

CTFS (1B)

- Continuous Time Fourier Series

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Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin nx dx$$

$$n = 1, 2, 3, \dots$$



one-sided spectrum
only positive frequencies

Trigonometric Identities

$$\cos \theta \cos \phi = \frac{1}{2} (\cos(\theta - \phi) + \cos(\theta + \phi))$$

$$\sin \theta \sin \phi = \frac{1}{2} (\cos(\theta - \phi) - \cos(\theta + \phi))$$

$$\sin \theta \cos \phi = \frac{1}{2} (\sin(\theta + \phi) + \sin(\theta - \phi))$$

$$\cos \theta \sin \phi = \frac{1}{2} (\sin(\theta + \phi) - \sin(\theta - \phi))$$

$$\frac{1}{2} (1 + \cos(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\frac{1}{2} (1 - \cos(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\frac{1}{2} (\sin(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\frac{1}{2} (\sin(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\int_{-\pi}^{+\pi} \cos nx \cos mx dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \sin mx dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \cos mx dx = 0$$

$$\int_{-\pi}^{+\pi} \cos nx \sin mx dx = 0$$

$$\int_{-\pi}^{+\pi} \cos nx \cos mx dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin nx \sin mx dx = \pi \quad (n = m)$$

Trigonometric Orthogonality

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} \underline{f(x) \cos nx} dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} \underline{f(x) \sin nx} dx$$

$n = 1, 2, 3, \dots$

$$\int_{-\pi}^{+\pi} \cos nx \cos mx dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \sin mx dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \cos mx dx = 0$$

$$\int_{-\pi}^{+\pi} \cos nx \sin mx dx = 0$$

$$\int_{-\pi}^{+\pi} \underline{\cos nx \cos mx} dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \underline{\sin nx \sin mx} dx = \pi \quad (n = m)$$

$$a_n \leftarrow \underline{f(x) \cdot \cos nx} = a_0 \cdot \cos nx + \sum_{m=1}^{\infty} (a_m \underline{\cos mx \cdot \cos nx} + b_m \sin mx \cdot \cos nx)$$

$$b_n \leftarrow \underline{f(x) \cdot \sin nx} = a_0 \cdot \sin nx + \sum_{m=1}^{\infty} (a_m \cos mx \cdot \sin nx + b_m \underline{\sin mx \cdot \sin nx})$$

Any Period $p = 2L$

$$g(v) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} g(v) dv$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \cos nv dv$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \sin nv dv$$

$n = 1, 2, \dots$

$$v: [-\pi, +\pi]$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

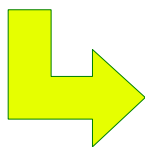
$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{n\pi x}{L} dx$$

$n = 1, 2, 3, \dots$

$$x: [-L, +L]$$



$$v = \frac{\pi}{L} x$$
$$dv = \frac{\pi}{L} dx$$



Time and Frequency

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{n\pi x}{L} dx$$

$n = 1, 2, 3, \dots$

$$x: [-L, +L]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi n}{T} t + b_n \sin \frac{2\pi n}{T} t \right)$$

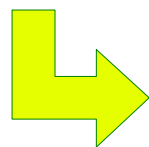
$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos \frac{2\pi n t}{T} dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin \frac{2\pi n t}{T} dt$$

$n = 1, 2, \dots$

$$t: [0, T]$$



$$2L = T$$



Continuous Time Periodic Signal $x(t)$

Harmonic Frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi n}{T} t + b_n \sin \frac{2\pi n}{T} t \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos \frac{2\pi n t}{T} dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin \frac{2\pi n t}{T} dt$$

$n = 1, 2, \dots$

$$t: [0, T]$$

resolution frequency

n-th harmonic frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(2\pi n f_0 t) dt \quad n = 1, 2, \dots$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(2\pi n f_0 t) dt \quad n = 1, 2, \dots$$

$$t: [0, T]$$

$$f_0 = \frac{1}{T}$$

$$f_n = n f_0 = n \frac{1}{T}$$

Radial Frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n2\pi f_0 t) + b_n \sin(n2\pi f_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n2\pi f_0 t) dt \quad n = 1, 2, \dots$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n2\pi f_0 t) dt \quad n = 1, 2, \dots$$

$$t: [0, T]$$

linear frequency

angular (radial) frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t) dt$$
$$n = 1, 2, \dots$$

$$t: [0, T]$$

f

$$\omega = 2\pi f$$

Complex Fourier Series Coefficients

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t) dt$$

$n = 1, 2, \dots$

$$t: [0, T]$$

Real coefficients

$$a_0, a_n, b_n, \quad n = 1, 2, \dots$$

Complex coefficients

$$A_0, A_n, B_n, \quad n = 1, 2, \dots$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} (A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t})$$

$$A_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$A_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$B_n = \frac{1}{T} \int_0^T x(t) e^{+jn\omega_0 t} dt$$

$$t: [0, T]$$

one-sided spectrum

only positive frequencies

two-sided spectrum

Both pos and neg frequencies

Euler Equation (1)

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t) dt$$

$n = 1, 2, \dots$

$$e^{+j\omega t} = \cos \omega t + j \sin \omega t$$

$$e^{-j\omega t} = \cos \omega t - j \sin \omega t$$

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$\begin{aligned} & a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \\ &= a_n \frac{1}{2} (e^{jn\omega_0 t} + e^{-jn\omega_0 t}) + b_n \frac{1}{2j} (e^{jn\omega_0 t} - e^{-jn\omega_0 t}) \\ &= \frac{(a_n - jb_n)}{2} e^{jn\omega_0 t} + \frac{(a_n + jb_n)}{2} e^{-jn\omega_0 t} \\ &= A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t} \end{aligned}$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} (A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t})$$

| | | | | | | |
|-----------|---|-------|---|----------------------------|---|---------------------------|
| zero freq | → | A_0 | = | a_0 | } | only positive frequencies |
| pos freq | → | A_n | = | $\frac{1}{2} (a_n - jb_n)$ | | |
| neg freq | → | B_n | = | $\frac{1}{2} (a_n + jb_n)$ | | |

Euler Equation (2)

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t) dt$$

$n = 1, 2, \dots$

$$x(t) = A_0 + \sum_{n=1}^{\infty} (A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t})$$

$$A_0 = a_0$$

$$A_n = \frac{1}{2} (a_n - jb_n)$$

$$B_n = \frac{1}{2} (a_n + jb_n)$$

$$A_n = \frac{1}{T} \int_0^T x(t) (\cos(n\omega_0 t) - j \sin(n\omega_0 t)) dt$$

$$B_n = \frac{1}{T} \int_0^T x(t) (\cos(n\omega_0 t) + j \sin(n\omega_0 t)) dt$$



$$A_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$B_n = \frac{1}{T} \int_0^T x(t) e^{jn\omega_0 t} dt$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} (A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t})$$



$$x(t) = \sum_{n=0}^{\infty} (A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t})$$

Complex Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t) dt$$
$$n = 1, 2, \dots$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} (A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t})$$

$$A_0 = a_0$$

$$A_n = \frac{1}{2} (a_n - jb_n)$$

$$B_n = \frac{1}{2} (a_n + jb_n)$$

$$n = 1, 2, \dots$$

$$x(t) = \sum_{n=0}^{\infty} (A_n e^{+jn\omega_0 t} + B_n e^{-jn\omega_0 t})$$

$$A_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$
$$n = 0, 1, 2, \dots$$

$$B_n = \frac{1}{T} \int_0^T x(t) e^{+jn\omega_0 t} dt$$
$$n = 1, 2, \dots$$

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{+jn\omega_0 t}$$

$$C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$
$$n = -2, -1, 0, +1, +2, \dots$$

$$C_n = \begin{cases} A_0 & (n = 0) \\ A_n & (n > 0) \\ B_n & (n < 0) \end{cases}$$

Single-Sided Spectrum

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t) dt$$

$n = 1, 2, \dots$

$$x(t) = g_0 + \sum_{n=1}^{\infty} g_n \cos(n\omega_0 t + \phi_n)$$

$$g_0 = a_0$$

$$g_n = \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = \tan^{-1} \left(-\frac{b_n}{a_n} \right)$$

$$n = 1, 2, \dots$$

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$$

$$g_n \cos(n\omega_0 t + \phi_n) = g_n \cos(\phi_n) \cos(n\omega_0 t) - g_n \sin(\phi_n) \sin(n\omega_0 t)$$

Two-Sided Spectrum

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t) dt$$

$n = 1, 2, \dots$

$$x(t) = A_0 + \sum_{n=1}^{\infty} (A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t})$$

$$A_0 = a_0$$

$$A_n = \frac{1}{2} (a_n - j b_n)$$

$$B_n = \frac{1}{2} (a_n + j b_n)$$

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$n = -2, -1, 0, +1, +2, \dots$$

$$C_n = \begin{cases} A_0 & (n = 0) \\ A_n & (n > 0) \\ B_n & (n < 0) \end{cases}$$

$$|C_n| = \frac{A_n}{2} \quad (n \neq 0)$$

$$\text{Arg}(C_n) = \begin{cases} +\phi_n = \tan^{-1} \left(-\frac{b_n}{a_n} \right) & (n > 0) \\ -\phi_n = \tan^{-1} \left(+\frac{b_n}{a_n} \right) & (n < 0) \end{cases}$$

CTFS of Impulse Train (1)

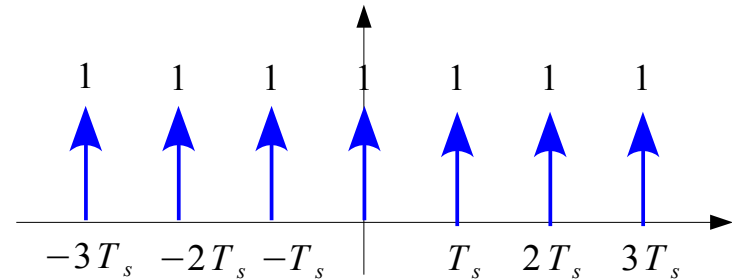
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Fourier Series Expansion of Impulse Train

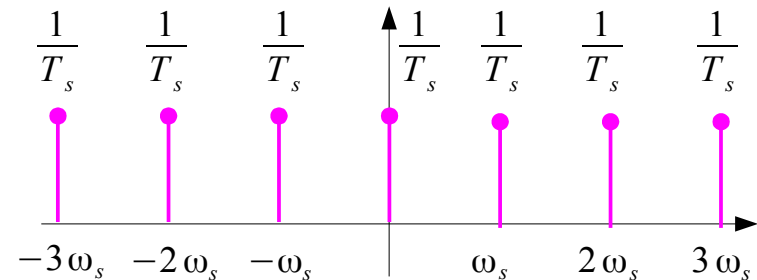
$$p(t) = \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t}$$

Fourier Series Coefficients

$$\begin{aligned} a_k &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jk\omega_s 0} dt \\ &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) dt = \frac{1}{T_s} \end{aligned}$$



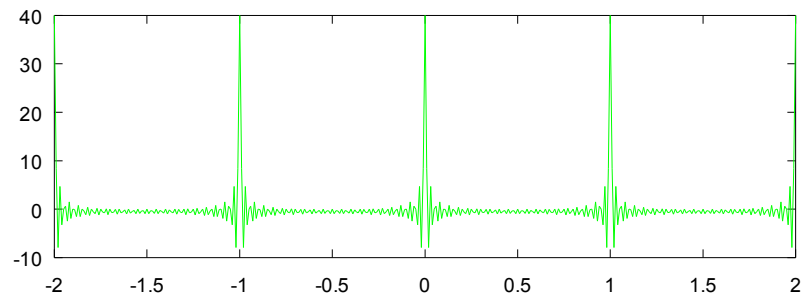
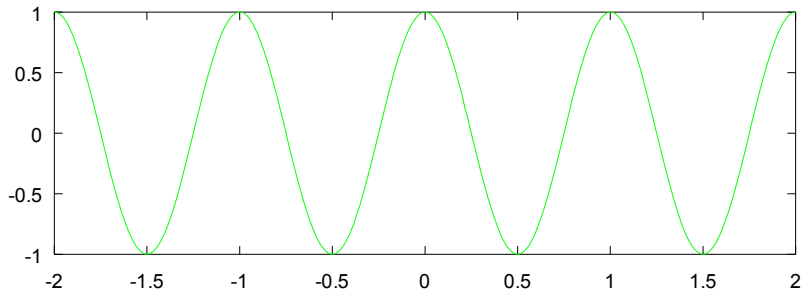
$$\omega_s = \frac{2\pi}{T_s}$$



CTFS of Impulse Train (2)

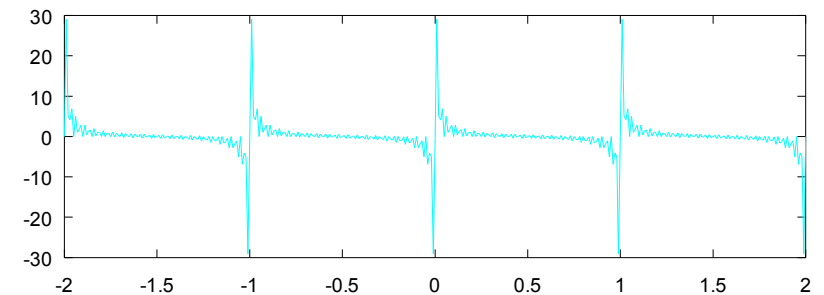
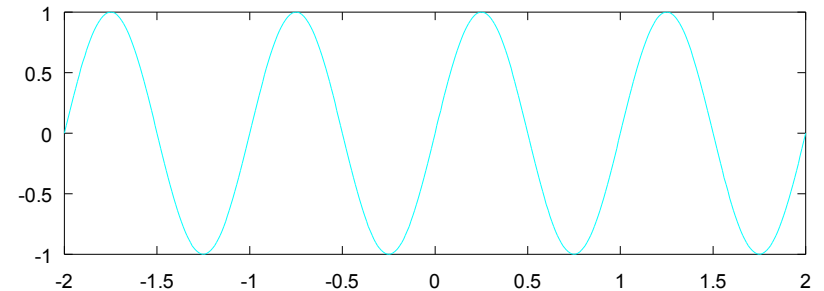
$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} (\cos k\omega_s t - j \sin k\omega_s t)$$

$\cos 2\pi \cdot 1 \cdot t$



$$\sum_{k=1}^{40} \cos 2\pi \cdot k \cdot t$$

$\sin 2\pi \cdot 1 \cdot t$

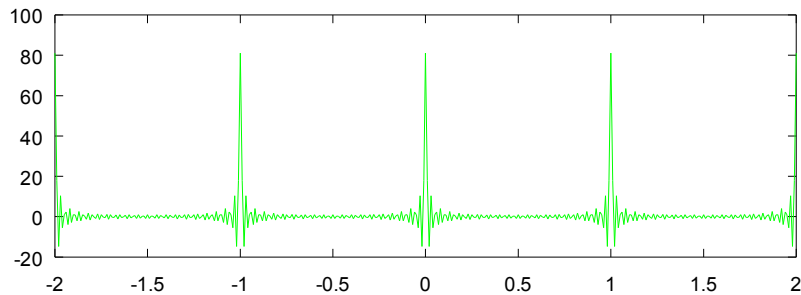
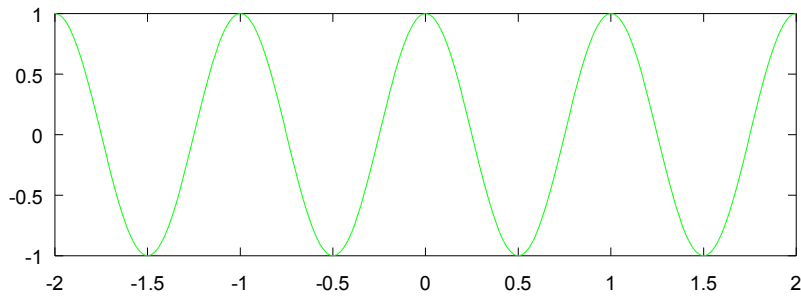


$$\sum_{k=1}^{40} \sin 2\pi \cdot k \cdot t$$

CTFS of Impulse Train (3)

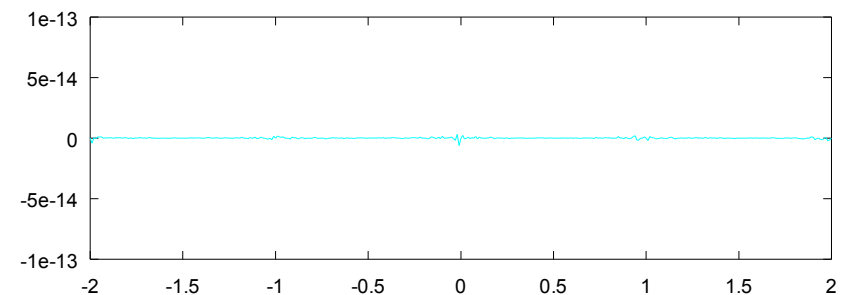
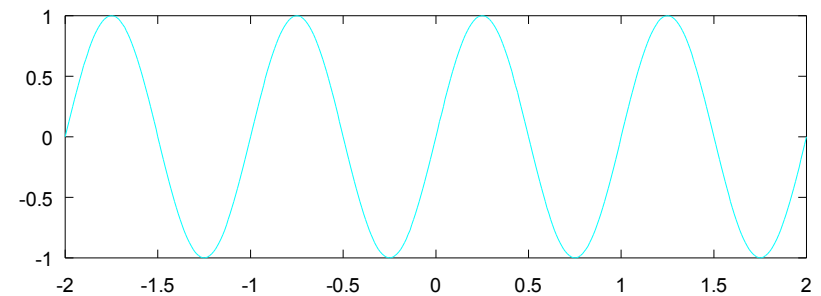
$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} (\cos k\omega_s t - j \sin k\omega_s t)$$

$\cos 2\pi \cdot 1 \cdot t$



$$\sum_{k=-40}^{40} \cos 2\pi \cdot k \cdot t$$

$\sin 2\pi \cdot 1 \cdot t$



$$\sum_{k=-40}^{40} \sin 2\pi \cdot k \cdot t$$

References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] G. Beale, http://teal.gmu.edu/~gbeale/ece_220/fourier_series_02.html
- [4] C. Langton, <http://www.complextoreal.com/chapters/fft1.pdf>