CTFT (2B)

- Continuous Time Fourier Transform

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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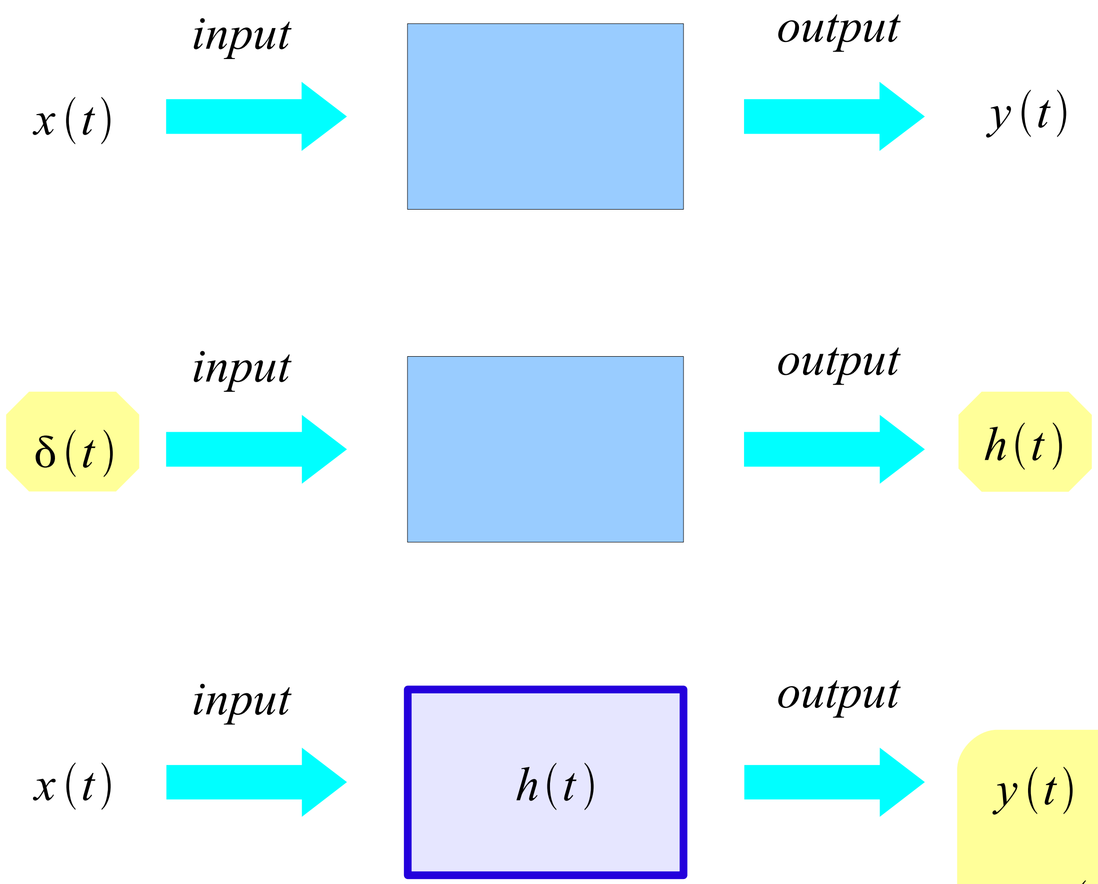
Impulse Response

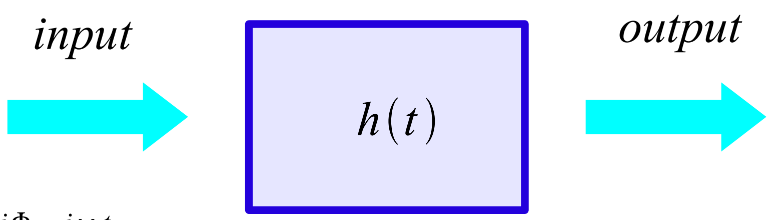


$$y(t)$$

$$= h(t) * x(t)$$

$$= \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Frequency Response



$$x(t) = \underline{A e^{j\Phi} e^{j\omega t}}$$

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau) x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau) A e^{j\Phi} e^{j\omega(t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau) \underline{A e^{j\Phi} e^{j\omega t}} e^{-j\omega\tau}\, d\tau$$

$$= \underline{A e^{j\Phi} e^{j\omega t}} \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau}\, d\tau$$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-jw\tau}\, d\tau$$

$$x(t) = A e^{j\Phi} e^{j\omega t}$$

$$y(t) = H(j\omega) A e^{j\Phi} e^{j\omega t}$$

CTFS and CTFT

Continuous Time Fourier Series

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_n\, e^{+jn\omega_0 t}$$

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

From CTFS to CTFT

Continuous Time Fourier **Series**

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_n\, e^{+jn\omega_0 t}$$

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jn\omega_0 t}\, dt \qquad x_{T_0}(t) = \sum_{n=0}^{\infty} C_n\, e^{+jn\omega_0 t} \cdot \frac{2\pi}{2\pi} \cdot \frac{T_0}{T_0}$$

$$C_n T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jn\omega_0 t}\, dt \qquad x_{T_0}(t) = \frac{1}{2\pi}\sum_{n=0}^{\infty} C_n T_0\, e^{+jn\omega_0 t} \cdot \frac{2\pi}{T_0}$$

$$T_0 \rightarrow \infty \qquad C_n T_0 \rightarrow X(j\omega) \qquad x_{T_0} \rightarrow x(t) \qquad \omega = \frac{2\pi}{T_0} \rightarrow d\omega$$

Continuous Time Fourier **Transform**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

CTFT of Time Domain Impulse

Continuous Time Fourier Transform

$$x(t) = A\delta(t) \Longleftrightarrow X(j\omega) = A$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(j\omega) = \int_{-\infty}^{+\infty} A\delta(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} A\delta(t) e^{0} dt$$

$$= A \int_{-\infty}^{+\infty} \delta(t) dt = A$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A\, e^{+j\omega t} d\omega$$

$$= \frac{A}{2\pi} \int_{-\infty}^{+\infty} e^{+j\omega t} d\omega = A\delta(t)$$

CTFT of Frequency Domain Impulse

Continuous Time Fourier Transform

$$X(j\omega) = 2\pi\,\delta(\omega) \iff x(t) = 1$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\,\delta(\omega) e^{+j\omega t}\, d\omega$$

$$= \int_{-\infty}^{+\infty} \delta(\omega) e^{0}\, d\omega = 1$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t}\, dt = 2\pi\,\delta(\omega)$$

CTFS of Impulse Train

Continuous Time Fourier <u>Series</u>

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$C_n = \frac{1}{T_s}\int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jn\omega_s t}\, dt$$

$$= \frac{1}{T_s}\int_{-T_s/2}^{+T_s/2} \delta(t)\, dt = \frac{1}{T_s}$$

$$x(t) = \sum_{n=0}^{\infty} C_n\, e^{+jn\omega_0 t}$$

$$p(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_s t}$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

CTFT of Impulse Train

Continuous Time Fourier Transform

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$P(j\omega) = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) e^{-jn\omega t} dt = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t} e^{-jn\omega t} dt$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} \int_{-\infty}^{+\infty} e^{-j(\omega - n\omega_s)t} dt$$

$$= \sum_{n=-\infty}^{+\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - n\omega_s)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$$

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - n\omega_s)\, e^{+j\omega t} d\omega = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} \int_{-\infty}^{+\infty} \delta(\omega - n\omega_s)\, e^{+j\omega t} d\omega$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t} = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t} d\omega$$

Other Convention

Continuous Time Fourier Transform {unitary, angular frequency}

$$X(j\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow$$

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Continuous Time Fourier Transform {non-unitary, angular frequency}

$$X(j\omega) = 1 \cdot \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow$$

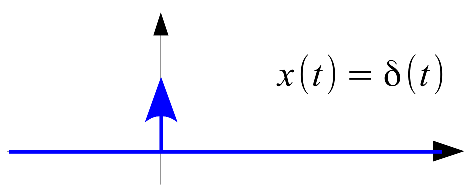
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

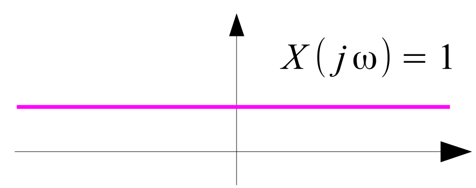
CTFT of Impulse

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$x(t) = A\,\delta(t)$$

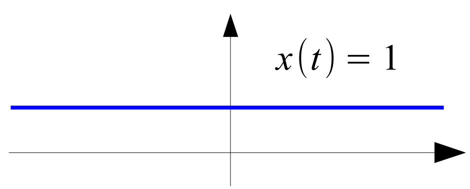
$$X(jw) = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\omega 0}\, dt = \int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$

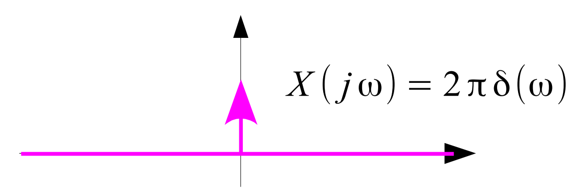




$$X(jw) = 2\pi\delta(\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega)\, e^{+j\omega t}\, d\omega = \int_{-\infty}^{+\infty} \delta(\omega)\, e^{+j0t}\, d\omega = \int_{-\infty}^{+\infty} \delta(\omega)\, d\omega = 1$$



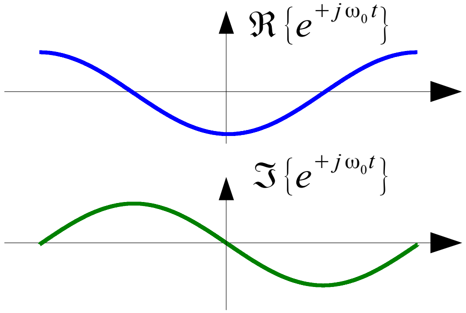


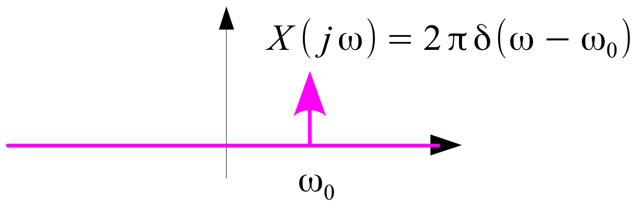
CTFT of Sinusoid

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(jw) = 2\pi\delta(\omega - \omega_0)$$

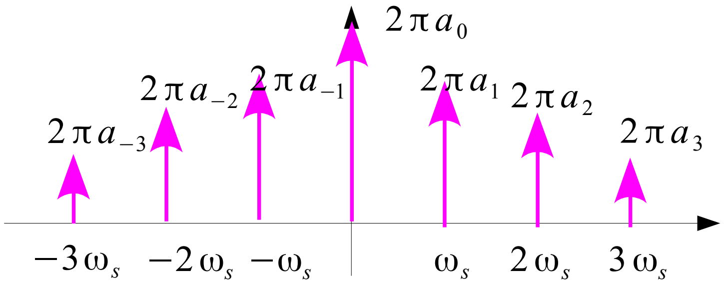
$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega-\omega_0)\, e^{+j\omega t}\, d\omega \\ &= \int_{-\infty}^{+\infty} \delta(\omega-\omega_0)\, e^{+j\omega_0 t}\, d\omega \\ &= e^{+j\omega_0 t}\int_{-\infty}^{+\infty} \delta(\omega)\, d\omega \\ &= e^{+j\omega_0 t} \\ &= \cos\omega_0 t + j\sin\omega_0 t \end{aligned}$$





CTFT of Periodic Signals

A general formula for the CTFT
of any <u>periodic</u> function
for which a CTFS exists



Period

$$T_s \Rightarrow \omega_s = \frac{2\pi}{T_s}$$

Fourier Series Expansion

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t}$$

Fourier Series Coefficients

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} x(t) \, e^{-jk\omega_s t} \, dt$$

Fourier Transform

$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{+\infty} e^{jk\omega_s t} \, e^{-j\omega t} \, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{+\infty} e^{-j(\omega - k\omega_s)t} \, dt$$

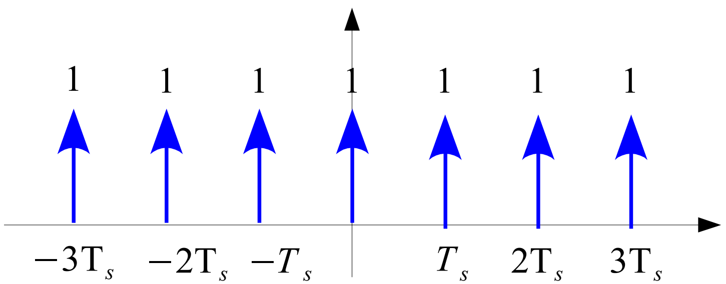
$$= \sum_{k=-\infty}^{\infty} 2\pi \, a_k \, \delta(\omega - k\omega_s)$$

CTFT of Impulse Train

A general formula for the CTFT
of any periodic function
for which a CTFS exists

Period

$$T_s \Rightarrow \omega_s = \frac{2\pi}{T_s}$$



Fourier Series Expansion

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t}$$

Fourier Transform

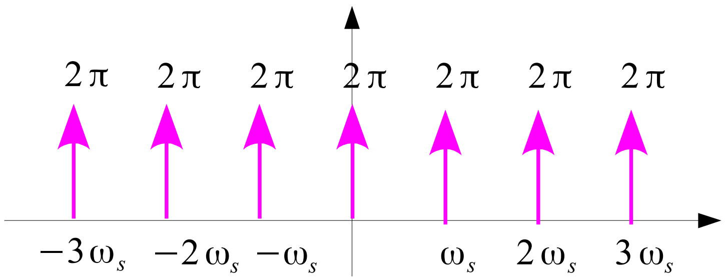
$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_s)$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - k\omega_s)$$

Fourier Series Coefficients

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} x(t) \, e^{-jk\omega_s t} \, dt$$

$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) \, dt = \frac{1}{T_s}$$



References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003