \max\left\{R_B(\zeta,\tau)[\Pi_B(v) + (1-q_B(v))W_B(v)] - K_B(\zeta,\tau), 0\right\} \tag{11}$$

where the first maximand represents the payoff for entry, the second represents the payoff for exiting. Solve (11) for $W_B(v)$, we obtain an equivalent ratio-form formula:

$$W_B(v) = \max\left\{\frac{\ell_B(\zeta,\tau)\Pi_B(v) - \kappa_B}{r+\ell_B(\zeta,\tau)q_B(v)}, 0\right\}. \tag{12}$$

Therefore, the buyers' sequentially optimal entry policy in the inactive state is

$$\chi_B(v) = I\left\{\ell_B(\zeta,\tau)\Pi_B(v) \geq \kappa_B\right\} \tag{13}$$

where $I(\cdot)$ is the indicator function. Note that (13) implicitly assumes that traders enter if they are indifferent between entering or not. This is only for expositional simplicity because it turns out that the set of such indifferent traders is of measure 0.

Complete parallel logic applies to the sellers' side. We can define Π_S, q_S, R_S and K_S similarly:

$$\Pi_S(c) = \alpha \int_{\tilde{v}(v)\geq p_S(c)} [p_S(c) - c]\, d\Phi(v) + (1-\alpha) \int_{p_B(v)\geq \tilde{c}(c)} [p_B(v) - c]\, d\Phi(v) \tag{14}$$

$$q_S(c) = \alpha \int_{\tilde{v}(v)\geq p_S(c)} d\Phi(v) + (1-\alpha) \int_{p_B(v)\geq \tilde{c}(c)} d\Phi(v) \tag{15}$$

[13]That is, the distribution function of waiting time t is $1-\exp(-\ell_B t)$.

$$R_S(\zeta,\tau) = \frac{\ell_S(\zeta,\tau)}{r+\ell_S(\zeta,\tau)}, \quad K_S(\zeta,\tau) = \frac{\kappa_S}{r+\ell_S(\zeta,\tau)}. \tag{16}$$

Then we have the recursive equation for W_S:

$$W_S(c) = \max\left\{R_S(\zeta,\tau)[\Pi_S(c) + (1-q_S(c))W_S(c)] - K_S(\zeta,\tau), 0\right\}, \tag{17}$$

and the sellers' sequentially optimal entry policy in the inactive state is

$$\chi_S(c) = I\left\{\ell_S(\zeta,\tau)\Pi_S(c) \geq \kappa_S\right\}. \tag{18}$$

This completes the description of the strategic part of a nontrivial steady-state equilibrium. To complete the description of nontrivial steady-state equilibrium, we turn to the steady state equations for the distributions of active buyer and seller types Φ and Γ and active trader masses B and S. In a steady-state market equilibrium, traders who once enter would not exit until they trade successfully. Therefore,

$$b\int_v^1 \chi_B(x)dF(x) = B\ell_B(\zeta,\tau)\int_v^1 q_B(x)d\Phi(x) \ \ \forall v \in [0,1] \tag{19}$$

$$s\int_0^c \chi_S(x)dG(x) = S\ell_S(\zeta,\tau)\int_0^c q_S(x)d\Gamma(x) \ \ \forall c \in [0,1], \tag{20}$$

which simply state that the inflow rate of every types of traders must be equal to the corresponding outflow rate.

These preparations allow us to formally define nontrivial steady-state equilibrium as follows.

Definition 1 *A collection $E \equiv \{\chi_B, \chi_S, p_B, p_S, \tilde{v}, \tilde{c}, B, S, \Phi, \Gamma\}$ is a nontrivial steady-state equilibrium if there exists a pair of equilibrium payoff functions $\{W_B, W_S\}$ such that the proposing strategies p_B and p_S, responding strategies $\tilde{v}$ and $\tilde{c}$, entry strategies χ_B and χ_S satisfy the sequential optimality conditions* (5)*,* (6)*,* (1)*,* (2)*,* (13) *and* (18)*, and the distributions of active buyer and seller types Φ and Γ and active trader masses B and S solve the steady-state equations* (19) *and* (20)*, and the payoff functions W_B and W_S solve the recursive equations* (11) *and* (17)*.*

3 Basic Properties of Equilibria

We will need the following equilibrium properties from our companion paper Shneyerov and Wong (2007). Here, we give a proof that is different because of its connection to some other results in this paper.

Lemma 2 *In any nontrivial steady-state equilibrium, $W_B(v)$ and $W_S(c)$ are absolutely continuous and convex. $W_B(v)$ is nondecreasing and $W_S(c)$ is nonincreasing. Moreover,*

$$W_B(v) = \int_{\underline{v}}^{v} \frac{\ell_B q_B(x)}{r+\ell_B q_B(x)}dx \quad \textit{for all } v \in [\underline{v}, 1] \tag{21}$$

$$W_S(c) = \int_{c}^{\bar{c}} \frac{\ell_S q_S(x)}{r+\ell_S q_S(x)}dx \quad \textit{for all } c \in [0, \bar{c}]\,. \tag{22}$$

The sets of active trader types are $A_B = [\underline{v}, 1]$ *and* $A_S = [0, \bar{c}]$. *The probabilities of trading are monotonic:* $q_B(v)$ *is nondecreasing in* v *on* A_B, *while* $q_S(c)$ *is nonincreasing in* c *on* A_S. *The dynamic types* $\tilde{v}(v) = v - W_B(v)$ *and* $\tilde{c}(c) = c + W_S(c)$ *are absolutely continuous and nondecreasing. Their slopes are*

$$\tilde{v}'(v) = \frac{r}{r + \ell_B q_B(v)} \quad (a.e.\ v \in A_B) \tag{23}$$

$$\tilde{c}'(c) = \frac{r}{r + \ell_S q_S(v)} \quad (a.e.\ c \in A_S). \tag{24}$$

Proof: We prove the results for buyers only. Rewrite the recursive equation for the buyers, considering a one-shot deviation to proposing and responding as type $\hat{v}$ would:

$$\begin{aligned} W_B(v) &= \max\left\{R_B \max_{\hat{v}\in[0,1]}[\hat{\Pi}_B(v,\hat{v}) + (1 - q_B(\hat{v}))W_B(v)] - K_B, 0\right\} \\ &= \max\left\{R_B \max_{\hat{v}\in[0,1]}[\hat{\Pi}_B(v - W_B(v),\hat{v}) + W_B(v)] - K_B, 0\right\} \end{aligned}$$

where

$$\hat{\Pi}_B(v,\hat{v}) \equiv q_B(\hat{v})v - t_B(\hat{v}),$$

$$t_B(v) \equiv (1-\alpha)\int_{\tilde{c}(c)\le p_B(v)} p_B(v)\, d\Gamma(c) + \alpha \int_{p_S(c)\le \tilde{v}(v)} p_S(c)\, d\Gamma(c).$$

If $R_B = 1$ (or $r = 0$), the recursive equation indicate that whenever $W_B(v) \neq 0$, we have $\max_{\hat{v}\in[0,1]} \hat{\Pi}_B(v - W_B(v), \hat{v}) = K_B > 0$ so that $v - W_B(v)$ must be some positive constant x. It is then easily seen that the recursive equation has a unique solution $W_B(v) = \max\{v - x, 0\}$, which is nondecreasing, continuous and convex.

Now suppose $R_B < 1$ (or $r > 0$). Then the right-hand side of the recursive equation can be regarded as a contraction mapping that assigns each W_B another function on the same domain. Applying standard techniques of discounted dynamic programming, we can see that the solution W_B is unique, nondecreasing, continuous and convex.

Then from continuity and monotonicity, $W_B(v)$ is absolutely continuous and hence differentiable almost everywhere. Whenever differentiable, we have by the envelope theorem[14]

$$W_B'(v) = \chi_B(v) R_B \left\{q_B(v)[1 - W_B'(v)] + W_B'(v)\right\}.$$

Solve for $W_B'(v)$ and simplify,

$$W_B'(v) = \chi_B(v)\frac{\ell_B q_B(v)}{r + \ell_B q_B(v)}.$$

It then follows from the convexity of W_B that q_B is nondecreasing on A_B.

For $v \in A_B$, the trading probability $q_B(v)$ must be strictly positive, otherwise the participation cost κ_B cannot be recovered. Thus $W_B(v)$ is strictly increasing on A_B and $A_B = [\underline{v}, 1]$.

[14] See e.g. Milgrom and Segal (2002).

In order to prove (21), it now suffices to show $W_B(\underline{v}) = 0$. Indeed, if $W_B(\underline{v}) \neq 0$, then either $\underline{v} = 0$ or $\underline{v} = 1$. We preclude the possibility of $\underline{v} = 1$ because we are looking at nontrivial equilibrium. $\underline{v} = 0$ is also impossible because in that case type 0 buyer cannot expect their participation cost recovered.

It follows from its definition that $\tilde{v}$ is absolutely continuous because W_B is. Its derivative, which exists almost everywhere on A_B, is given by (23), which are non-negative. Q.E.D.

Shneyerov and Wong (2007) also show that the proposing strategies are monotone.

Lemma 3 *In any nontrivial steady-state equilibrium, the proposing strategies p_B and p_S are nondecreasing on A_B and A_S respectively.*

Since the dynamic opportunity costs of trading for marginal entrants are zero (i.e. $W_B(\underline{v}) = W_S(\bar{c}) = 0$), we can see that the marginal participating types are equal to the corresponding dynamic types:

$$\bar{c} = \tilde{c}(\bar{c}), \quad \underline{v} = \tilde{v}(\underline{v}).$$

The sellers' minimum acceptable price $\underline{c}$ and the buyers' maximum acceptable price $\bar{v}$ are defined by:

$$\begin{aligned} \underline{c} &\equiv \inf_c \{\tilde{c}(c) : c \in A_S\} = \tilde{c}(0), \\ \bar{v} &\equiv \sup_v \{\tilde{v}(v) : v \in A_B\} = \tilde{v}(1), \end{aligned}$$

which, taken together define what we call the *acceptance interval* $[\underline{c}, \bar{v}]$. The smallest and largest offers by buyers and sellers are

$$\begin{aligned} \underline{p}_B &\equiv \inf_v \{p_B(v) : v \in A_B\} = p_B(\underline{v}), \\ \bar{p}_B &\equiv \sup_v \{p_B(v) : v \in A_B\} = p_B(\bar{v}), \\ \underline{p}_S &\equiv \inf_c \{p_S(c) : c \in A_S\} = p_S(\underline{c}), \\ \bar{p}_S &\equiv \sup_c \{p_S(c) : c \in A_S\} = p_S(\bar{c}). \end{aligned}$$

We define the *price interval* as $[\underline{p}_B, \bar{p}_S]$.

Lemma 4 *In any nontrivial steady-state equilibrium, $\tilde{c}(c) < p_S(c)$ and $p_B(v) < \tilde{v}(v)$ for all $c \in [0, \bar{c}]$ and all $v \in [\underline{v}, 1]$. (They imply $\underline{p}_B < \underline{v}$ and $\bar{c} < \bar{p}_S$.) Moreover, if $r > 0$, then $\underline{c} < \underline{v} \leq \underline{p}_S \leq \bar{p}_S < \bar{v}$ and $\underline{c} < \underline{p}_B \leq \bar{p}_B \leq \bar{c} < \bar{v}$, while if $r = 0$, then $\underline{c} < \underline{v} = \underline{p}_S = \bar{p}_S = \bar{v}$ and $\underline{c} = \underline{p}_B = \bar{p}_B = \bar{c} < \bar{v}$.*

The proof of Lemma 4 is again in Shneyerov and Wong (2007). In particular, in equilibrium, the buyers' offers must be lower than their dynamic opportunity valuation, and the sellers' offers must be higher than their dynamic opportunity cost. Moreover, buyers never propose anything below the lowest acceptable price of sellers $\underline{c}$, and sellers never propose anything above the highest acceptable price of buyers $\bar{v}$. In other words, $\underline{c} \leq p_B(v) <$

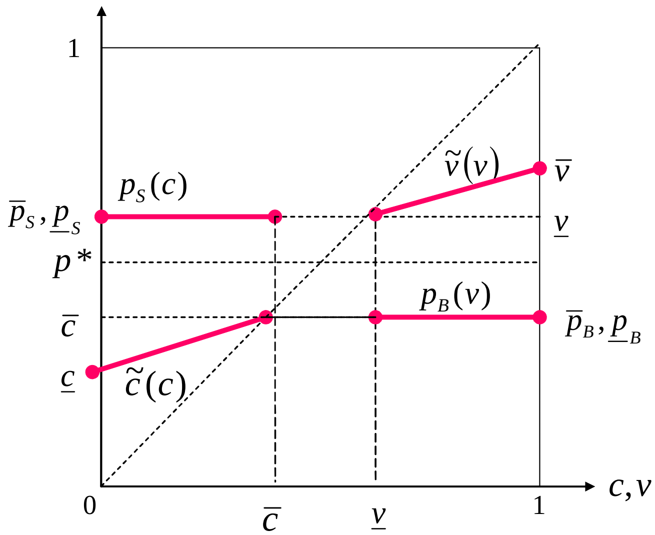

Figure 2: A full-trade equilibrium

$\tilde{v}(v)$ and $\bar{v} \geq p_S(c) > \tilde{c}(c)$. Furthermore, in order for the marginal entrants to recover participation costs, we must also have $\underline{c} < \underline{v}$ and $\bar{c} < \bar{v}$.

In our companion paper Shneyerov and Wong (2007), we identify two kinds of equilibria: full-trade equilibria and non-full-trade equilibria. In a full-trade equilibrium, every meeting results in a trade. It can be defined as a nontrivial steady-state equilibrium with $\underline{p}_B = \bar{c}$ and $\bar{p}_S = \underline{v}$, see Figure 2. In a non-full-trade equilibrium, not every meeting results in a trade; see Figure 3. In Shneyerov and Wong (2007), we prove the existence of both kinds of equilibria.

4 Rate of Convergence for Strategies

Before we prove our general rate of convergence theorem, we show how the linear rate is obtained for full-trade equilibrium. This can be done in a simple manner because a full-trade equilibrium admits a simple characterization.

The marginal participating types $\underline{v}$ and $\bar{c}$ must satisfy the following two equations

$$\ell_B(\zeta, \tau)(1-\alpha)\left(\underline{v} - \bar{c}\right) = \kappa_B, \tag{25}$$

$$\ell_S(\zeta, \tau)\alpha\left(\underline{v} - \bar{c}\right) = \kappa_S. \tag{26}$$

The intuition here is as follows. In the left-hand sides of equations (25) and (26) we have marginal traders' expected profits from trading, gross of participation costs, over a short period dt, divided by the length of the period. To see the intuition behind equation (25), note that a marginal participating buyer $\underline{v}$ makes positive profit only if he meets a seller, proposes, and his offer is accepted (the combined probability is $\ell_B \cdot (1-\alpha)$), and conditional on that, the profit is equal to the difference between his valuation and the price he proposes,

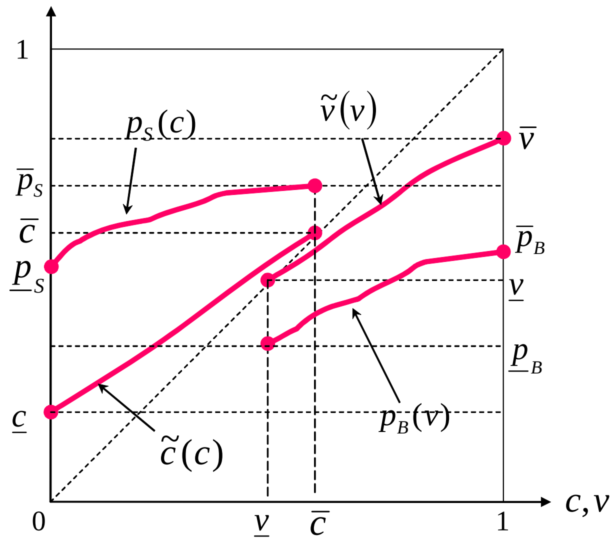

Figure 3: A non-full-trade equilibrium

$\underline{v} - \underline{p}_B$. Similar logic applies to equation (26).[15]

Since traders always exit in pairs, in steady state the marginal participating types also satisfy the mass balance condition

$$b[1 - F(\underline{v})] = sG(\bar{c}), \tag{27}$$

which states that inflow rates of buyers and sellers are equal.

Equations (25)-(27) fully characterize a full-trade equilibrium, if it ever exists.[16] Noticing that $\ell_S(\zeta,\tau)/\ell_B(\zeta,\tau) = \zeta$, the entry equations (25) and (26) can be easily solved for ζ and $\underline{v} - \bar{c}$:

$$\zeta = \frac{1-\alpha}{\alpha}\frac{\kappa_S}{\kappa_B} \equiv z, \tag{28}$$

$$\underline{v} - \bar{c} = \tau \cdot K(z), \tag{29}$$

where

$$K(\zeta) \equiv \frac{\kappa_B \zeta + \kappa_S}{m(\zeta)}$$
$$\left(= \frac{\kappa_B}{\ell_B(\zeta,1)} + \frac{\kappa_S}{\ell_S(\zeta,1)} \right).$$

It follows from (29) that the entry gap $\underline{v} - \bar{c}$ converges to 0 at the linear rate in τ. From Lemma 2, since $q_B(v) = q_S(c) = 1$ and $\zeta = z$ in the full-trade equilibrium, the slopes of

[15] These equations follow from Lemma 7 below.

[16] However, to verify that the solution of (25)-(27) is really an equilibrium, one has to ensure that (a) $\tau \cdot K(z) < 1$, (b) buyers do not have an incentive to propose higher than $\bar{c}$, and (c) sellers do not have an incentive to propose below $\underline{v}$. See Shneyerov and Wong (2007) for details.

responding strategies also converge to 0 linearly in τ. Consequently, the acceptance interval $\bar{v} - \underline{c}$ converges at that rate as well.

Our main contribution is to show that *all* equilibria (i.e. also non-full-trade) converge at the linear rate in τ. Our main result is the following theorem.

Theorem 5 (Rate of convergence for strategies) *In any nontrivial steady-state equilibrium, we have*

$$\tau \cdot K(z) \leq \bar{p}_S - \underline{p}_B \leq \bar{v} - \underline{c} \leq \tau \cdot K(z) \left(1 + \frac{2r}{\kappa}\right)^3,$$

where $\kappa \equiv \min\{\kappa_B, \kappa_S\}$.

Notice that the upper bound provided in Theorem 5 converges to the lower bound as r gets small relative to $\kappa \equiv \min\{\kappa_B, \kappa_S\}$. It indicates that our bounds are tight at least when the discount rate is small relative to the search costs. In the Appendix, we provide completely analogous rate of convergence result in the context of full information. In particular, we show that the rate of convergence is exactly the same under full information bargaining. Furthermore, the results in the next section also hold under full information.

The main difficulty with the proof is that there is no simple characterization of a non-full-trade equilibrium, even though, it is relatively easy to show that, as in the full-trade equilibrium, the entry gap is $O(\tau)$.[17]

Lemma 6 (Rate of convergence for entry gap) *In any nontrivial steady-state equilibrium, we have*

$$\max\{\underline{v} - \bar{c}, 0\} \leq \tau \cdot K(z). \tag{30}$$

Nevertheless, we know of no simple argument that would show that the slopes of responding strategies converge to 0 as $\tau \to 0$ uniformly over all types. A difficulty arises because we would need to bound the trading probabilities $\ell_B(\zeta, \tau) q_B(v)$ and $\ell_S(\zeta, \tau) q_S(c)$ away from 0 uniformly as $\tau \to 0$. Our proof avoids this difficulty by looking at a carefully chosen set of types where such bounding can be done.

To continue, we need the following lemma that describes the indifference conditions for marginal entrants.

Lemma 7 *In any nontrivial steady-state equilibrium,*

$$\ell_B(\zeta, \tau)(1-\alpha)\tilde{\Gamma}(\underline{p}_B)(\underline{v} - \underline{p}_B) = \kappa_B \tag{31}$$

$$\ell_S(\zeta, \tau)\alpha[1 - \tilde{\Phi}(\bar{p}_S)](\bar{p}_S - \bar{c}) = \kappa_S. \tag{32}$$

Proof: Notice that by Lemma 4, $\underline{v} \leq \underline{p}_S$ and therefore a $\underline{v}$-buyer could make positive profit only when he is the proposer. When he is a proposer, his offer $\underline{p}_B$ will be accepted only if the seller's dynamic type $\tilde{c}(c) \leq \underline{p}_B$. The entry condition (13) then implies (31). Similar logic leads to (32). Q.E.D.

[17] Lemma 6 and Lemma 7 are from Shneyerov and Wong (2007); we give the proofs here for the sake of completeness. We postpone the proof of Lemma 6 till we prove Lemma 7.

Proof of Lemma 6: We will apply a revealed preference argument to the entry conditions (31) and (32). Consider the deviations in which the $\underline{v}$-buyers propose $\bar{c}$ and $\bar{c}$-sellers propose $\underline{v}$. Those marginal entrants must not be able to have strictly positive payoff from such deviations:

$$\begin{aligned}(1-\alpha)\,\ell_B\,(\underline{v}-\bar{c}) &\leq \kappa_B, \\ \alpha\ell_S\,(\underline{v}-\bar{c}) &\leq \kappa_S.\end{aligned}$$

It follows that

$$\begin{aligned}\underline{v}-\bar{c} &\leq \min\left\{\frac{\kappa_B}{(1-\alpha)\,\ell_B},\frac{\kappa_S}{\alpha\ell_S}\right\} \\ &\leq \tau\cdot K\,(z)\,.\end{aligned} \tag{33}$$

The last inequality follows from the fact that $\frac{\kappa_S}{\ell_S\alpha}$ is nonincreasing and $\frac{\kappa_B}{\ell_B(1-\alpha)}$ is nondecreasing in ζ, and that they are equal if and only if $\zeta=z$, at which both of them equal $\tau\cdot K\,(z)$. Q.E.D.

Proof of Theorem 5:
Step 1: We claim that

$$\text{(a): } \frac{\underline{v}-\underline{c}}{\bar{v}-\underline{c}}\geq\frac{\kappa_B}{r+\kappa_B}$$

$$\text{(b): } \frac{\bar{v}-\bar{c}}{\bar{v}-\underline{c}}\geq\frac{\kappa_S}{r+\kappa_S}.$$

We provide the proof for part (a) only. The proof for part (b) is the flip of that for part (a). Apply (31) and notice that $q_B(v)\geq(1-\alpha)\tilde{\Gamma}(\underline{p}_B)>0$ whenever $v\in[\underline{v},1]$, and that $\underline{v}-\underline{c}\geq\underline{v}-\underline{p}_B>0$, we have $\ell_B q_B(v)(\underline{v}-\underline{c})\geq\kappa_B$ whenever $v\in[\underline{v},1]$. Then for almost all $v\in[\underline{v},1]$,

$$\tilde{v}'(v)=\frac{r}{r+\ell_B q_B(v)}\leq\frac{r}{r+\kappa_B/(\underline{v}-\underline{c})}.$$

Hence

$$\bar{v}-\underline{v}=\int_{\underline{v}}^{1}\tilde{v}'(v)dv\leq\frac{r}{r+\kappa_B/(\underline{v}-\underline{c})},$$

$$\frac{\bar{v}-\underline{v}}{\underline{v}-\underline{c}}\leq\frac{r}{(\underline{v}-\underline{c})r+\kappa_B}\leq\frac{r}{\kappa_B},$$

$$\frac{\underline{v}-\underline{c}}{\bar{v}-\underline{c}}=\frac{1}{1+(\bar{v}-\underline{v})/(\underline{v}-\underline{c})}\geq\frac{1}{1+\frac{r}{\kappa_B}}=\frac{\kappa_B}{r+\kappa_B}.$$

Step 2: We claim that

$$\text{(a): } \min\{\underline{v},\bar{c}\}-\underline{c}\leq\frac{4r\,(r+\kappa_B)}{\ell_S\alpha\kappa_B}$$

$$\text{(b): } \bar{v}-\max\{\underline{v},\bar{c}\}\leq\frac{4r(r+\kappa_S)}{\ell_B(1-\alpha)\kappa_S}.$$

Again by symmetry, we only provide a proof for (a). Let $y\equiv\min\{\underline{v},\bar{c}\}-\underline{c}$.

Consider a type c seller with $\tilde{c}(c) \leq \underline{c} + y/2$, then

$$\begin{aligned} \left[1 - \tilde{\Phi}(p_S(c))\right][p_S(c) - \tilde{c}(c)] &= \max_{\lambda}\left\{\left[1 - \tilde{\Phi}(\lambda)\right][\lambda - \tilde{c}(c)]\right\} \\ &\geq \underline{v} - \left(\underline{c} + \frac{y}{2}\right) \geq \frac{\underline{v} - \underline{c}}{2}. \end{aligned}$$

Consequently, that seller's probability of trade in a given meeting, $q_S(c)$, is bounded from below by

$$q_S(c) \geq \alpha[1 - \tilde{\Phi}(p_S(c))] \geq \frac{\alpha}{p_S(c) - \tilde{c}(c)} \frac{\underline{v} - \underline{c}}{2} \geq \frac{\alpha}{2} \frac{\underline{v} - \underline{c}}{\bar{v} - \underline{c}} \geq \frac{\alpha \kappa_B}{2(r + \kappa_B)}.$$

The last inequality is from step 1(a).

Then from (24) in Lemma 2,

$$\tilde{c}'(c) = \frac{r}{r + \ell_S q_S(c)} \leq \frac{2r(r + \kappa_B)}{2r(r + \kappa_B) + \ell_S \alpha \kappa_B} \leq \frac{2r(r + \kappa_B)}{\ell_S \alpha \kappa_B}.$$

Now we can see that

$$\frac{y}{2} = \int_{\tilde{c}(c) \in [\underline{c}, \underline{c} + \frac{y}{2}]} \tilde{c}'(c)\, dc \leq \frac{2r(r + \kappa_B)}{\ell_S \alpha \kappa_B},$$

which is the same as (a).

Step 3: Let κ be $\min\{\kappa_B, \kappa_S\}$. We claim that

$$\bar{v} - \underline{c} \leq \tau \cdot \min\left\{\frac{\kappa_B}{(1 - \alpha)\ell_B}, \frac{\kappa_S}{\alpha \ell_S}\right\} \cdot \left(1 + \frac{r}{\kappa}\right)\left(1 + \frac{2r}{\kappa}\right)^2.$$

To prove it, first notice that from step 2(a) and (33), we have

$$\underline{v} - \underline{c} = \min\{\underline{v}, \bar{c}\} - \underline{c} + \max\{\underline{v} - \bar{c}, 0\} \leq \frac{4r(r + \kappa_B)}{\ell_S \alpha \kappa_B} + \frac{\kappa_S}{\ell_S \alpha}.$$

Then from step 1(a),

$$\begin{aligned} \bar{v} - \underline{c} &\leq \frac{r + \kappa_B}{\kappa_B}(\underline{v} - \underline{c}) \leq \frac{r + \kappa_B}{\ell_S \alpha \kappa_B}\left[\frac{4r(r + \kappa_B)}{\kappa_B} + \kappa_S\right] \\ &= \frac{\kappa_S}{\ell_S \alpha}\left(1 + \frac{r}{\kappa_B}\right)\left[1 + \frac{4r}{\kappa_S}\left(1 + \frac{r}{\kappa_B}\right)\right] \\ &\leq \frac{\kappa_S}{\ell_S \alpha}\left(1 + \frac{r}{\kappa}\right)\left[1 + \frac{4r}{\kappa}\left(1 + \frac{r}{\kappa}\right)\right] = \frac{\kappa_S}{\ell_S \alpha}\left(1 + \frac{r}{\kappa}\right)\left(1 + \frac{2r}{\kappa}\right)^2. \end{aligned}$$

Similarly, from step 2(b), inequality (33) and step 1(b),

$$\bar{v} - \underline{c} \leq \frac{\kappa_B}{\ell_B(1 - \alpha)}\left(1 + \frac{r}{\kappa}\right)\left(1 + \frac{2r}{\kappa}\right)^2.$$

We get our claim by combining these two upper bounds of $\bar{v} - \underline{c}$.

Step 4: We claim that

$$\bar{v} - \underline{c} \geq \bar{p}_S - \underline{p}_B \geq \max\left\{\frac{\kappa_S}{\ell_S \alpha}, \frac{\kappa_B}{\ell_B (1-\alpha)}\right\}.$$

To prove it, observe that (31), (32) and Lemma 4 imply

$$\kappa_B \leq \ell_B(1-\alpha)(\underline{v} - \underline{p}_B) \leq \ell_B(1-\alpha)(\bar{p}_S - \underline{p}_B),$$

$$\kappa_S \leq \ell_S \alpha(\bar{p}_S - \bar{c}) \leq \ell_S \alpha(\bar{p}_S - \underline{p}_B),$$

and $\bar{p}_S - \underline{p}_B \leq \bar{v} - \underline{c}$.

Step 5: Notice that $\frac{\kappa_S}{\ell_S \alpha}$ is nonincreasing and $\frac{\kappa_B}{\ell_B(1-\alpha)}$ is nondecreasing in ζ, and that they are equal if and only if $\zeta = z$, at which both of them equal $\tau \cdot K(z)$. Thus we have

$$\min\left\{\frac{\kappa_S}{\ell_S \alpha}, \frac{\kappa_B}{\ell_B (1-\alpha)}\right\} \leq \tau \cdot K(z) \leq \max\left\{\frac{\kappa_S}{\ell_S \alpha}, \frac{\kappa_B}{\ell_B (1-\alpha)}\right\}.$$

Then from steps 3 and 4,

$$\tau \cdot K(z) \leq \bar{p}_S - \underline{p}_B \leq \bar{v} - \underline{c} \leq \tau \cdot K(z)\left(1 + \frac{r}{\kappa}\right)\left(1 + \frac{2r}{\kappa}\right)^2 \leq \tau \cdot K(z)\left(1 + \frac{2r}{\kappa}\right)^3.$$

Q.E.D.

We define the Walrasian price p^* as the price that clears the flows of the arriving cohorts:

$$b[1 - F(p^*)] = sG(p^*).$$

Then as a corollary of Theorem 5, traders' proposing and responding strategies must converge to the Walrasian price at no slower than linear convergence rate.

Corollary 8 *Fix $\kappa_B, \kappa_S > 0$ and $r \geq 0$. For any sequence of nontrivial steady-state equilibria parametrized by τ such that $\tau \to 0$, the price interval $[\underline{p}_{B\tau}, \bar{p}_{S\tau}]$ and acceptance interval $[\underline{c}_\tau, \bar{v}_\tau]$ collapse to the Walrasian price $\{p^*\}$ at no slower than linear convergence rate. More precisely,*

$$\left|\underline{p}_{B\tau} - p^*\right|, \left|\bar{p}_{S\tau} - p^*\right|, \left|\underline{c}_\tau - p^*\right|, \left|\bar{v}_\tau - p^*\right| < \tau \cdot K(z)\left(1 + \frac{2r}{\kappa}\right)^3.$$

Proof: It follows from (27) that the marginal participating types $\underline{v}$ and $\bar{c}$ must be on different sides of the Walrasian price p^*, i.e. either $\bar{c} \leq p^* \leq \underline{v}$ or $\underline{v} \leq p^* \leq \bar{c}$. Thus Lemma 4 implies that p^* must always fall within the acceptance interval, i.e. $p^* \in (\underline{c}, \bar{v})$. Finally, Theorem 5 gives the claim. Q.E.D.

5 Rate Optimality

To provide a benchmark for our results, we define traders' Walrasian utilities in the usual manner, as

$$W_B^*(v) = \max\{v - p^*, 0\}, \qquad W_S^*(c) = \max\{p^* - c, 0\}.$$

The total surplus is then

$$W^{0*} = bW_B^{0*} + sW_S^{0*}$$

where

$$W_B^{0*} = \int_{p^*}^{1} (v - p^*)\, dF(v), \qquad W_S^{0*} = \int_0^{p^*} (p^* - c)\, dG(c).$$

The following lemma shows that the traders' interim payoffs converge no slower than the length of acceptance interval $\bar{v} - \underline{c}$.

Lemma 9 *In any nontrivial steady-state equilibrium,* $|W_i^*(v) - W_i(v)| < \bar{v} - \underline{c}$ *for* $i = B, S$.

Proof. We will only prove the result for buyers. That for sellers can be proved by a symmetric argument. Observe that if $v \geq \underline{v}$ then $W_B(v) = v - \tilde{v}(v)$; and if $v < \underline{v}$ then $W_B(v) = 0$. Consequently,

$$W_B(v) - W_B^*(v) = W_B(v) - \max\{v - p^*, 0\} = \begin{cases} p^* - \tilde{v}(v) & \text{if} \quad v \geq \underline{v} \text{ and } v \geq p^* \\ v - \tilde{v}(v) & \text{if} \quad v \geq \underline{v} \text{ and } v < p^* \\ p^* - v & \text{if} \quad v < \underline{v} \text{ and } v \geq p^* \\ 0 & \text{if} \quad v < \underline{v} \text{ and } v < p^* \end{cases}.$$

In any of the four cases,

$$|W_B(v) - W_B^*(v)| \leq \sup_v \{|\tilde{v}(v) - p^*| : v \in A_B\} < \bar{v} - \underline{c}.$$

Q.E.D.

Combine Lemma 9 and Theorem 5, we obtain the following rate of convergence theorem for interim utilities.

Theorem 10 (Rate of convergence for interim utilities) *Fix* $\kappa_B, \kappa_S > 0$ *and* $r \geq 0$. *Then the interim utilities* $W_{B\tau}(v)$, $W_{S\tau}(c)$ *converge to their Walrasian counterparts* $W_B^*(v)$ *and* $W_S^*(c)$ *at least as fast as linear rate, as* $\tau \to 0$. *More precisely,*

$$|W_B^*(v) - W_{B\tau}(v)|, |W_S^*(c) - W_{S\tau}(c)| \leq \tau \cdot K(z) \left(1 + \frac{2r}{\kappa}\right)^3,$$

Remark 11 *In Theorem 10, absolute values for both* $W_B^*(v) - W_{B\tau}(v)$ *and* $W_S^*(c) - W_{S\tau}(c)$ *are needed because they are not guaranteed to be positive. Indeed, if* $\underline{v}_\tau < p^*$, *then buyers with type* $v \in (\underline{v}_\tau, p^*]$ *would have strictly positive utilities in equilibrium but have* 0 *Walrasian utilities. We also do not have a positive lower bound for* $\frac{1}{\tau}|W_B^*(v) - W_{B\tau}(v)|$ *and* $\frac{1}{\tau}|W_S^*(c) - W_{S\tau}(c)|$. *Indeed, for some types* v, *we could have* $W_B^*(v) = W_{B\tau}(v) = 0$.

Our baseline model assumes a take-it-or-leave-it bargaining game. But the treatment can be straightforwardly extended to any bargaining protocol as long as traders' types are private information. In particular Lemma 2 holds for the double auction bargaining protocol as well, although, as shown later, our convergence results fail for arbitrary protocol.

We now show that no bargaining mechanism can converge at a faster rate, regardless of whether information is full or private. Any bargaining game played in each meeting results in a trading probability $q(v,c)$ and payment $t(v,c)$ from the buyer to the seller, as functions of traders' types. In steady-state equilibrium, the bargaining outcomes q and t are unchanged over time. Then buyers' and sellers' utilities under this bargaining mechanism are again given by the recursive equations (11) and (17), as long as we redefine q_B, q_S, Π_B and Π_S as follows:

$$q_B(v) \equiv \int q(v,c)d\Gamma(c), \quad q_S(c) \equiv \int q(v,c)d\Phi(v),$$

$$t_B(v) \equiv \int t(v,c)d\Gamma(c), \quad t_S(c) \equiv \int t(v,c)d\Phi(v),$$

$$\Pi_B(v) \equiv q_B(v)v - t_B(v), \quad \Pi_S(c) \equiv t_S(c) - q_S(c)c,$$

where $q_B(v)$ and $q_B(c)$ are, as before, the expected probabilities of trade; $t_B(v)$, $t_S(c)$ are the expected payments; and $\Pi_B(v)$ and $\Pi_S(c)$ are, again as before, the expected bargaining profits. Let $\chi_B(v)$ and $\chi_S(c)$ be the buyers' and sellers' entering probabilities respectively. Individual rationality requires

$$\begin{array}{ll} \ell_B[q_B(v)v - t_B(v)] \geq \kappa_B & \text{if} \quad \chi_B(v) > 0, \\ \ell_S[t_S(c) - q_S(c)c] \geq \kappa_S & \text{if} \quad \chi_S(c) > 0. \end{array} \tag{34}$$

The ex-ante utilities of traders in any such mechanism (including the take-it-or-leave-it bargaining mechanism) is the sum of surpluses of all buyers and sellers in a cohort,

$$W^0 = bW_B^0 + sW_S^0$$

where W_B^0 and W_S^0 are the buyer's and the seller's ex-ante utilities,

$$W_B^0 = \int \chi_B(v)W_B(v)\,dF(v), \qquad W_S^0 = \int \chi_S(c)W_S(c)\,dG(c). \tag{35}$$

We now prove that no individually rational bargaining mechanism can have a faster-than-linear rate of convergence, by establishing an explicit lower bound on $W^{0*} - W^0$.

Theorem 12 *For any individually rational bargaining protocol, in steady-state equilibrium we have*

$$W^{0*} - W^0 \geq \tau \cdot \mu \cdot \min_{\zeta>0} K(\zeta), \tag{36}$$

where μ is the equilibrium mass of buyers (or sellers) that enters the market per unit of time.

Proof: Rewrite the Walrasian total surplus W^{0*}:

$$\begin{aligned} W^{0*} &= b\int_{p^*}^{1}(v-p^*)dF(v)+s\int_{0}^{p^*}(p^*-c)dG(c) \\ &= \max_{\chi_B,\chi_S}\left\{\begin{array}{c} b\int \chi_B(v)vdF(v)-s\int\chi_S(c)cdG(c) \\ \text{s.t. } b\int\chi_B(v)dF(v)=s\int\chi_S(c)dG(c), \\ 0\le\chi_B(v)\le 1, \qquad 0\le\chi_S(c)\le 1 \end{array}\right\}. \end{aligned} \tag{37}$$

Then the total surplus $W^0 = bW_B^0 + sW_S^0$ for this mechanism can be bounded as follows. For all active buyer types v (i.e. $\chi_B(v)\neq 0$), $\ell_B[q_B(v)v - t_B(v)] \geq \kappa_B$, so from (12), we have for those v:

$$\begin{aligned} W_B(v) &= \frac{\ell_B[q_B(v)v-t_B(v)]-\kappa_B}{r+\ell_B q_B(v)} \\ &\le \frac{\ell_B[q_B(v)v-t_B(v)]-\kappa_B}{\ell_B q_B(v)} \\ &= v-\frac{\kappa_B}{\ell_B q_B(v)}-\frac{t_B(v)}{q_B(v)}, \end{aligned}$$

Similarly, for all active seller types c,

$$W_S(c)\le -c-\frac{\kappa_S}{\ell_S q_S(c)}+\frac{t_S(c)}{q_S(c)}.$$

Substituting these bounds into the definitions (35),

$$\begin{aligned} W^0 \le\ & b\int\chi_B(v)vdF(v)-s\int\chi_S(c)cdG(c) \\ & -b\int\chi_B(v)\frac{\kappa_B}{\ell_B}dF(v)-s\int\chi_S(c)\frac{\kappa_S}{\ell_S}dG(c) \\ & -b\int\chi_B(v)\frac{t_B(v)}{q_B(v)}dF(v)+s\int\chi_S(c)\frac{t_S(c)}{q_S(c)}dG(c). \end{aligned} \tag{38}$$

(In the second line, we used $q_B(v)\le 1$ and $q_S(c)\le 1$.) In view of (37), the terms on the right hand side of the first line do not exceed the Walrasian surplus W^{0*}. Also, since the steady-state condition implies that $b\chi_B(v)dF(v) = B\ell_B q_B(v)d\Phi(v)$ and similarly for sellers, and the transfers are balanced,

$$\int t_B(v)d\Phi(v)=\int t_S(c)d\Gamma(c),$$

the last line in (38) is 0.

Taking all these into account, we have

$$W^0\le W^{0*}-b\frac{\kappa_B}{\ell_B}\int\chi_B(v)dF(v)-s\frac{\kappa_S}{\ell_S}\int\chi_S(c)dG(c)$$

and therefore

$$W^{0*}-W^0\ge\left(\frac{\kappa_B}{\ell_B}+\frac{\kappa_S}{\ell_S}\right)\mu.$$

where

$$\mu \equiv s \int \chi_S(c) dG(c)$$

is the equilibrium mass of buyers (or sellers) that enters the market per unit of time. Furthermore,

$$\frac{\kappa_B}{\ell_B(\zeta,\tau)} + \frac{\kappa_S}{\ell_S(\zeta,\tau)} = \tau \cdot K(\zeta) \geq \tau \cdot \min_{\zeta>0} K(\zeta).$$

(36) in the statement of the proposition follows. Q.E.D.

As $\tau \to 0$, we must have $\mu_\tau \to sG(p^*)$ whenever $W_\tau^0 \to W^{0*}$. We therefore have the following corollary.

Corollary 13 *No individually rational bargaining mechanism can attain a faster-than-linear convergence rate for the traders' total ex-ante surplus W_τ^0 as $\tau \to 0$.*

It follows that baseline mechanism is worst-case asymptotic optimal. The intuition for why no other bargaining mechanism can attain a faster rate for the ex-ante surplus is that matching delays will still be present regardless of the efficiency of bargaining. Even if only the buyers with $v \geq p^*$ and sellers with $c \leq p^*$ enter and always trade to full efficiency, there still will be welfare loss at rate τ because of costly participation (and discounting), since the expected time between matches is proportional to τ.

Remark 14 *Since Theorem 12 does not require incentive compatibility, it in particular implies that even with full information, as in Mortensen and Wright (2002), convergence cannot be faster than linear.*

Remark 15 *Theorem 10 and Theorem 12 together imply that the traders' total ex-ante surplus W_τ^0 in our baseline model converges to W^{0*} at exactly linear rate as τ tends to 0.*

6 Results for k-Double Auction

It is interesting to know if convergence, or more strongly convergence at the linear rate, can be proved for other bargaining mechanisms. Although we have not been able to prove a general characterization theorem in this direction, we have a theorem showing that another well-studied trading mechanism, the double auction, does not have robust convergence properties. In other words, some equilibria do not converge to perfect competition.

Recall the rules of the bilateral k-double auction: once a meeting occurs, the buyer and the seller simultaneously and independently submit a bid price p_B and an ask price p_S respectively, and then trade occurs if and only if the buyer's bid is at least as high as the seller's ask, at the weighted average price $(1-k)p_S + kp_B$, where $k \in (0,1)$.

We maintain the notation as before except that the interpretations for $\tilde{v}(v)$, $\tilde{c}(c)$, $p_B(v)$ and $p_S(c)$ are now different. In the k-double auction, the strategies $p_B(v)$ and $p_S(c)$ are the strategies of submitting bids and asks respectively to the auctioneer. There is no responding strategies under double auction, but we still interpret $\tilde{v}(v)$ and $\tilde{c}(c)$ as dynamic types.

The definition for nontrivial steady-state equilibria can be obtained as a straightforward revision of the take-it-or-leave-it offering case. In particular, Lemma 2 still holds here. The proof goes through almost word-by-word, with trading probability function replaced with

$$q_B(v) \equiv \int_{p_S(c) \le p_B(v)} d\Gamma(c)$$

and expected payment function replaced with

$$t_B(v) \equiv \int_{p_S(c) \le p_B(v)} \left[k p_B\left(v\right) + (1-k) p_S\left(c\right)\right] d\Gamma(c).$$

As in the take-it-or-leave-it offering case, a k-double auction equilibrium could be either full-trade or non-full-trade. We claim that the full-trade class of double auction equilibria includes equilibria that are very inefficient, even with arbitrarily small frictions. (But at the same time, this class also includes equilibria that converge to perfect competition.)

The set of full-trade equilibria is even easier to characterize for the double auction case. In particular, from Lemma 2, the active sets A_B, A_S of types are still intervals $[\underline{v}, 1]$ and $[0, \bar{c}]$ for some marginal types $\underline{v}$ and $\bar{c}$; and we also have $\tilde{v}(v) < v$ and $\tilde{c}(c) > c$ for all $v > \underline{v}$ and all $c < \bar{c}$. Since all active traders' trading probabilities are strictly positive, they must in equilibrium submit serious bids/asks, and therefore, we must have $p_B(v) \le \tilde{v}(v) < v$ and $p_S(c) \ge \tilde{c}(c) > c$ for all $v > \underline{v}$ and all $c < \bar{c}$. Now it is clear that for an equilibrium to be full-trade, we must have $\bar{c} \le \underline{v}$, and all traders submit a common bid/ask p and hence every matched pair trades at the price p. Clearly, the bargaining outcome of this class of equilibria is ex-post efficient, in the sense that every meeting results in a trade if and only if $\tilde{v}(v) \ge \tilde{c}(c)$. Furthermore, $\underline{v}$-buyers and $\bar{c}$-sellers have to recover their participation costs, thus in any full-trade equilibrium we have $\bar{c} < p < \underline{v}$ for some $p \in (0, 1)$. See Figure 4.

Any full-trade equilibrium for the double auction case must satisfy the following entry equations and mass balance equation:

$$\ell_B\left(\zeta, \tau\right)\left(\underline{v} - p\right) = \kappa_B, \tag{39}$$

$$\ell_S\left(\zeta, \tau\right)\left(p - \bar{c}\right) = \kappa_S, \tag{40}$$

$$b[1 - F\left(\underline{v}\right)] = sG\left(\bar{c}\right). \tag{41}$$

Unlike the take-it-or-leave-it offering case[18], it is easy to see that the converse is also true, i.e. any quadruple $\{p, \zeta, \underline{v}, \bar{c}\}$ satisfying (39), (40), (41) and $\tau \cdot K\left(\zeta\right) < 1$ must characterize a full-trade equilibrium. In particular, any trader's best-response bid/ask strategy is p, given that all other active traders submit p.[19]

$>$From equations (39) and (40), it follows that the entry gap is

$$\underline{v} - \bar{c} = \tau \cdot K\left(\zeta\right). \tag{42}$$

The next proposition shows that $\underline{v} - \bar{c}$ can be arbitrary close to 1 for all τ (such that an equilibrium exist), so that equilibrium outcomes are arbitrary far from efficiency even

[18] see footnote 16.

[19] Similar logic implies that a full-trade equilibrium continues to be characterized by equations (39)-(41) even if information is complete.

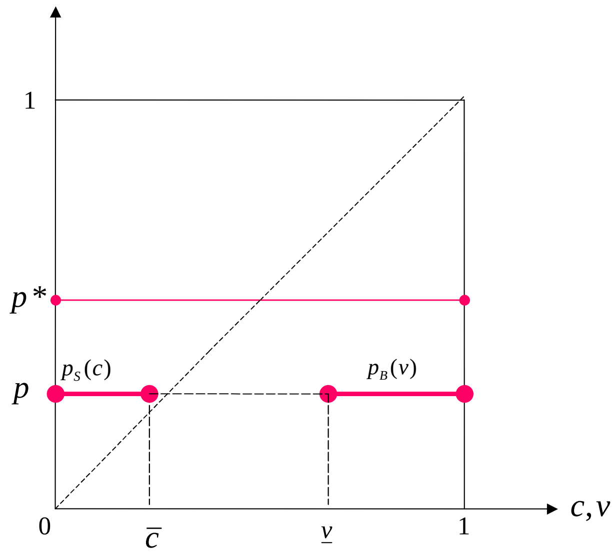

Figure 4: A full-trade equilibrium under double auction

with small frictions. The set of equilibrium entry gaps converges to the full-range $(0, 1)$ as frictions disappear, so the set of full-trade equilibria ranges from perfectly competitive to almost perfectly inefficient. Moreover, the set of equilibrium prices also converges to the full-range $(0, 1)$ as frictions disappear. Thus indeterminacy grows rather vanishes with competition, contrary to the results in the static double auction literature.

Proposition 16 *A full-trade equilibrium exists if and only if the minimal equilibrium entry gap* $\underline{v} - \bar{c}$ *is less than* 1*:*

$$\tau \cdot \min_{\zeta > 0} K(\zeta) < 1. \tag{43}$$

The set of equilibrium values of $\underline{v} - \bar{c}$ *in full-trade equilibria is an interval* $[\tau \cdot \min_{\zeta>0} K(\zeta), 1)$*. As* $\tau \to 0$*, this set and the set of equilibrium prices converge to* $(0, 1)$ *In particular, there exists a sequence of full-trade equilibria that converges to perfect competition, but also sequences that do not converge.*

Proof: The proof follows the graphical argument shown in Figure 5. Given τ, the right panel shows the marginal types $\underline{v}$ and $\bar{c}$ in a steady-state equilibrium. The left panel shows the supportable values of the market tightness $\underline{\zeta}$ and $\bar{\zeta}$ that correspond to the given gap $\underline{v} - \bar{c} < 1$. (In general, there can be one, two or more such values.)

Our assumption $\lim_{B \to 0} M(B, S) = \lim_{S \to 0} M(B, S) = 0$ implies $\lim_{\zeta \to \infty} \ell_B(\zeta) = \lim_{\zeta \to 0} \ell_S(\zeta) = 0$. It in turn implies

$$\lim_{\zeta \to 0} K(\zeta) = \lim_{\zeta \to \infty} K(\zeta) = \infty, \tag{44}$$

as depicted in the left panel.

Given that (44) holds, a solution ζ to the equation $\tau K(\zeta) = \underline{v} - \bar{c}$ exists if and only if $\underline{v} - \bar{c} \in [\tau \cdot \min_{\zeta > 0} K(\zeta), 1)$. Since $\lim_{\tau \to 0} \tau K(\zeta) = 0$ for any $\zeta > 0$, we also must have

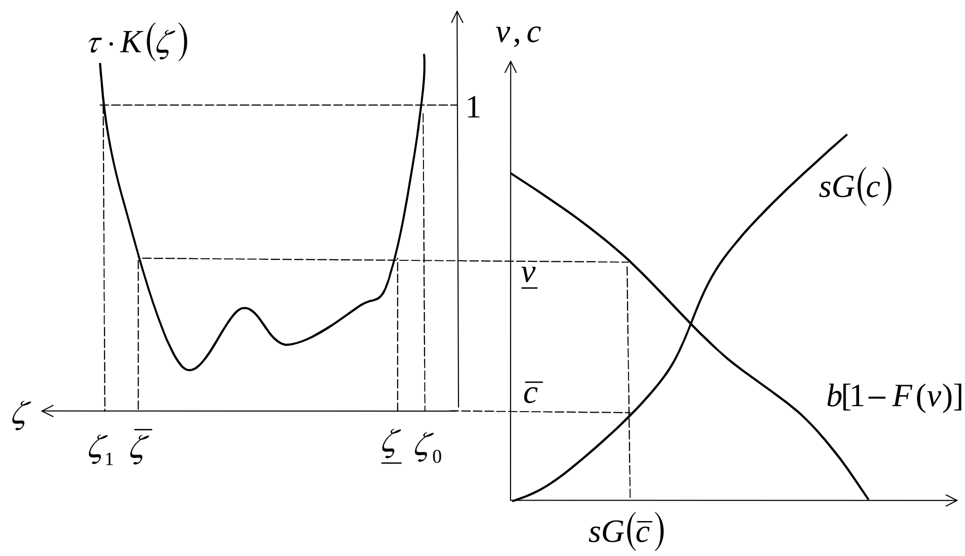

Figure 5: Construction of a double auction full-trade equilibrium

$\tau \cdot \min_{\zeta>0} K(\zeta) \rightarrow 0$ as $\tau \rightarrow 0$. It proves that the set of supportable entry gap $\underline{v} - \bar{c}$ converges to the interval $(0, 1)$.

Now fix any τ such that $\tau \cdot \min_{\zeta>0} K(\zeta) < 1$. Consider the longest interval $[\zeta_0, \zeta_1]$ such that $\tau K(\zeta_0) = \tau K(\zeta_1) = 1$ and $\tau K(\zeta) < 1$ for $\zeta \in (\zeta_0, \zeta_1)$. For any $\zeta \in (\zeta_0, \zeta_1)$, $\underline{v}$ and $\bar{c}$ can be found uniquely from (42) and (41) (graphically shown in Figure 5). Denote $\underline{v}_\tau(\zeta)$ and $\bar{c}_\tau(\zeta)$ as the results. The equilibrium price p can also be found uniquely from equation (39) or equation (40):

$$p_\tau(\zeta) \equiv \bar{c}_\tau(\zeta) + \frac{\kappa_S}{\ell_S(\zeta, \tau)} \tag{45}$$

$$(= \underline{v}_\tau(\zeta) - \frac{\kappa_B}{\ell_B(\zeta, \tau)}). \tag{46}$$

This formally defines a continuous mapping $p_\tau(\cdot)$ of $[\zeta_0, \zeta_1]$ into $\mathbb{R}_+$. Consequently, its image is a closed interval that contains the points $p(\zeta_0)$ and $p(\zeta_1)$; and the set of supportable equilibrium price contains this interval. The definitions of ζ_0 and ζ_1 imply that $\zeta_0 \rightarrow 0$ and $\zeta_1 \rightarrow \infty$ as $\tau \rightarrow 0$. Now $\bar{c}_\tau(\zeta_1) = 0$ for all τ and $\ell_S(\zeta_1, \tau) \rightarrow \infty$ as $\tau \rightarrow 0$, therefore (45) implies that $\lim_{\tau\rightarrow 0} p_\tau(\zeta_1) = 0$. Similarly, $\underline{v}_\tau(\zeta_0) = 1$ for all τ and $\ell_B(\zeta_0, \tau) \rightarrow \infty$ as $\tau \rightarrow 0$, so that (46) implies that $\lim_{\tau\rightarrow 0} p_\tau(\zeta_0) = 1$. It proves that the set of supportable equilibrium price converges to $(0, 1)$. Q.E.D.

It is not hard to see that the condition $\tau \cdot \min_{\zeta>0} K(\zeta) < 1$ is also necessary for any nontrivial steady-state equilibrium to exist. We thus have the following corollary.

Corollary 17 *There exists a nontrivial steady-state equilibrium (either full-trade or non-full-trade) if and only if* $\tau \cdot \min_{\zeta>0} K(\zeta) < 1$.

Proof: Having Proposition 16, it now suffices to claim the necessity of $\kappa_B/\ell_B(\zeta,\tau) + \kappa_S/\ell_S(\zeta,\tau) < 1$ for a nontrivial equilibrium to exist. Recall the notation for a general bargaining game introduced in Section 5. Individual rationality (34) implies

$$\frac{\kappa_B}{\ell_B(\zeta,\tau)} \le \int\int [q(v,c)v - t(v,c)]d\Phi(v)d\Gamma(c),$$

$$\frac{\kappa_S}{\ell_S(\zeta,\tau)} \le \int\int [t(v,c) - q(v,c)c]d\Phi(v)d\Gamma(c),$$

and hence

$$\frac{\kappa_B}{\ell_B(\zeta,\tau)} + \frac{\kappa_S}{\ell_S(\zeta,\tau)} \le \int\int (v-c)d\Phi(v)d\Gamma(c) < 1.$$

Q.E.D.

Remark 18 *This necessary and sufficient condition is weaker than the one under take-it-or-leave-it offering, which is $\tau \cdot K(z) < 1$, as shown in Shneyerov and Wong (2007).*

Proposition 16 shows that the set of double-auction equilibria, even if we restrict attention to the full-trade ones, is very large. For more intuition, re-write the first two characterizing equations of the double-auction full-trade equilibrium in a parallel way to the baseline model:

$$\ell_B(\zeta,\tau)(1-\alpha_{DA})(\underline{v}-\bar{c}) = \kappa_B,$$
$$\ell_S(\zeta,\tau)\alpha_{DA}(\underline{v}-\bar{c}) = \kappa_S,$$

where

$$\alpha_{DA} \equiv \frac{p-\bar{c}}{\underline{v}-\bar{c}}$$

is what we may call the relative bargaining power under double-auction full-trade equilibrium. These equations are the same as the first two equations (25) and (26) that characterize a full-trade equilibrium in our benchmark model, with the only difference that the exogenous bargaining power α is now replaced with the endogenous bargaining power α_{DA}. (The remaining mass balance equation is the same in both models.) If $\alpha_{DA} = \alpha$, the equilibria in both models have the same marginal types $\underline{v}$ and $\bar{c}$, and once these are solved for, the price p is uniquely determined from the equation $\alpha_{DA} = \alpha$, or equivalently $p = \bar{c} + (\underline{v}-\bar{c})\alpha$. In other words, to any $\alpha \in (0,1)$ there corresponds a double-auction full-trade equilibrium with $\alpha_{DA} = \alpha$ and the same marginal types $\underline{v}$ and $\bar{c}$ as in a take-it-or-leave-it full-trade equilibrium.[20]

The above discussion has the following two implications. First, since α_{DA} can be arbitrary, in the double auction model, there is a great multiplicity of equilibria.[21] Second, since we know that full-trade equilibria of the take-it-or-leave-it game converge at the linear rate, it follows immediately that there is a sequence of double-auction equilibria that also converges at the linear rate to perfect competition. We state this finding as a corollary.

[20] Rigorously speaking, we should say "the solution of (25)-(27)" instead of "the take-it-or-leave-it full-trade equilibrium", because this solution might not be an equilibrium. See footnote 16.

[21] The nature of indeterminacy here is analogous to that in the Nash demand game. As is well-known, the outcome of double auction is highly indeterminate even when information is complete.

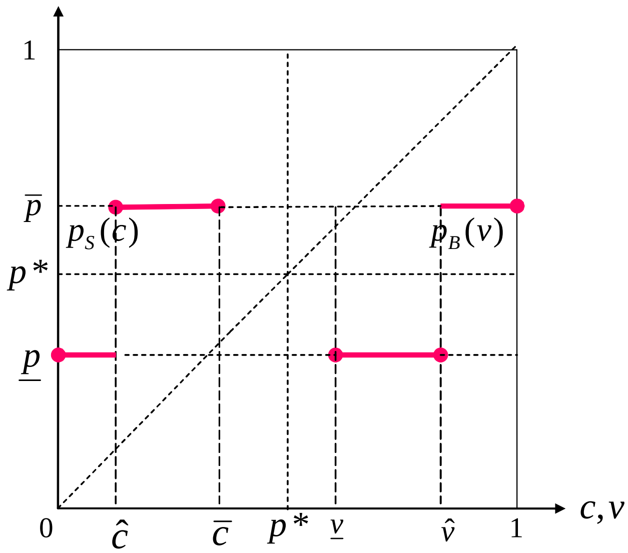

Figure 6: A two-step equilibrium that fails to converge

Corollary 19 *There are double-auction full-trade equilibria that converge, in terms of the ex-ante utilities, at the linear rate in* τ*.*

The above discussion explains why double auction full-trade equilibria can have non-Walrasian limit while it cannot be the case in the take-it-or-leave-it offering analogue. But Figure 5 and the logic in the proof of Proposition 16 also make clear that for the double auction full-trade equilibria to be non-convergent to Walrasian limit, we have to let the bargaining power of one side vanish (i.e. either $\underline{v} - p \to 0$ or $p - \bar{c} \to 0$ as $\tau \to 0$) and also let the market become unbalanced (i.e. either $\zeta \to 0$ or $\zeta \to \infty$ as $\tau \to 0$). One might wonder if all equilibria (e.g. non-full-trade) will converge to the Walrasian outcome if we preclude that class of equilibria, which is perhaps a natural assumption to make. It turns out that this is not so, as we show next.

We construct a non-full-trade equilibrium of the following nature (see Figure 6). There are two seller cutoff types $\hat{c} \in (0,1)$ and $\bar{c} \in (0,1)$ with $\hat{c} < \bar{c}$, and two buyer cutoff types $\hat{v} \in (0,1)$ and $\underline{v} \in (0,1)$ with $\hat{v} > \underline{v}$. The sellers with $c \in [0, \hat{c})$ enter and submit $p_S(c) = \underline{p}$, where $\underline{p}$ is some constant strictly below p^*. The sellers with $c \in [\hat{c}, \bar{c}]$ enter and submit $p_S(c) = \bar{p}$, where $\bar{p} > p^*$. The sellers with $c \in (\bar{c}, 1]$ do not enter. Similarly, the buyers with $v \in (\hat{v}, 1]$ enter and submit $\bar{p}$, the buyers with $v \in [\underline{v}, \hat{v}]$ enter and submit $\underline{p}$, and the buyers with $v \in [0, \underline{v})$ do not enter. We call the equilibria of this kind *two-step equilibria*.

The following theorem gives our non-convergence result for the two-step (non-full-trade) equilibria.[22]

Theorem 20 *For any* $a \in (0,1)$*, there exist* $r_0 > 0$*,* $\tau_0 > 0$ *and* $\bar{W} < W^{0*}$ *such that for all* $r \in [0, r_0)$ *and* $\tau \in (0, \tau_0)$*, there exists a two-step equilibrium in which the price spread is larger than* a*, i.e.* $\bar{p} - \underline{p} > a$*, and the total ex-ante surplus is smaller than* $\bar{W}$*, i.e.* $W^0 < \bar{W}$*.*

[22] As a by-product, we also prove the existence of non-full-trade equilibrium for small τ and r.

Proof: We derive a system of equations characterizing the set of two-step equilibria. But before doing so, it is convenient to introduce some notations. In a two-price equilibrium, the buyers with $v > \hat{v}$ who submit the high bid price $\bar{p}$, trade with any seller they meet. Buyers with $v \in [\underline{v}, \hat{v}]$, who submit the low bid price $\underline{p}$, trade only with those sellers with $c < \hat{c}$, who submit $\underline{p}$; their probability of trading is equal to $\Gamma(\hat{c})$. Similarly sellers with $c < \hat{c}$ trade with any buyer they meet, and sellers with $c \in [\hat{c}, \bar{c}]$ trade only with those buyers with $v > \hat{v}$; their probability of trading is equal to $1 - \Phi(\hat{v})$.

In our constructed equilibria $\Gamma(\hat{c})$ and $1 - \Phi(\hat{v})$ will converge to 0 as τ goes to 0, so it is convenient to divide them by τ:

$$\pi_B \equiv \frac{1 - \Phi(\hat{v})}{\tau}, \qquad \pi_S \equiv \frac{\Gamma(\hat{c})}{\tau}.$$

Since $\underline{v}$-buyers and $\bar{c}$-sellers are indifferent between entering or not, we have

$$\ell_B \tau \pi_S (\underline{v} - \underline{p}) = \kappa_B \tag{47}$$

$$\ell_S \tau \pi_B (\bar{p} - \bar{c}) = \kappa_S. \tag{48}$$

Since $\hat{v}$-buyers are indifferent between biding $\underline{p}$ or $\bar{p}$, and $\hat{c}$-sellers are indifferent between asking $\underline{p}$ or $\bar{p}$, we have

$$\tau \pi_S [\tilde{v}(\hat{v}) - \underline{p}] = \tau \pi_S \left\{ \tilde{v}(\hat{v}) - [(1-k)\underline{p} + k\bar{p}] \right\} + (1 - \tau \pi_S) [\tilde{v}(\hat{v}) - \bar{p}] \tag{49}$$

$$\tau \pi_B [\bar{p} - \tilde{c}(\hat{c})] = \tau \pi_B \left\{ [(1-k)\underline{p} + k\bar{p}] - \tilde{c}(\hat{c}) \right\} + (1 - \tau \pi_B) [\underline{p} - \tilde{c}(\hat{c})]. \tag{50}$$

Since the utility equations (21), (22) still hold here, we have

$$\hat{W}_B = (\hat{v} - \underline{v}) \frac{m(\zeta)\pi_S}{\zeta r + m(\zeta)\pi_S} \tag{51}$$

$$\hat{W}_S = (\bar{c} - \hat{c}) \frac{m(\zeta)\pi_B}{r + m(\zeta)\pi_B}. \tag{52}$$

where we denoted $\hat{W}_B \equiv W_B(\hat{v})$ and $\hat{W}_S \equiv W_S(\hat{c})$.

To complete the description of the two-step equilibrium, the indifference conditions are supplemented with steady-state mass balance conditions for each interval of types. Here, it suffices to require that the total inflows into the intervals $[\underline{v}, 1]$ and $[0, \bar{c}]$ are balanced with outflows,

$$b\left[1 - F(\underline{v})\right] = Sm(\zeta)\left[\pi_S + \pi_B(1 - \tau\pi_S)\right], \tag{53}$$

$$sG(\bar{c}) = Sm(\zeta)\left[\pi_B + \pi_S(1 - \tau\pi_B)\right] \tag{54}$$

and that the inflows into the intervals $v \in [\hat{v}, 1]$ and $[0, \hat{c}]$ are also balanced with outflows,

$$b[1 - F(\hat{v})] = Sm(\zeta)\pi_B, \tag{55}$$

$$sG(\hat{c}) = Sm(\zeta)\pi_S. \tag{56}$$

(Observe that the matching rate is $Sm(\zeta)/\tau$ for both buyers and sellers, and that τ cancels out.) We also define the price spread,

$$a_0 \equiv \bar{p} - \underline{p}.$$

Then equations (47) through (56) form a 10-equation system with 12 endogenous variables $\{\underline{p}, a_0, \zeta, \underline{v}, \bar{c}, \hat{v}, \hat{c}, \pi_B, \pi_S, S, \hat{W}_B, \hat{W}_S\}$. This system does characterize an equilibrium. Indeed, one can easily see that buyers with $v \in (\hat{v}, 1]$ strictly prefer to bid $\bar{p}$, buyers with $v \in (\underline{v}, \hat{v})$ strictly prefer to bid $\underline{p}$, and buyers with $v \in [0, \underline{v})$ strictly prefer not to enter. Similar remark applies for sellers.

Since we have two degrees of freedom, we can fix some $\zeta > 0$ and $a_0 \in (a, 1)$ and then let equations (47) - (56) determine $\{\underline{p}, \underline{v}, \bar{c}, \hat{v}, \hat{c}, \pi_B, \pi_S, S, \hat{W}_B, \hat{W}_S\}$. We claim that solution exists for small enough τ and r. To see this, one can check that when $\tau = r = 0$, we have a (unique) solution with $\underline{p}$ implicitly determined by $b[1 - F(\underline{p} + a_0)] = sG(\underline{p})$, and all other variables given by

$$\bar{c} = \underline{p}, \ \ \underline{v} = \bar{p} = \underline{p} + a_0, \ \ \pi_B = \frac{\kappa_S}{m(\zeta)a_0}, \ \ \pi_S = \frac{\kappa_B\zeta}{m(\zeta)a_0}, \ \ S = \frac{sG(\underline{p})a_0}{\kappa_B\zeta + \kappa_S},$$

$$1 - F(\hat{v}) = \frac{[1 - F(\bar{p})]\kappa_S}{\kappa_B\zeta + \kappa_S}, \ \ G(\hat{c}) = \frac{G(\underline{p})\kappa_B\zeta}{\kappa_B\zeta + \kappa_S}, \ \ \hat{W}_B = \hat{v} - \bar{p}, \ \ \hat{W}_S = \underline{p} - \hat{c}.$$

One can also check that the Jacobian evaluated at $\tau = r = 0$ is not zero.[23] Therefore the Implicit Function Theorem applies. Because $\bar{p} - \underline{p} \equiv a_0 > a$, there exists a two-step equilibrium with $\bar{p} - \underline{p} > a$ when τ and r are small enough. Moreover, since $\underline{v} \to \bar{p}$ and $\bar{c} \to \underline{p}$ as $(\tau, r) \to (0, 0)$, the spread $\underline{v} - \bar{c}$ is also bounded below by a. It follows that the associated total ex-ante surplus W^0 is bounded away from the Walrasian total ex-ante surplus W^{0*}. Q.E.D.

Unlike Proposition 16, the construction in the proof of Theorem 20 treats buyers and sellers symmetrically. In particular, ζ could be fixed at any value. Then why does the double auction mechanism has non-Walrasian limit equilibria while the take-it-or-leave-it offering mechanism does not? One can verify that the dynamic types do collapse to singletons even in the two-step non-convergent equilibria. Thus to fix the idea, let us simply suppose the discount rate r is 0 so that the ultimate trading probabilities are 1 and therefore the dynamic types are constant and equal $\tilde{c} = \bar{c} \to \underline{p}$ and $\tilde{v} = \underline{v} \to \bar{p}$. Also suppose τ is very small. Then all buyers have dynamic types approximately $\underline{p}$ and all sellers have dynamic types approximately $\bar{p}$. Unlike under take-it-or-leave-it offering, the dynamic types are no longer the acceptance levels. Effectively the bids/asks also play this role. A seller submitting an ask lower than the dynamic types of all buyers does not guarantee herself a successful trade. To guarantee a trade, she has to ask lower than all buyers' bids. Consider a seller with $c < \hat{c}$. This seller's equilibrium ask price is $\underline{p}$. She realizes fully that the buyer's dynamic willingness-to-pay is always $\bar{p}$ approximately, and would like to demand that much if acceptance is guaranteed, as it would be under the take-it-or-leave-it offering. However, demanding that much under the double auction protocol runs into the risk of being countered with the buyer's bid of $\underline{p}$, resulting in no trade. In our equilibrium with τ small, most of the active buyers bid $\underline{p}$. Weighing these trade-offs carefully, the seller decides to submit $\underline{p}$ and not $\bar{p}$. Similar logic applies to the buyers.

[23] The Mathematica® notebook that contains the evaluation of the Jacobian is available at www.econ.ubc.ca/sart/homepage.htm.

Now consider a seller with $c = \underline{p} + \varepsilon$ where $\varepsilon > 0$ is small. Although her type (or dynamic type) is significantly lower than buyers' dynamic types, which is $\bar{p}$ approximately, she chooses not to enter even when the expected participation costs incurred to obtain a meeting is very small as τ becomes very small. It is again because most of the active buyers bid $\underline{p}$, making her prospect of trade meager. Similar logic applies to the buyers.

Finally, to complete our logic, we explain why the fraction of active buyers bidding $\underline{p}$ is very high relative to the fraction bidding $\bar{p}$. It is because, in our equilibrium, buyers bidding $\underline{p}$ can only trade with those sellers asking $\underline{p}$, which makes their outflow rate tiny. On the contrary, buyers bidding $\bar{p}$ trade in any meeting. Thus in steady state, the buyers who bid $\underline{p}$ accumulate and dominate the buyers' side of the market. Similar logic applies to the sellers. These arguments together explain why marginal traders do not enter to quest the significant size of the unexploited surplus $\underline{v} - \bar{c}$, keeping a positive gap between $\underline{v}$ and $\bar{c}$.

The rules of the double auction do not provide a tight connection between the dynamic types and actual acceptance levels as would be the case under the take-it-or-leave-it bargaining. Here, a bid/ask is both an offer and an acceptance level. On the contrary, under take-it-or-leave-it offering, proposing strategies and responding strategies are separate decisions because traders are clear about who is proposer and who is responder. Ex-post, the bargaining power is given to one party, and thus well-defined. Therefore the responder is always held to her acceptance level, which creates strong incentive to enter. Ex-ante, both parties could have the full bargaining power. Therefore the incentives to enter are evenly distributed over both sides of the market, driving the marginal participating types close to each other and to the Walrasian price, and leading to rapid convergence.

7 Concluding remarks

In this paper, we have shown that the take-it-or-leave-it matching and bargaining mechanism converges to perfect competition at the linear rate in τ, a number proportional to the length of the waiting period until the next meeting. No other individually rational mechanism can attain a faster convergence rate, and some mechanisms have equilibria that may not be convergent. In particular, we show that the double auction mechanism has non-convergent equilibria.

One caveat is that the non-convergent examples we have constructed for the double auction are all quite special in that they require a great deal of coordination among the traders. Additional assumptions (e.g. continuity of strategies and boundedness of the ratio of buyers to sellers) could be imposed to restrict the set of equilibria with the purpose of proving their convergence at the linear rate, but this has not yielded to our efforts. In addition, allowing a multilateral matching technology may also restore convergence of all equilibria of the double auction mechanism. Also, we have only explored specific bargaining protocols. Characterizing a set of protocols for which the rate of convergence is (optimal) linear is an open question.[24]

[24] In particular, the general techniques recently developed by Lauermann (2006) could be useful.

Appendix: Rate of Convergence under Full Information

This appendix proves the rate of convergence result of Section 4, i.e. Theorem 5 and Corollary 8, in the context of full information bargaining. Once they are proved, all results in Section 5 straightforwardly go through. Indeed, Theorem 12 has nothing to do with whether information is private or public; the proof of Lemma 9 is valid, word by word, under full information; and Theorem 10 only need an obvious modification for the upper bound in the statement.

We assume that, as in our baseline model, traders bargaining using the take-it-or-leave-it offering protocol with bargaining weight $\alpha \in (0,1)$. In this environment of full information, one could also equivalently assume that the bargaining outcome of a matched pair is given by the generalized Nash bargaining solution with seller's relative bargaining power being α.[25]

A v-buyer and a c-seller, when they meet, trade if and only if the matching surplus $v - c - W_B(v) - W_S(c)$ is non-negative, or equivalently $\tilde{v}(v) \geq \tilde{c}(c)$, where $\tilde{v}$ and $\tilde{c}$ are still defined by (1) and (2).

The recursive equations for buyers' utility $W_B(v)$ in the searching state is

$$W_B(v) = \max\left\{R_B\left[(1-\alpha)\int \max\{v - \tilde{c}(c), W_B(v)\}\, d\Gamma(c) + \alpha W_B(v)\right] - K_B, 0\right\}. \quad (57)$$

The explanation of (57) is similarly to that of its private information counterpart, except that now the buyer would capture all, if any, the matching surplus by proposing at his partner's acceptance level $\tilde{c}(c)$ whenever he is the proposer; and he would get none of the matching surplus when either there is no such surplus or he is the responder. Similarly, the recursive equations for sellers' utility $W_S(c)$ in the searching state is

$$W_S(c) = \max\left\{R_S\left[\alpha\int \max\{\tilde{v}(v) - c, W_S(c)\}\, d\Phi(v) + (1-\alpha) W_S(c)\right] - K_S, 0\right\}. \quad (58)$$

And buyers' and sellers' trading probabilities $q_B(v)$ and $q_S(c)$ conditional on a meeting are

$$q_B(v) \equiv \int_{\tilde{v}(v)\geq\tilde{c}(c)} d\Gamma(c) \text{ and } q_S(c) \equiv \int_{\tilde{v}(v)\geq\tilde{c}(c)} d\Phi(v).$$

Solving for $W_B(v)$ and $W_S(c)$, we have

$$W_B(v) = \max\left\{\frac{\ell_B(\zeta,\tau)(1-\alpha)\int_{\tilde{v}(v)\geq\tilde{c}(c)}[v - \tilde{c}(c)]\, d\Gamma(c) - \kappa_B}{r + \ell_B(\zeta,\tau)(1-\alpha) q_B(v)}, 0\right\},$$

$$W_S(c) = \max\left\{\frac{\ell_S(\zeta,\tau)\alpha\int_{\tilde{v}(v)\geq\tilde{c}(c)}[\tilde{v}(v) - c]\, d\Phi(v) - \kappa_S}{r + \ell_S(\zeta,\tau)\alpha q_S(c)}, 0\right\}.$$

[25] Despite the equivalence, assuming that traders bargain mechanically according to generalized Nash bargaining solution allows us to formally get rid of the proposing and responding strategies.

Thus the entry strategies are

$$\chi_B(v) = I\left[\ell_B(\zeta,\tau)(1-\alpha)\int_{\tilde{v}(v)\geq\tilde{c}(c)}[v-\tilde{c}(c)]\,d\Gamma(c)\geq\kappa_B\right], \tag{59}$$

$$\chi_S(c) = I\left[\ell_S(\zeta,\tau)\alpha\int_{\tilde{v}(v)\geq\tilde{c}(c)}[\tilde{v}(v)-c]\,d\Phi(v)\geq\kappa_S\right]. \tag{60}$$

Finally, the steady-state equations (19) and (20) have nothing to be changed.

Definition 21 *A collection $E \equiv \{\chi_B, \chi_S, B, S, \Phi, \Gamma\}$ is a nontrivial steady-state equilibrium under full information if there exists a pair of equilibrium payoff functions $\{W_B, W_S\}$ such that the entry strategies χ_B and χ_S satisfy* (59) *and* (60)*, the distributions of active buyer and seller types Φ and Γ and active trader masses B and S solve the steady-state equations* (19) *and* (20)*, and the payoff functions W_B and W_S solve the recursive equations* (57) *and* (58)*.*

Moreover, the definitions of $\underline{v}$, $\bar{v}$, $\underline{c}$, $\bar{c}$, A_B and A_S are not changed here. The price of any successful trade must lie in the interval $[\underline{c}, \bar{v}]$. Our convergence result will be based on the difference $\bar{v} - \underline{c}$.

We will need the following lemmas. The first one is parallel to Lemma 2.

Lemma 22 *In any nontrivial steady-state equilibrium , $W_B(v)$ and $W_S(c)$ are absolutely continuous and convex. $W_B(v)$ is nondecreasing and $W_S(c)$ is nonincreasing. Moreover,*

$$W_B(v) = \int_{\underline{v}}^{v}\frac{\ell_B(1-\alpha)q_B(x)}{r+\ell_B(1-\alpha)q_B(x)}dx \quad \text{for all } v\in[\underline{v},1] \tag{61}$$

$$W_S(c) = \int_{c}^{\bar{c}}\frac{\ell_S\alpha q_S(x)}{r+\ell_S\alpha q_S(x)}dx \quad \text{for all } c\in[0,\bar{c}]\,. \tag{62}$$

The sets of active trader types are $A_B = [\underline{v}, 1]$ and $A_S = [0, \bar{c}]$. The probabilities of trading are monotonic: $q_B(v)$ is nondecreasing in v on A_B, while $q_S(c)$ is nonincreasing in c on A_S. The dynamic types $\tilde{v}(v) = v - W_B(v)$ and $\tilde{c}(c) = c + W_S(c)$ are absolutely continuous and nondecreasing. Their slopes are

$$\tilde{v}'(v) = \frac{r}{r+\ell_B(1-\alpha)q_B(v)} \quad (\text{a.e. } v\in A_B) \tag{63}$$

$$\tilde{c}'(c) = \frac{r}{r+\ell_S\alpha q_S(v)} \quad (\text{a.e. } c\in A_S)\,. \tag{64}$$

Proof: We prove the results for buyers only. If $R_B = 1$ (or $r = 0$), the recursive equation indicate that whenever $W_B(v) \neq 0$, we have $(1-\alpha)\int\max\{v - W_B(v) - \tilde{c}(c), 0\}\,d\Gamma(c) = K_B > 0$ so that $v - W_B(v)$ must be some positive constant x. It is then easily seen that the recursive equation has a unique solution $W_B(v) = \max\{v - x, 0\}$, which is nondecreasing, continuous and convex.[26]

Now suppose $R_B < 1$ (or $r > 0$). Then the right hand side of the recursive equation can be regarded as a contraction mapping that assigns each W_B another function on the same domain. Applying standard techniques of discounted dynamic programming, we can see that the solution W_B is unique, nondecreasing, continuous and convex.

Then from continuity and monotonicity, $W_B(v)$ is absolutely continuous and hence differentiable almost everywhere. Whenever differentiable, we have

$$\begin{aligned} W_B'(v) &= \chi_B(v)R_B \cdot \frac{d}{dv}\left[(1-\alpha)\int \max\{v - \tilde{c}(c), W_B(v)\}\,d\Gamma(c) + \alpha W_B(v)\right] \\ &= \chi_B(v)R_B \cdot \frac{d}{dv}\left[(1-\alpha)\int \max\{v - W_B(v) - \tilde{c}(c), 0\}\,d\Gamma(c) + W_B(v)\right] \\ &= \chi_B(v)R_B\left\{(1-\alpha)\int_{v-W_B(v)\geq\tilde{c}(c)}[1 - W_B'(v)]d\Gamma(c) + W_B'(v)\right\} \\ &= \chi_B(v)R_B\left\{(1-\alpha)q_B(v)[1 - W_B'(v)] + W_B'(v)\right\}. \end{aligned}$$

Solve for $W_B'(v)$ and simplify,

$$W_B'(v) = \chi_B(v)\frac{\ell_B(1-\alpha)q_B(v)}{r + \ell_B(1-\alpha)q_B(v)}.$$

It then follows from the convexity of W_B that q_B is nondecreasing on A_B.

For $v \in A_B$, the trading probability $q_B(v)$ must be strictly positive, otherwise the participation cost κ_B cannot be recovered. Thus $W_B(v)$ is strictly increasing on A_B and $A_B = [\underline{v}, 1]$.

In order to prove (61), it now suffices to show $W_B(\underline{v}) = 0$. Indeed, if $W_B(\underline{v}) \neq 0$, then either $\underline{v} = 0$ or $\underline{v} = 1$. We preclude the possibility of $\underline{v} = 1$ because we are looking at nontrivial equilibrium. $\underline{v} = 0$ is also impossible because in that case type 0 buyer cannot expect their participation cost recovered.

It follows from its definition that $\tilde{v}$ is absolutely continuous because W_B is. Its derivative, which exists almost everywhere on A_B, is given by (63), which is non-negative. Q.E.D.

Corollary 23 *In any nontrivial steady-state equilibrium , $\tilde{v}(\underline{v}) = \underline{v}$ and $\tilde{c}(\bar{c}) = \bar{c}$. Moreover,*

$$\ell_B(1-\alpha)\int \max\{\underline{v} - \tilde{c}(c), 0\}\Gamma(c) = \kappa_B \tag{65}$$

$$\ell_S\alpha\int \max\{\tilde{v}(v) - \bar{c}, 0\}\Phi(v) = \kappa_S. \tag{66}$$

[26] If $v - x < 0$ then $W_B(v)$ cannot be $v - x$ and hence $W_B(v) = 0$. If $v - x > 0$ then $W_B(v)$ cannot be 0 because the first maximand at $W_B(v) = 0$ is

$$(1-\alpha)\int \max\{v - \tilde{c}(c), 0\}d\Gamma(c) - K_B > (1-\alpha)\int \max\{x - \tilde{c}(c), 0\}d\Gamma(c) - K_B = 0.$$

Proof. They are implications of $W_B(\underline{v}) = W_S(\bar{c}) = 0$. Q.E.D.

Lemma 24 *In any nontrivial steady-state equilibrium , $\underline{c} < \underline{v}$ and $\bar{c} < \bar{v}$.*

Proof: We prove $\underline{c} < \underline{v}$ only. Suppose $\underline{v} \leq \underline{c}$. Then buyer with type $\underline{v}$ must prefer not enter as she cannot recover the participation cost. Q.E.D.

Theorem 25 *In any nontrivial steady-state equilibrium , we have*

$$\tau \cdot K(z) \leq \bar{v} - \underline{c} \leq \tau \cdot K(z)\left(1 + \frac{r}{\kappa}\right)^2,$$

where $\kappa \equiv \min\{\kappa_B, \kappa_S\}$.

Notice that the upper bound provided in Theorem 25 converges to the lower bound as r gets small relative to $\kappa \equiv \min\{\kappa_B, \kappa_S\}$. It indicates that our bounds are tight at least when the discount rate is small relative to the search costs.

Proof of Theorem 25:

Step 1: We claim that

$$\text{(a):} \quad \frac{\underline{v} - \underline{c}}{\bar{v} - \underline{c}} \geq \frac{\kappa_B}{r + \kappa_B} \quad \text{and} \quad \text{(b):} \quad \frac{\bar{v} - \bar{c}}{\bar{v} - \underline{c}} \geq \frac{\kappa_S}{r + \kappa_S}.$$

We provide the proof for part (a) only. The proof for part (b) is the flip of that for part (a). Apply (65),

$$\frac{\kappa_B}{\ell_B(1-\alpha)} = \int_{\underline{v} \geq \tilde{c}(c)} [\underline{v} - \tilde{c}(c)]\Gamma(c) \leq \int_{\underline{v} \geq \tilde{c}(c)} [\underline{v} - \underline{c}]\Gamma(c) = q_B(\underline{v})(\underline{v} - \underline{c}).$$

Thus for any $v \geq \underline{v}$, we have $\ell_B(1-\alpha)q_B(v) \geq \kappa_B/(\underline{v} - \underline{c})$. Then for almost all $v \in [\underline{v}, 1]$,

$$\tilde{v}'(v) = \frac{r}{r + \ell_B(1-\alpha)q_B(v)} \leq \frac{r}{r + \kappa_B/(\underline{v} - \underline{c})}.$$

Hence

$$\bar{v} - \underline{v} = \int_{\underline{v}}^{1} \tilde{v}'(v)dv \leq \frac{r}{r + \kappa_B/(\underline{v} - \underline{c})},$$

$$\frac{\bar{v} - \underline{v}}{\underline{v} - \underline{c}} \leq \frac{r}{(\underline{v} - \underline{c})r + \kappa_B} \leq \frac{r}{\kappa_B},$$

$$\frac{\underline{v} - \underline{c}}{\bar{v} - \underline{c}} = \frac{1}{1 + (\bar{v} - \underline{v})/(\underline{v} - \underline{c})} \geq \frac{1}{1 + \frac{r}{\kappa_B}} = \frac{\kappa_B}{r + \kappa_B}.$$

Step 2: We claim that

$$\text{(a):} \quad \min\{\underline{v}, \bar{c}\} - \underline{c} \leq \frac{r}{\ell_S \alpha} \quad \text{and} \quad \text{(b):} \quad \bar{v} - \max\{\underline{v}, \bar{c}\} \leq \frac{r}{\ell_B(1-\alpha)}.$$

Again by symmetry, we only provide a proof for (a). It is clear that $q_S(c) = 1$ if $\tilde{c}(c) \le \min\{\underline{v}, \bar{c}\}$. Thus,

$$\min\{\underline{v}, \bar{c}\} - \underline{c} = \int_{\tilde{c}(c) \le \min\{\underline{v}, \bar{c}\}} \tilde{c}'(c) dc \le \frac{r}{r + \ell_S \alpha} \le \frac{r}{\ell_S \alpha}.$$

Step 3: We claim that

$$\max\{\underline{v} - \bar{c}, 0\} \le \min\left\{\frac{\kappa_S}{\ell_S \alpha}, \frac{\kappa_B}{\ell_B(1-\alpha)}\right\}.$$

To prove it, apply (66), we have

$$\kappa_S \ge \ell_S \alpha \int \max\{\underline{v} - \bar{c}, 0\} \Gamma(c) = \ell_S \alpha \max\{\underline{v} - \bar{c}, 0\}.$$

Similarly, we also have $\kappa_B \ge \ell_B(1-\alpha)\max\{\underline{v} - \bar{c}, 0\}$.

Step 4: We claim that

$$\bar{v} - \underline{c} \le \min\left\{\frac{\kappa_B}{\ell_B(1-\alpha)}, \frac{\kappa_S}{\ell_S \alpha}\right\}\left(1 + \frac{r}{\kappa_B}\right)\left(1 + \frac{r}{\kappa_S}\right).$$

To prove it, first notice that from step 2(a) and step 3, we have

$$\underline{v} - \underline{c} = \min\{\underline{v}, \bar{c}\} - \underline{c} + \max\{\underline{v} - \bar{c}, 0\} \le \frac{r}{\ell_S \alpha} + \frac{\kappa_S}{\ell_S \alpha} = \frac{\kappa_S}{\ell_S \alpha}\left(1 + \frac{r}{\kappa_S}\right).$$

Then from step 1(a),

$$\bar{v} - \underline{c} \le \frac{r + \kappa_B}{\kappa_B}(\underline{v} - \underline{c}) \le \frac{\kappa_S}{\ell_S \alpha}\left(1 + \frac{r}{\kappa_B}\right)\left(1 + \frac{r}{\kappa_S}\right).$$

Similarly, from step 2(b) and step 3 and 1(b),

$$\bar{v} - \underline{c} \le \frac{\kappa_B}{\ell_B(1-\alpha)}\left(1 + \frac{r}{\kappa_B}\right)\left(1 + \frac{r}{\kappa_S}\right).$$

Step 5: We claim that

$$\bar{v} - \underline{c} \ge \max\left\{\frac{\kappa_S}{\ell_S \alpha}, \frac{\kappa_B}{\ell_B(1-\alpha)}\right\}.$$

To prove it, observe that (65), (66) and Lemma 24 imply

$$\kappa_B \le \ell_B(1-\alpha)(\bar{v} - \underline{c})$$

$$\kappa_S \le \ell_S \alpha(\bar{v} - \underline{c}).$$

Step 6: Notice that $\frac{\kappa_S}{\ell_S \alpha}$ is nonincreasing and $\frac{\kappa_B}{\ell_B(1-\alpha)}$ is nondecreasing in ζ, and that they are equal if and only if $\zeta = z$, at which both of them equal $\tau K(z)$. Thus we have

$$\min\left\{\frac{\kappa_S}{\ell_S \alpha}, \frac{\kappa_B}{\ell_B(1-\alpha)}\right\} \le \tau K(z) \le \max\left\{\frac{\kappa_S}{\ell_S \alpha}, \frac{\kappa_B}{\ell_B(1-\alpha)}\right\}.$$

Then from steps 4 and 5,

$$\tau \cdot K(z) \leq \bar{v} - \underline{c} \leq \tau \cdot K(z) \left(1 + \frac{r}{\kappa_B}\right)\left(1 + \frac{r}{\kappa_S}\right) \leq \tau \cdot K(z) \left(1 + \frac{r}{\kappa}\right)^2.$$

Q.E.D.

Corollary 26 *For any sequence of nontrivial steady-state equilibria parametrized by τ such that $\tau \to 0$, the price interval $[\underline{c}_\tau, \bar{v}_\tau]$ collapses to the Walrasian price $\{p^*\}$ at no slower than linear convergence rate. More precisely,*

$$|\underline{c}_\tau - p^*|, |\bar{v}_\tau - p^*| < \tau \cdot K(z) \left(1 + \frac{r}{\kappa}\right)^2.$$

Proof: The same as the proof of Corollary 8, except that the applications of Lemma 4 and Theorem 5 are replaced by their full-information counterparts Lemma 24 and Theorem 25. Q.E.D.

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