CLTI Differential Equation

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Causal LTI Systems (1)

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^N} + \cdots + a_1 \frac{d\,y(t)}{dt} + a_0\, y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{d\,x(t)}{dt} + b_0\, x(t)$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = \boxed{b_{N-M}} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

$$Q(D)\, y(t) = P(D)\, x(t)$$

$$M = N$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = \boxed{b_0} \frac{d^N x(t)}{dt^N} + b_1 \frac{d^{N-1} x(t)}{dt^{N-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

$$Q(D)\, y(t) = P(D)\, x(t)$$

Causal LTI Systems (2)

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^N} + \cdots + a_1 \frac{d\,y(t)}{dt} + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^M} + \cdots + b_1 \frac{d\,x(t)}{dt} + b_0 x(t)$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

$$Q(D)\, y(t) = P(D)\, x(t)$$

- **Zero Input Response**
- **Zero State Response (Convolution with h(t))**

- **Natural Response (Homogeneous Solution)**
- **Forced Response (Particular Solution)**

Zero Input Response $y_0(t)$ – (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$Q(D)\, y_0(t) = 0 \quad \Rightarrow \quad (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_0(t) = 0$$

linear combination of $y_0(t)$ and its derivatives = 0

$\Leftrightarrow$ iff

$$y_0(t) = ce^{\lambda t}$$
$$y_0^{(1)}(t) = c\lambda e^{\lambda t}$$
$$y_0^{(2)}(t) = c\lambda^2 e^{\lambda t}$$
$$\cdots$$

$$\Downarrow$$

$$Q(\lambda) = 0 \quad \Leftrightarrow \quad \underbrace{(\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1} \lambda + a_N)}_{=\ 0} \ \underbrace{ce^{\lambda t}}_{\neq\ 0} = 0$$

Zero Input Response $y_0(t)$ – (2)

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$Q(D) y_0(t) = 0 \quad \Rightarrow \quad (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y_0(t) = 0$$

$$Q(\lambda) = 0 \quad \Leftrightarrow \quad \underbrace{(\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1} \lambda + a_N)}_{= 0} \; \underbrace{ce^{\lambda t}}_{\neq 0} = 0$$

$$Q(\lambda) = (\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1} \lambda + a_N) = 0$$

$$Q(\lambda) = (\lambda - \boldsymbol{\lambda}_1)(\lambda - \boldsymbol{\lambda}_2) \cdots (\lambda - \boldsymbol{\lambda}_N) = 0$$

$\boldsymbol{\lambda}_i$ characteristic roots

$$\boxed{y_0(t)} = c_1 e^{\boldsymbol{\lambda}_1 t} + c_2 e^{\boldsymbol{\lambda}_2 t} \cdots + c_N e^{\boldsymbol{\lambda}_N t}$$

$e^{\boldsymbol{\lambda}_i t}$ characteristic modes

ZIR: a linear combination of the characteristic modes of the system

Zero State Response y(t) – (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) \;=\; b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot\; y(t) \;=\; \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot\; x(t)$$

$$Q(D) \quad \cdot \quad y(t) \;= \qquad P(D) \quad \cdot \quad x(t)$$

All initial conditions are zero $\quad y(0^-) \;=\; \dot{y}(0^-) \;=\; \ddot{y}(0^-) \;\cdots\; = \; y^{(N-2)}(0^-) \;=\; y^{(N-1)}(0^-) \;=\; 0$

Zero State Response $\quad y(t) \;=\; x(t) * h(t) \;=\; \int_{-\infty}^{+\infty} x(\tau)\, y(t-\tau)\, d\tau$

Impulse response $\quad h(t)$

causal system: *response cannot begin before the input*

causal input *x(t)*: *The input starts at t=0* $\qquad h(\tau) = 0 \quad \tau < 0$

causal *h(t)*: *The causal system's response to a unit impulse cannot begin before t=0*

$$h(t-\tau) = 0 \quad t - \tau < 0$$

Causality $\quad y(t) \;=\; \int_{0^-}^{t} x(\tau)\, y(t-\tau)\, d\tau \quad , \quad t \ge 0$

Total Response

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$y(t) = \underline{\sum_{k=1}^{N} c_k e^{\lambda_k t}} + \underline{x(t) * h(t)}$$

Zero Input Response

Zero State Response

$$y(t) = \underline{y_n(t)} + \underline{y_\Phi(t)}$$

Natural Response

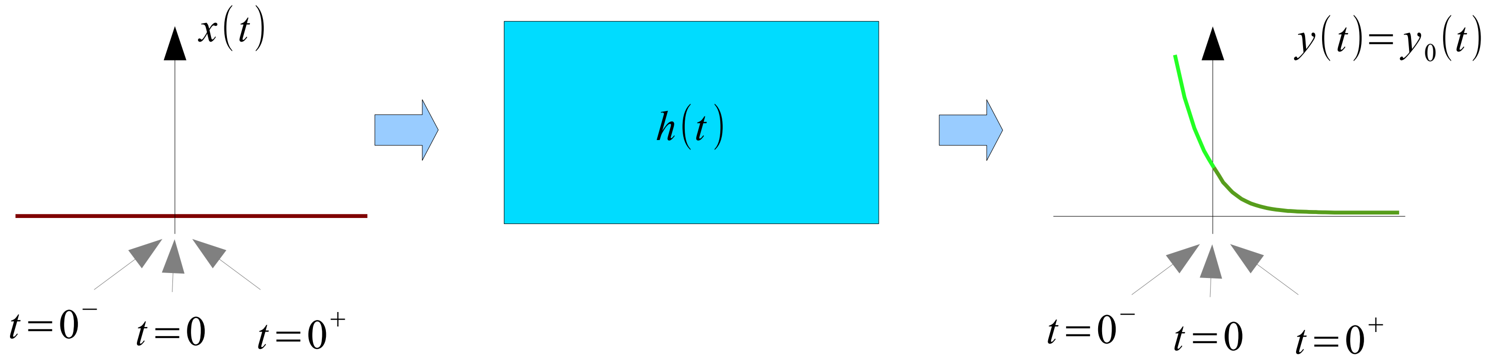
Forced Response

Zero Input Response

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$



Input is zero

Only initial conditions drives the system

$$y_0(0^-) = y_0(0) = y_0(0^+)$$

$$\dot{y}_0(0^-) = \dot{y}_0(0) = \dot{y}_0(0^+)$$

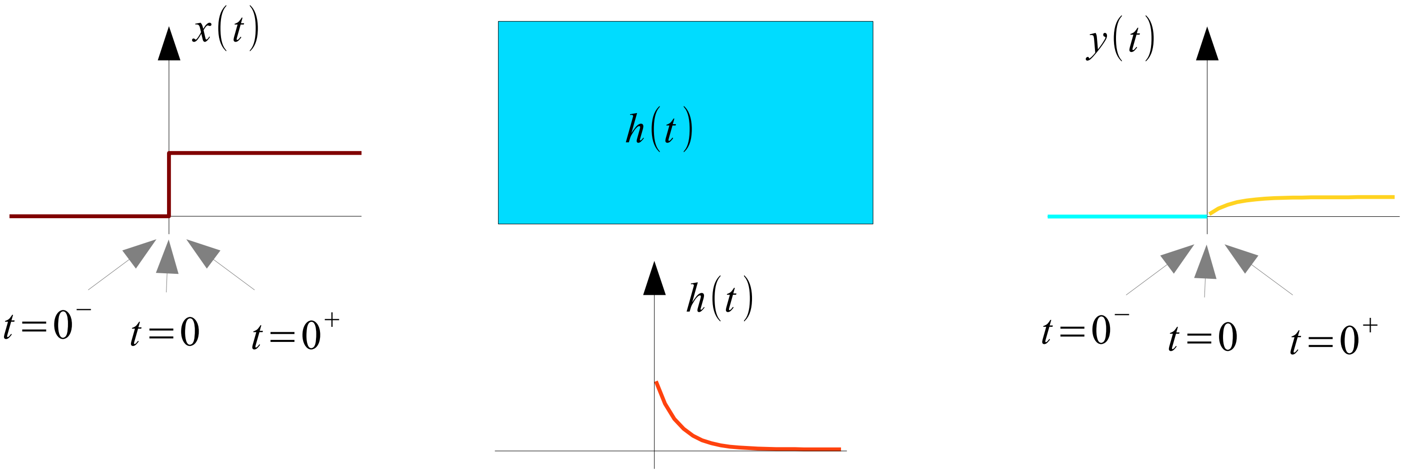
$$\ddot{y}_0(0^-) = \ddot{y}_0(0) = \ddot{y}_0(0^+)$$

Zero State Response

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) \;=\; b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) \;=\; \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \quad \cdot \quad y(t) \;=\; P(D) \quad \cdot \quad x(t)$$



All initial conditions are zero

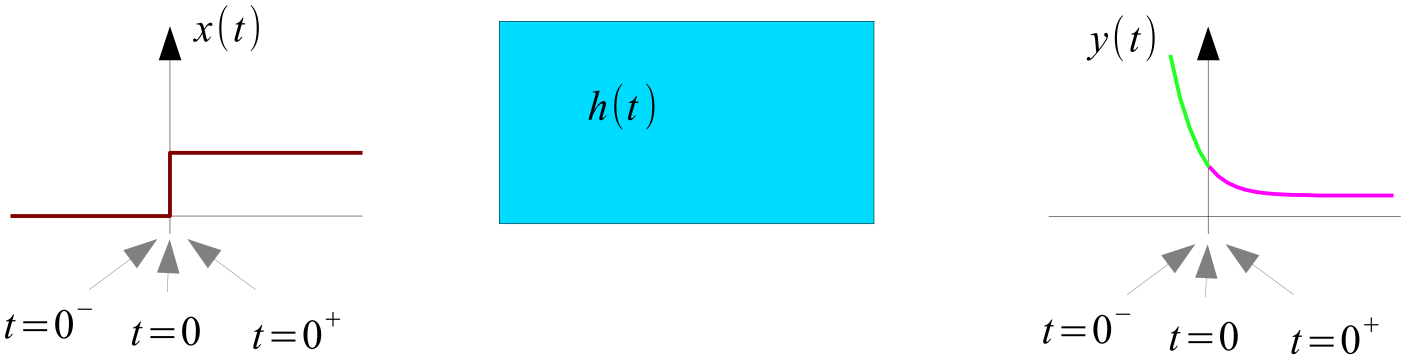
$$y(0^-) \;=\; \dot{y}(0^-) \;=\; \ddot{y}(0^-) \;\cdots\; = \; y^{(N-2)}(0^-) \;=\; y^{(N-1)}(0^-) \;=\; 0$$

Total Response y(t)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\boxed{\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right)} \cdot y(t) = \boxed{\left(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N\right)} \cdot x(t)$$

$$Q(D) \quad \cdot \quad y(t) = \quad P(D) \quad \cdot \quad x(t)$$



zero input response

+

zero state response

$y(t) = y_0(t) \leftarrow t \le 0^-$

because the input has not started yet

$y(0^-) = y_0(0^-)$

$\dot{y}(0^-) = \dot{y}_0(0^-)$

in general, the total response

$y(0^-) \neq y(0^+)$

$\dot{y}(0^-) \neq \dot{y}(0^+)$

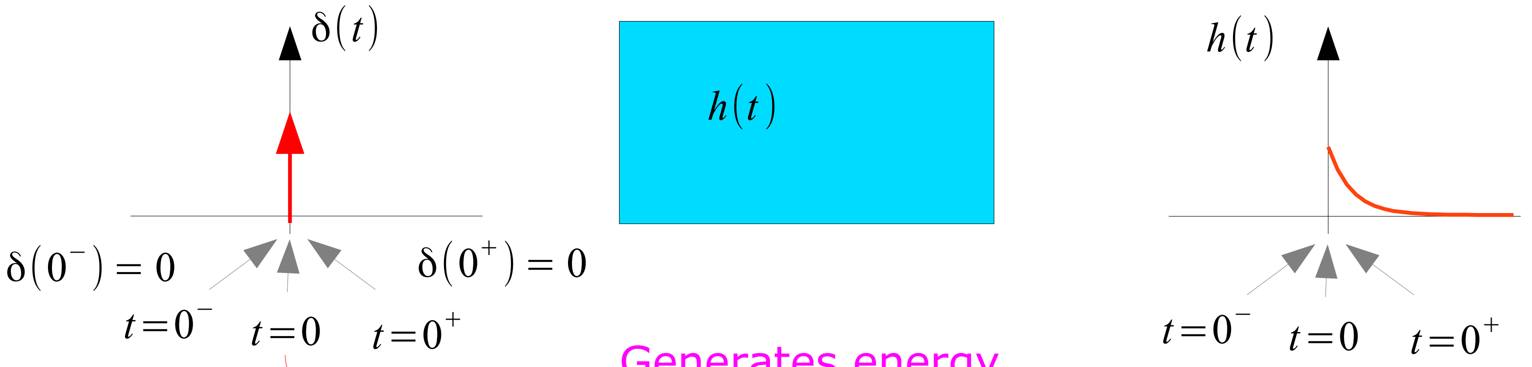
possible discontinuity at t = 0

Impulse Response h(t)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$



Generates energy storage creates nonzero initial condition at $t=0^+$

All initial conditions are zero at $t=0^-$

$$y(0^-) = y^{(1)}(0^-) = \cdots = y^{(N-2)}(0^-) = y^{(N-1)}(0^-) = 0$$

$$y(0^-) = y^{(1)}(0^-) = \cdots = y^{(N-2)}(0^-) = 0, \quad y^{(N-1)}(0^-) = 1$$

$t \geq 0^+$ $(t \neq 0)$	h(t) = characteristic mode terms
$t = 0$	h(t) can have at most an impulse $A_0\delta(t)$

$$h(t) = A_0\delta(t) + char\ mode\ terms \quad t \geq 0$$

h(t) can have at most a δ(t)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_0 \frac{d^N x(t)}{dt^N} + b_1 \frac{d^{N-1} x(t)}{dt^{N-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

$M = N$

$$Q(D)\, y(t) = P(D)\, x(t)$$

$$\underline{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, h(t)} = \boxed{(b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\,\delta(t)}$$

If $\delta^{(1)}(t)$ is included in $h(t)$, then the highest order term

$\delta^{(N+1)}(t) \neq \delta^{(N)}(t)$ contradiction

h(t) cannot contain $\delta^{(i)}$(t) at all ➡ *h(t)* can contain at most δ(t) $M \le N$

$$\frac{d^N y(t)}{dt^N} = \delta^{(N)}(t)$$

New Initial Condition created by δ(t)

$$\frac{d^N y(t)}{d t^N} + a_1 \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{d t} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{d t^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{d t^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{d t} + b_N\, x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$\frac{d^N y(t)}{d t^N} + a_1 \frac{d^{N-1} y(t)}{d t^{N-1}} + a_2 \frac{d^{N-2} y(t)}{d t^{N-2}} + \cdots + a_{N-1} \frac{d\, y(t)}{d t} + a_N\, y(t) = \boxed{\delta(t)}$$

$$y_n^{(N)}(t) = \delta(t)$$

$$\delta(t) \qquad u(t) \qquad t\, u(t) \qquad \frac{1}{(N-2)!} t^{(N-2)} u(t) \qquad \frac{1}{(N-1)!} t^{(N-1)} u(t)$$

integration *integration* *integration* *integration*

$$\boxed{y_n^{(N-1)}(0) = 1}$$

$$\underline{y_n^{(N-2)}(0) = \cdots \; y_n^{(2)} = y_n^{(1)}(0) = y_n(0)} = 0$$

unit jump discontinuity at t = 0

no jump discontinuity is allowed at t = 0

Simplified Impulse Matching Method (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \quad \cdot \quad y(t) = \qquad P(D) \quad \cdot \quad x(t)$$

$$h(t) = b_0 \delta(t) + [P(D) y_n(t)] u(t)$$

$y_n(t)$ *linear combination of characteristic modes*
with the following initial conditions

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) \;\cdots\; = y_n^{(N-2)}(0) = 0 \qquad \boxed{y_n^{(N-1)}(0) = 1}$$

$$Q(D)\, y(t) = \boxed{P(D)\, x(t)}$$

$$\Downarrow$$

$$Q(D)\, w(t) = \boxed{x(t)}$$

$$\Downarrow$$

$$Q(D)\, y_n(t) = \boxed{\delta(t)}$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_n(t) = \delta(t)$$

$$y_n^{(N)}(t) + a_1 y_n^{(N-1)}(t) + \cdots + a_{N-1} y_n^{(1)}(t) + y_n(t) = \delta(t)$$

$$Q(D)\, w(t) = x(t)$$

$$Q(D) \boxed{P(D)\, w(t)} = P(D)\, x(t)$$

$$Q(D) \boxed{y(t)} = P(D)\, x(t)$$

$$y(t) = P(D)\, w(t)$$

Simplified Impulse Matching Method (2)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y_n(t) = \delta(t)$$

$$y_n^{(N)}(t) + a_1 y_n^{(N-1)}(t) + \cdots + a_{N-1} y_n^{(1)}(t) + y_n(t) = \delta(t)$$

$$h(t) = P(D)[y_n(t)u(t)]$$

$$h(t) = b_o \delta(t) + P(D) y_n(t), \qquad t \ge 0$$

$$h(t) = b_o \delta(t) + [P(D) y_n(t)] u(t)$$

$$Q(D) w(t) = x(t)$$

$$Q(D) \boxed{P(D) w(t)} = \boxed{P(D) x(t)}$$

$$y(t) = P(D) w(t)$$

$$Q(D) y_n(t) = \delta(t)$$

$$Q(D) \boxed{P(D) y_n(t)} = \boxed{P(D) \delta(t)}$$

$$h(t) = P(D) y_n(t)$$

causal $y_n(t)u(t)$

$$h(t) = P(D)[y_n(t)u(t)]$$

Classical Solution (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \quad \cdot \quad y(t) = \qquad P(D) \quad \cdot \quad x(t)$$

When all the ***characteristic mode terms*** *of the total system response together, they form the system's* ***natural response*** $y_n(t)$ *(homogeneous, complementary solution)*

$$y(t) = y_n(t) + y_\Phi(t)$$

$$Q(D)\,[y_n(t) + y_\Phi(t)] = P(D)\,x(t)$$

The remaining portion of noncharacteristic mode terms form the system's ***forced response*** *(particular solution)* $y_\Phi(t)$

$$Q(D)\,y_n(t) = 0$$

$$Q(D)\,y_\Phi(t) = P(D)\,x(t)$$

Classical Solution (2)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

- *linear combination of the* ***characteristic modes****.* $y_n(t)$
- *the same form as that of the* ***zero input response***
- *only its constants are different*
- *these constants are determined from the* ***auxiliary conditions***
- *initial conditions at* $t = 0^+$
- *at* $t = 0^-$ *only the* ***zero input response***
- *initial condition at* $t = 0^-$ $\rightarrow$ *applied to* ***zero input response***
- ***zir*** *and* ***zsr*** *cannot be separated*

IC – Zero Input Response (2)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$Q(\lambda) = (\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1} \lambda + a_N) = 0$$

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N) = 0$$

ZIR: a linear combination of the characteristic modes of the system

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \cdots + c_N e^{\lambda_N t}$$

In practice, these initial conditions are known

$$y_0(0^-), \ \dot{y}_0(0^-), \ \ddot{y}_0(0^-), \ \cdots$$

But ZIR is not affected by the input.
Therefore, the following conditions are met

$$y_0(0^-) = y_0(0) = y_0(0^+)$$

$$\dot{y}_0(0^-) = \dot{y}_0(0) = \dot{y}_0(0^+)$$

$$\ddot{y}_0(0^-) = \ddot{y}_0(0) = \ddot{y}_0(0^+)$$

$$\cdots = \cdots = \cdots$$

IC – Impulse Response (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$t \geq 0^+$ h(t) = characteristic mode terms

$t \geq 0$ h(t) = $A_0 \delta(t)$ + characteristic mode terms

Simplified Impulse Matching Method → $h(t) = b_0 \delta(t) + [P(D)\, y_n(t)]u(t)$

$y_n(t)$ *linear combination of characteristic modes*
with the following initial conditions

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) \ \cdots \ = y_n^{(N-2)}(0) = 0 \qquad \boxed{y_n^{(N-1)}(0) = 1}$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} \underline{+ \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t)} = \delta(t)$$

⬇ ⬇ ⬇

$\delta(t)$ $u(t)$ *no jump discontinuity is allowed at t = 0*

IC – Impulse Response (2)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$y_n(t)$ *linear combination of characteristic modes*
with the following initial conditions

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) \cdots = y_n^{(N-2)}(0) = 0 \qquad \boxed{y_n^{(N-1)}(0) = 1}$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \underline{a_2 \frac{d^{N-2} y(t)}{dt^{N-2}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t)} = \delta(t)$$

⬇ $\delta(t)$ ⬇ $u(t)$ ⬇ *no jump discontinuity is allowed at t = 0* *unit jump discontinuity at t = 0*

integration

⬇

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) \cdots = y_n^{(N-2)}(0) = 0$$

⬇

$$\boxed{y_n^{(N-1)}(0) = 1}$$

$$y_n^{(N)}(t) = \delta(t)$$

IC – Classical Solution

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$Q(\lambda) = (\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1} \lambda + a_N) = 0$$

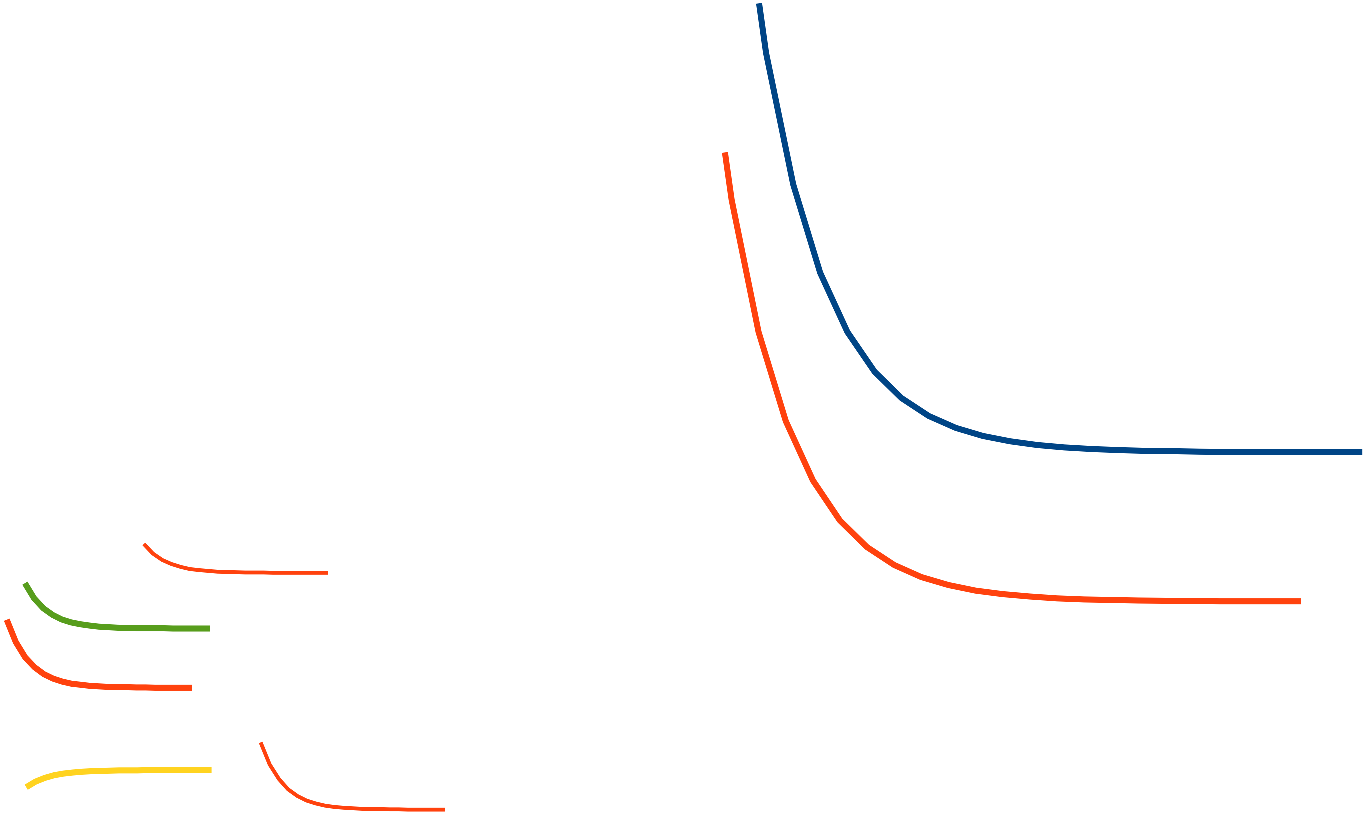
$y_n(t)$ *linear combination of characteristic modes with the following initial conditions*

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N) = 0$$

$y_0(0^+),\ \dot{y}_0(0^+),\ \ddot{y}_0(0^+),\ \cdots$

- *at* $t = 0^-$ *only the* ***zero input response***
- *initial condition at* $t = 0^-$ *only can be applied to* ***zero input response***
- ***zir*** *and* ***zsr*** *cannot be separated in the classical solution*
- *Therefore initial condition at* $t = 0^+$ *must be used.*

Impulse Response h(t)



References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] B.P. Lathi, Linear Systems and Signals (2nd Ed)