The Two Dimensional Heat Equation

Consider the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{4}\frac{\partial u}{\partial t} \text{ for } 0 < x < 3,\ 0 < y < 5,\ 0 < t$$
$$u(x,0,t) = 0,\ u(x,5,t) = 0,\quad < x < 3,\quad 0 < t$$
$$u(0,y,t) = 0,\ u(3,y,t) = 0,\quad < y < 5,\quad 0 < t$$
$$u(x,y,t) = \sin(\tfrac{\pi}{3}x)\sin(\tfrac{\pi}{5}y), \text{ for } 0 < x < 3,\ 0 < y < 5$$

Let $u(x,y,t) = \Phi(x,y)T(t)$. We then have

$$\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right)T = \frac{1}{4}\Phi T'.$$

Separating variables yields

$$\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right)\frac{1}{\Phi} = \frac{1}{4}\frac{T}{T}' = -\lambda^2.$$

So

$$\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) = -\lambda^2\Phi \quad \text{for } 0 < x < 3,\ 0 < y < 5, \text{ and}$$
$$T' + 4\lambda^2 T = 0 \text{ for } 0 < t$$

The boundary conditions lead to

$$\Phi(x,0)T(t) = 0$$
$$\Phi(x,5)T(t) = 0$$
$$\Phi(0,y)T(t) = 0$$
$$\Phi(3,y)T(t) = 0$$

This means either T(t)=0 (which can't happen) or the $\Phi(x,y) = 0$ on the boundary. That is

$$\Phi(x,0) = 0$$
$$\Phi(x,5) = 0$$
$$\Phi(0,y) = 0$$
$$\Phi(3,y) = 0$$

We now have two problems to worry about

$$\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) = -\lambda^2\Phi$$

with

$$\Phi(x,0) = 0$$
$$\Phi(x,5) = 0$$
$$\Phi(0,y) = 0$$
$$\Phi(3,y) = 0$$

Here we again try $\Phi(x,y) = X(x)Y(y)$. This means that

$$X''Y + Y''X = -\lambda^2 XY \Rightarrow$$
$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2 \text{ for } 0 < x < 3, \ \ 0 < y < 5$$

Since $\frac{X''}{X}$ is a function of x and $\frac{Y''}{Y}$ is a function of y, and their sum is a constant, this means that $\frac{X''}{X}$ =constant, and $\frac{Y''}{Y}$ = constant. Now the boundary conditions become

$$\Phi(x,0) = 0 \Rightarrow X(x)Y(0) = 0 \text{ for } 0 < x < 3$$
$$\Phi(x,5) = 0 \Rightarrow X(x)Y(5) = 0 \text{ for } 0 < x < 3$$
$$\Phi(0,y) = 0 \Rightarrow X(0)Y(y) = 0 \text{ for } 0 < y < 5$$
$$\Phi(3,y) = 0 \Rightarrow X(3)Y(y) = 0 \text{ for } 0 < y < 5$$

Since neither X nor Y can be identically equal to zero this means

$$Y(0) = 0$$
$$Y(5) = 0$$
$$X(0) = 0$$
$$X(3) = 0$$

It seems clear (from our previous experience) that $\frac{X''}{X}$ and $\frac{Y''}{Y}$ must be negative constants. Thus we will let $\frac{X''}{X} = -\mu^2$ and $\frac{Y''}{Y} = -\nu^2$. We now end up with

$$\frac{X''}{X} + \frac{Y''}{Y} = -\mu^2 + (-\nu^2) = -\lambda^2 \Rightarrow$$
$$\mu^2 + \nu^2 = \lambda^2$$

When we consider

$$\frac{X''}{X} = -\mu^2 \text{ for } 0 < x < 3 \Rightarrow$$
$$X_m(x) = \sin(\frac{m\pi x}{3}) \text{ with } \mu^2 = \left(\frac{m\pi}{3}\right)^2 \ \ m = 1,2,3\cdots$$

and

$$\frac{Y''}{Y} = -\nu^2 \text{ for } 0 < y < 5 \Rightarrow$$
$$Y_n(y) = \sin(\frac{n\pi y}{5}) \text{ with } \nu^2 = \left(\frac{n\pi}{5}\right)^2 \ \ n = 1,2,3\cdots$$

Thus,

$$\Phi_{mn}(x,y) = X_m(x)Y_n(y) = \sin(\frac{m\pi x}{3})\sin(\frac{n\pi y}{5}) \text{ with}$$

$$\lambda_{mn}^2 = \left(\frac{m\pi}{3}\right)^2 + \left(\frac{n\pi}{5}\right)^2 \Rightarrow$$

$$T_{mn}(t) = e^{(-\lambda_{mn}^2 t)} = e^{-\left(\left(\frac{m\pi}{3}\right)^2+\left(\frac{n\pi}{5}\right)^2\right)t} \Rightarrow$$

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin(\frac{m\pi x}{3})\sin(\frac{n\pi y}{5})e^{-\left(\left(\frac{m\pi}{3}\right)^2+\left(\frac{n\pi}{5}\right)^2\right)t}$$

Now at $t = 0$

$$u(x,y,0) = f(x,y) = \sin(\frac{\pi}{3}x)\sin(\frac{\pi}{5}y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin(\frac{m\pi x}{3})\sin(\frac{n\pi y}{5}) \text{ for } 0 < x < 3,\ 0 < y < 5$$

Notice that

$$\int_0^a\int_0^b \sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})\sin(\frac{p\pi x}{a})\sin(\frac{q\pi x}{b})dxdy =$$

$$\frac{ab(-1)^{n+m}\sin(p\pi)m\sin(q\pi)n}{\pi^2(m^2n^2 - m^2q^2 - p^2n^2 + q^2p^2)} = 0$$

However, if $p = m$ and $q = m$ then

$$\int_0^a\int_0^b \sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})dydx = \frac{ab}{4}$$

Thus, we can use the idea of Fourier series to get

$$a_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin(\frac{m\pi x}{a})\sin(\frac{n\pi x}{b})dydx$$

In our case we have

$$a_{mn} = \frac{4}{15}\int_0^5\int_0^3 \sin(\frac{\pi x}{3})\sin(\frac{\pi y}{5})\sin(\frac{m\pi x}{3})\sin(\frac{n\pi y}{5})dxdy = 0 \text{ if } m \neq 1, n \neq 1$$

$$a_{11} = \frac{4}{15}\int_0^5\int_0^3 \sin(\frac{\pi x}{3})\sin(\frac{\pi y}{5})\sin(\frac{\pi x}{3})\sin(\frac{\pi y}{5})dxdy = 1$$

Thus,

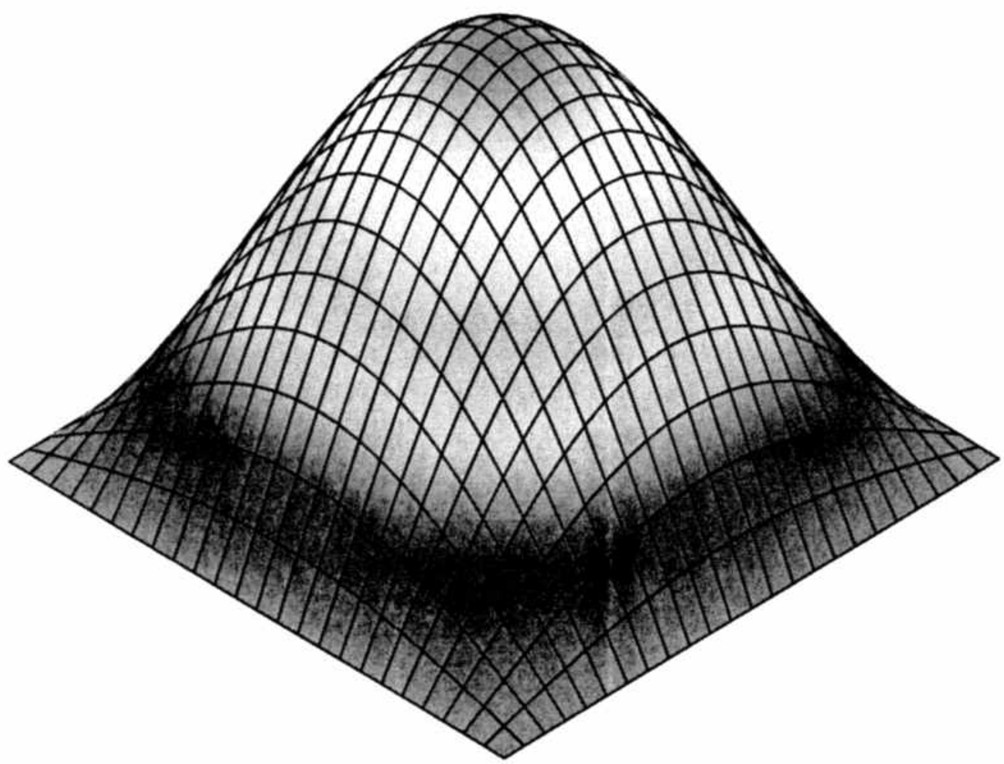
$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin(\frac{m\pi x}{3})\sin(\frac{n\pi y}{5})e^{-\left(\left(\frac{m\pi}{3}\right)^2+\left(\frac{n\pi}{5}\right)^2\right)t}$$

$$u(x,y,t) = \sin(\frac{\pi x}{3})\sin(\frac{\pi y}{5})e^{-\left(\left(\frac{\pi}{3}\right)^2+\left(\frac{\pi}{5}\right)^2\right)t}$$

```
> restart:
> uxyt1:=sin(Pi*x/3)*sin(y*Pi/5)*exp(-((Pi/3)^2+(Pi/5)^2)*t);
```

$$uxyt1 := \sin\left(\frac{1}{3}\,\pi\,x\right)\,\sin\left(\frac{1}{5}\,y\,\pi\right)\,\mathrm{e}^{-\frac{34}{225}\,\pi^2\,t} \tag{1}$$

```
> with(plots):
> animate3d(uxyt1,x=0..3,y=0..5,t=0..2,frames=100,shading=zhue);
```



more advanced

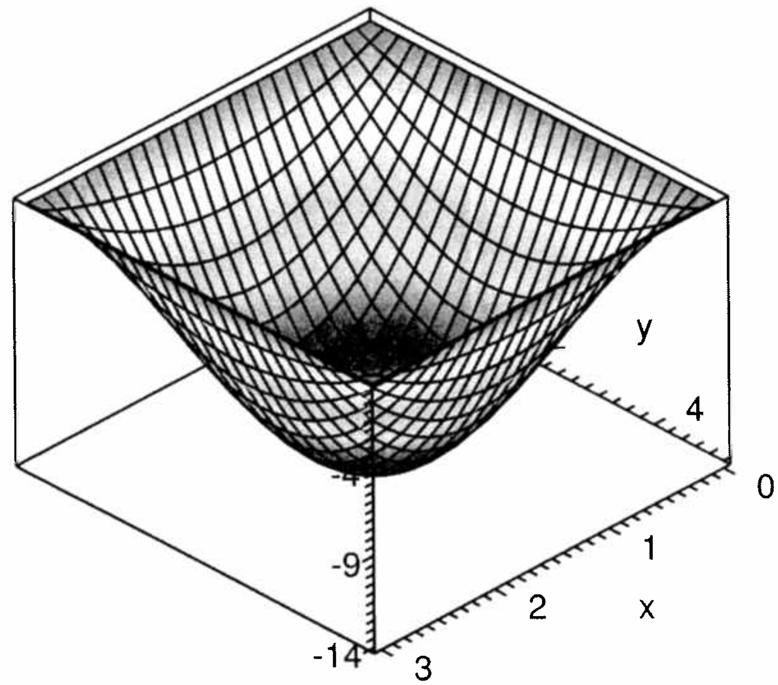
```
> restart:
> f:=x*(x-3)*y*(5-y);
```

$$f := x\,(x-3)\,y\,(5-y) \tag{1.1}$$

```
> fn:=f*sin(m*Pi*x/3)*sin(n*y*Pi/5);
```

$$fn := x\,(x-3)\,y\,(5-y)\,\sin\left(\frac{1}{3}\,m\,\pi\,x\right)\,\sin\left(\frac{1}{5}\,n\,y\,\pi\right) \tag{1.2}$$

```
> plot3d(f,x=0..3,y=0..5,axes=box,shading=zhue);
```




```
> anm:=(4/15)*int(int(fn,x=0..3),y=0..5);
```

$$anm := -\frac{1}{m^3 \pi^6 n^3}\Big(900\,\big(4 - 2\,m\,\pi\sin(m\,\pi) - 4\cos(m\,\pi) + m\,\pi^2\sin(m\,\pi)\,n\sin(n\,\pi) - 4\cos(n\,\pi) - 2\,n\,\pi\sin(n\,\pi) + 4\cos(m\,\pi)\cos(n\,\pi) + 2\,m\,\pi\sin(m\,\pi)\cos(n\,\pi) + 2\cos(m\,\pi)\,n\,\pi\sin(n\,\pi)\big)\Big) \tag{1.3}$$

```
> assume(n,integer);
> assume(m,integer);
> anm;
```

$$-\frac{900\,\big(4 - 4\,(-1)^{m\sim} - 4\,(-1)^{n\sim} + 4\,(-1)^{m\sim}(-1)^{n\sim}\big)}{m\sim^3\,\pi^6\,n\sim^3} \tag{1.4}$$

```
> uxyt5:=sum(sum(anm*sin(m*Pi*x/3)*sin(n*y*Pi/5)*exp((-(m*Pi/3)
  ^2-(n*Pi/5)^2)*t),m=1..5),n=1..5);
```

$$uxyt5 := -\frac{14400\sin\left(\frac{1}{3}\,\pi\,x\right)\sin\left(\frac{1}{5}\,y\,\pi\right)\mathrm{e}^{-\frac{34}{225}\pi^2 t}}{\pi^6} - \frac{1600}{3}\,\frac{\sin(\pi\,x)\sin\left(\frac{1}{5}\,y\,\pi\right)\mathrm{e}^{-\frac{26}{25}\pi^2 t}}{\pi^6} - \frac{576}{5}\,\frac{\sin\left(\frac{5}{3}\,\pi\,x\right)\sin\left(\frac{1}{5}\,y\,\pi\right)\mathrm{e}^{-\frac{634}{225}\pi^2 t}}{\pi^6} \tag{1.5}$$

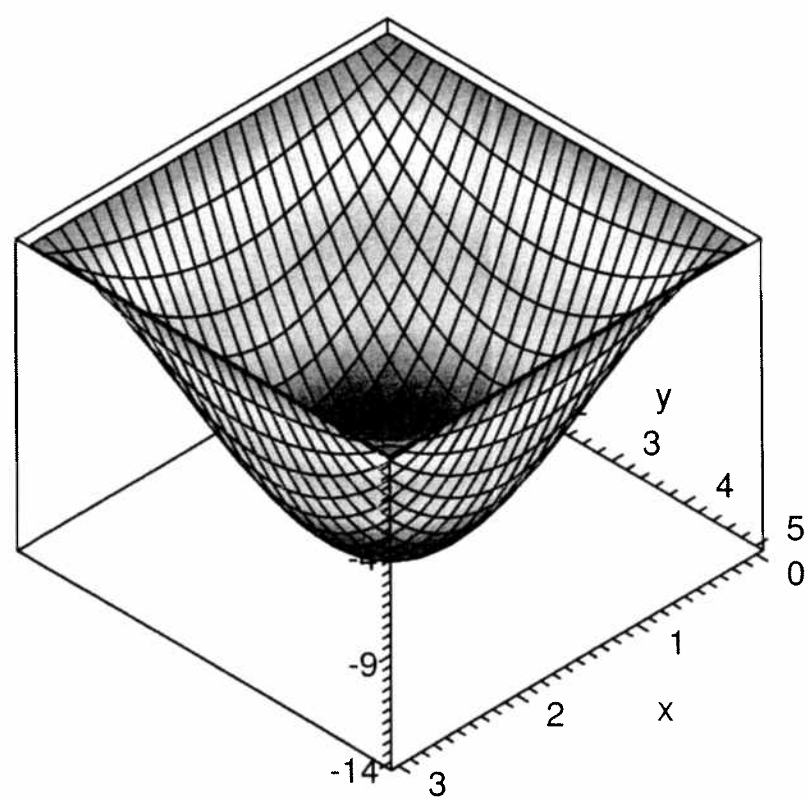
$$-\frac{1600}{3}\,\frac{\sin\left(\frac{1}{3}\pi x\right)\sin\left(\frac{3}{5}y\pi\right)e^{-\frac{106}{225}\pi^2 t}}{\pi^6}$$

$$-\frac{1600}{81}\,\frac{\sin(\pi x)\sin\left(\frac{3}{5}y\pi\right)e^{-\frac{34}{25}\pi^2 t}}{\pi^6}-\frac{64}{15}\,\frac{\sin\left(\frac{5}{3}\pi x\right)\sin\left(\frac{3}{5}y\pi\right)e^{-\frac{706}{225}\pi^2 t}}{\pi^6}$$

$$-\frac{576}{5}\,\frac{\sin\left(\frac{1}{3}\pi x\right)\sin(y\pi)\,e^{-\frac{10}{9}\pi^2 t}}{\pi^6}-\frac{64}{15}\,\frac{\sin(\pi x)\sin(y\pi)\,e^{-2\pi^2 t}}{\pi^6}$$

$$-\frac{576}{625}\,\frac{\sin\left(\frac{5}{3}\pi x\right)\sin(y\pi)\,e^{-\frac{34}{9}\pi^2 t}}{\pi^6}$$

```
> with(plots):
> animate3d(uxyt5,x=0..3,y=0..5,t=0..2,frames=100,axes=box,
  shading=zhue);
```




```
> restart:
> f:=x*y;
```

$$f := x\,y \qquad \textbf{(1.6)}$$

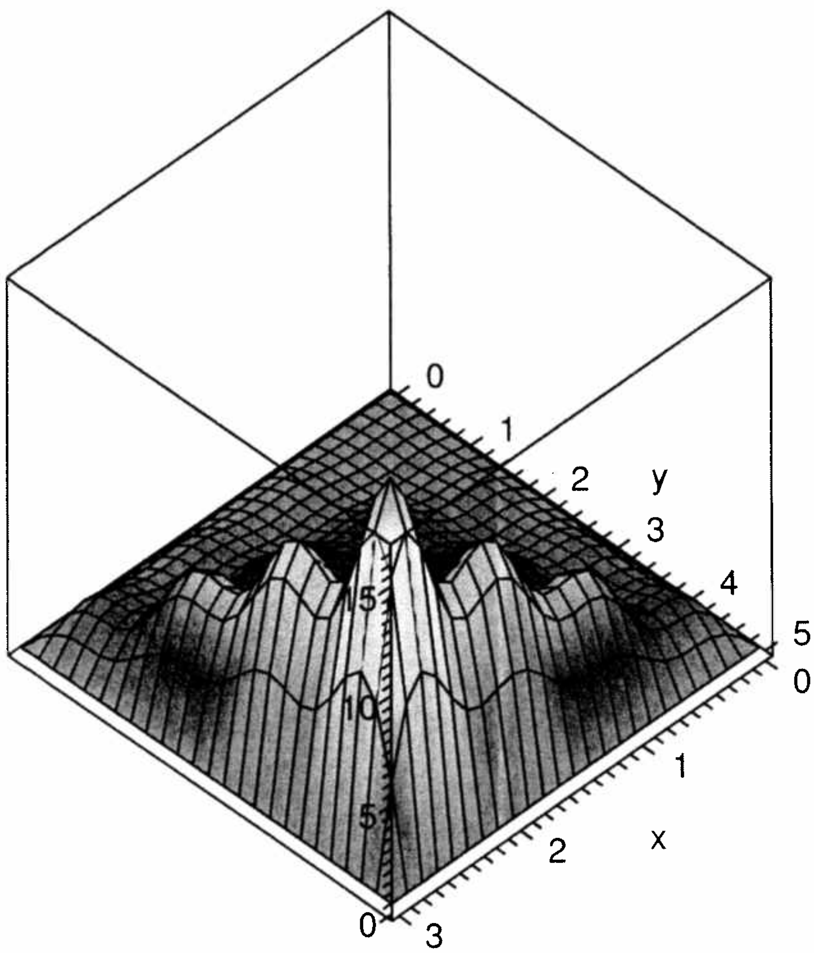
```
> fn:=f*sin(m*Pi*x/3)*sin(n*y*Pi/5);
```

$$fn := x\,y\sin\left(\frac{1}{3}\,m\,\pi\,x\right)\sin\left(\frac{1}{5}\,n\,y\,\pi\right) \tag{1.7}$$

```
> anm:=4/15*int(int(fn,x=0..3),y=0..5);
```

$$anm := -\frac{1}{\pi^4 m^2 n^2}\left(60\left(-\sin(\pi m)\sin(n\pi) + \cos(\pi m)\,\pi\,m\sin(n\pi) + \sin(\pi m)\cos(n\pi)\,n\,\pi - \cos(\pi m)\,\pi^2\,m\cos(n\pi)\,n\right)\right) \tag{1.8}$$

```
> assume(n,integer);assume(m,integer);
> uxyt5:=sum(sum(anm*sin(m*Pi*x/3)*sin(n*y*Pi/5)*exp(-((m*Pi/3)
  ^2+(n*Pi/5)^2)*t),m=1..7),n=1..7):
> with(plots):
> animate3d(uxyt5,x=0..3,y=0..5,t=0..0.5,frames=200,shading=
  zhue,axes=box);
```




```
>
```