Matched Filter (3B)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Gaussian Random Process

Thermal Noise zero-mean white Gaussian random process

$n(t)$ random function
the value at time t is characterized by
Gaussian probability density function

➡ $z(t) = a + n(t)$

$$p(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{n}{\sigma}\right)^2\right]$$

➡ $$p(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a}{\sigma}\right)^2\right]$$

σ^2 variance of n

$\sigma = 1$ normalized (standardized)
Gaussian function

Central Limit Theorem sum of statistically independent random variables
approaches Gaussian distribution
regardless of individual distribution functions

White Gaussian Noise (1)

$n($

Thermal Noise

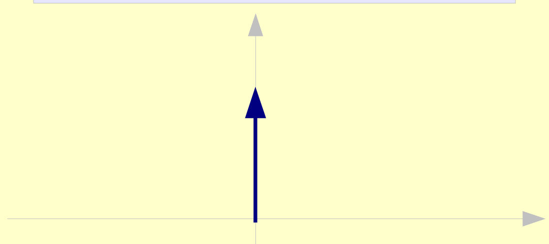
power spectral density is the same for all frequencies

$$G_n(f) = \frac{N_0}{2}$$ watts / hertz

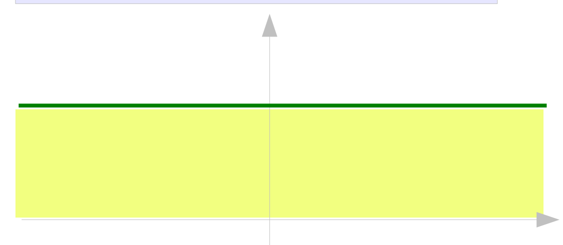
equal amount of noise power per unit bandwidth

uniform spectral density → White Noise

$$R_n(t) = \frac{N_0}{2}\delta(t)$$



$$G_n(f) = \frac{N_0}{2}$$



$\delta(t)$ totally uncorrelated, noise samples are independent

memoryless channel

White Gaussian Noise (2)

Thermal Noise

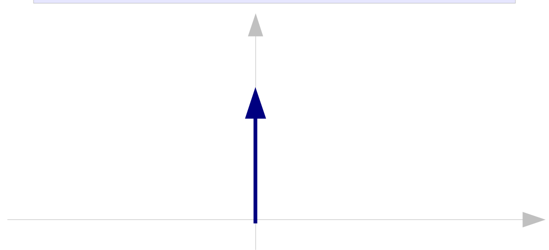
power spectral density is the same for all frequencies

$$G_n(f) = \frac{N_0}{2}$$ watts / hertz

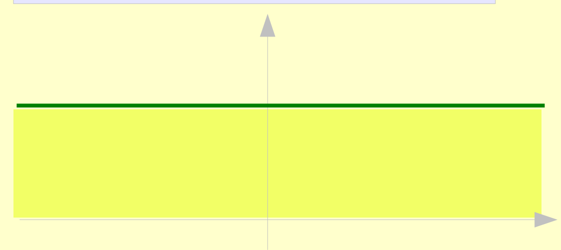
equal amount of noise power per unit bandwidth

uniform spectral density ➡ White Noise

$$R_n(t) = \frac{N_0}{2}\delta(t)$$



$$G_n(f) = \frac{N_0}{2}$$



average power

$$P_x^T = \frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)\,dt = \int_{-\infty}^{+\infty} G_x(f)\,df$$

$$P_n = \int_{-\infty}^{+\infty} \frac{N_0}{2}\,df = \infty$$

White Gaussian Noise (3)

$n($

Additive White Gaussian Noise (AWGN)

additive and no multiplicative mechanism

average power

$P_n = \infty$

$R_n(t) = \frac{N_0}{2}\delta(t)$

$\Updownarrow$

$G_n(f) = \frac{N_0}{2}$

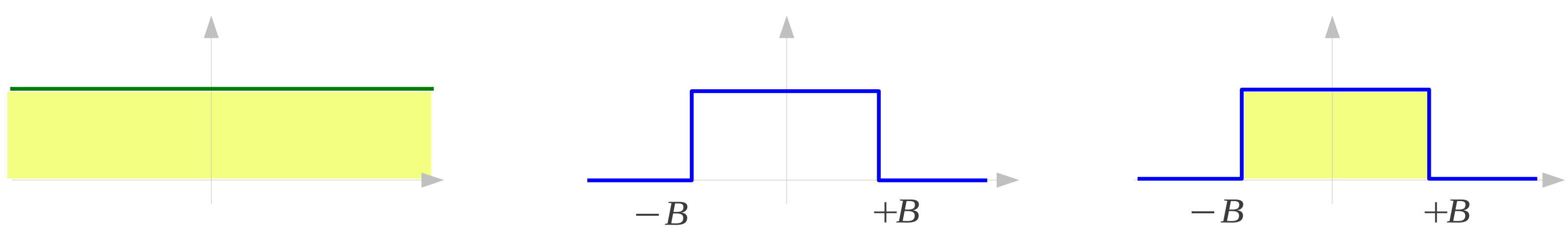
R.V			R.V
$X(t)$	→	$h(t)$	→ $Y(t)$
$R_{xx}(\tau)$	→	$h(t) * h^*(-t)$	→ $R_{yy}(\tau)$
$\Updownarrow$			$\Updownarrow$
$S_{xx}(\omega)$	→	$H(\omega)H^*(\omega)$	→ $S_{yy}(\omega)$

average power

$$P_{n0} = \int_{-B}^{+B} \frac{N_0}{2}\, df$$

$$= \frac{N_0}{2} \cdot 2B$$

$$= N_0 B$$



White Gaussian Noise (4)

$$n_0(t) = n(t) * h(t)$$

$$G_n(f) = \frac{N_0}{2}$$

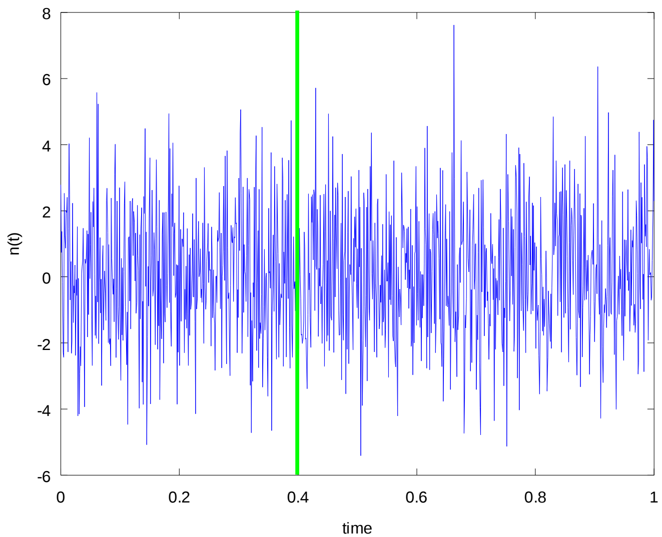
$$G_{n0}(f) = G_n(f)\,|H(f)|^2 = \begin{cases} \frac{N_0}{2}\,|H(f)|^2 & \text{for } |f| < f_u \\ 0 & \text{otherwise} \end{cases}$$

Average output noise power $\quad \sigma_0^2 = \overline{n_0^2(t)} = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H(f)|^2 \, df$

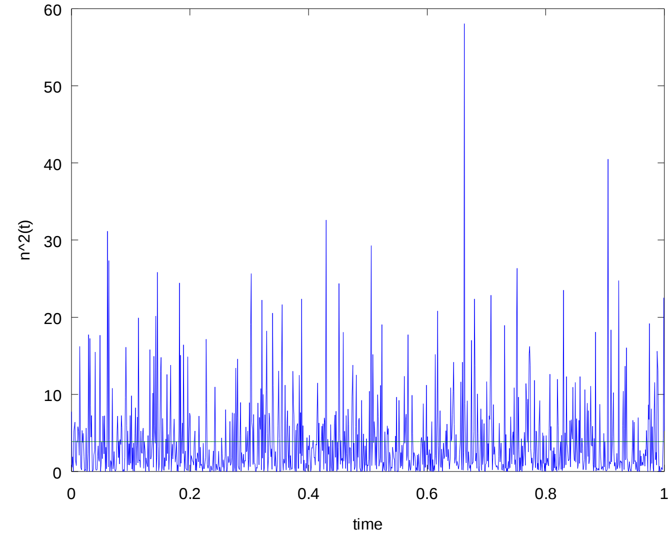
RMS $\quad \sigma_0 = \sqrt{\overline{n_0^2(t)}} = \sqrt{\frac{1}{T} \int_{-T/2}^{+T/2} n_0^2(t)\, dt}$

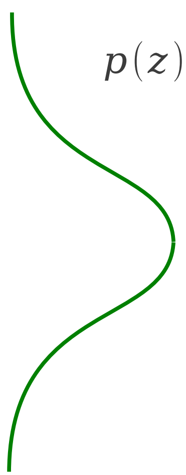
Gaussian Random Process

$n(t)$



$n^2(t)$

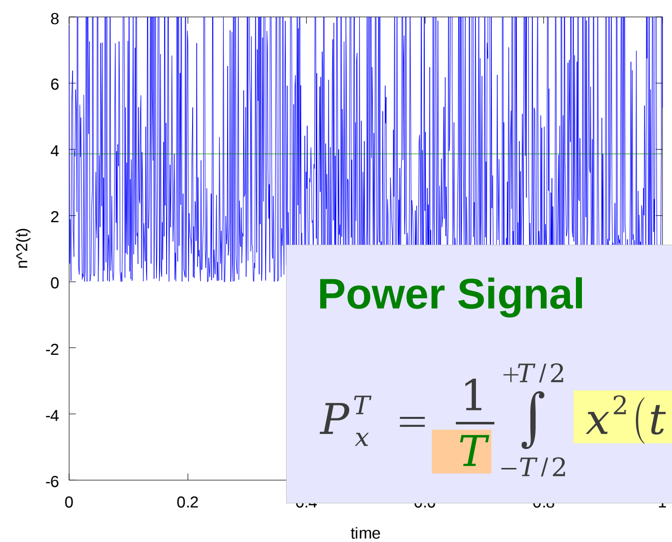




$$p(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2\right]$$

$m = 0$

$\sigma^2 \neq 0$



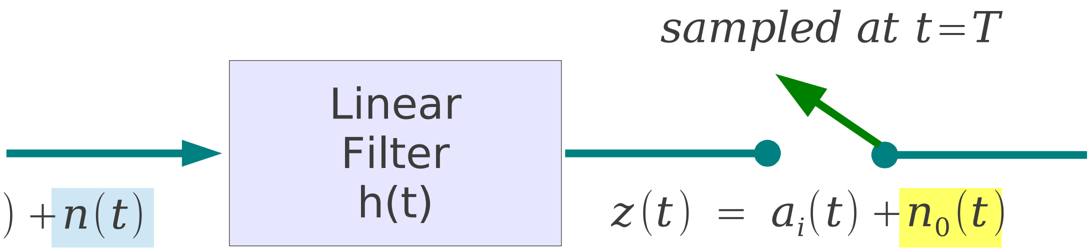
$n^2(t)$

Power Signal

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

Matched Filter (1)

<u>to find</u> a filter h(t) that gives max signal-to-noise ratio



$r(t) = s_i(t) + n(t)$

$z(t) = a_i(t) + n_0(t)$

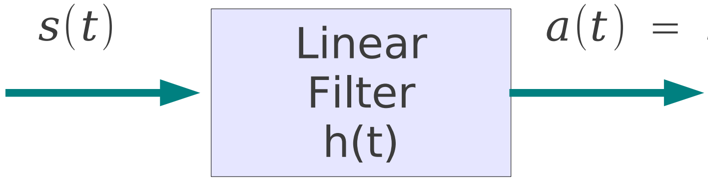
variance of $n_0(t)$ ➡ σ_0^2 avg noise power

$$\frac{a_i^2(T)}{\overline{n_0^2(t)}}$$

$$\frac{\text{instantaneous signal power}}{\text{average noise power}} \Rightarrow \left(\frac{S}{N}\right)_T = \frac{{a_i}^2}{{\sigma_0}^2}$$

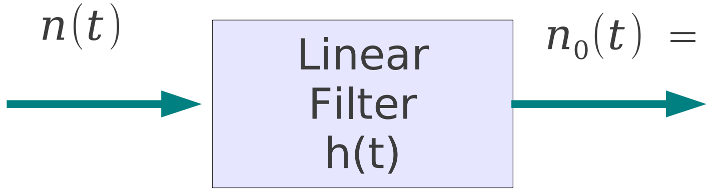
assume $H_0(f)$ a filter transfer function that maximizes $\left(\frac{S}{N}\right)_T$

Matched Filter (2)



$s(t)$ $\qquad a(t) = s(t) * h(t)$

$S(f)$ $\qquad A(f) = S(f)H(f) \quad \Leftrightarrow \quad a(t) = \int_{-\infty}^{+\infty} S(f)H(f) e^{j2\pi f t}\, df$



$n(t)$ $\qquad n_0(t) = n(t) * h(t)$

$$G_n(f) = \frac{N_0}{2}$$

$$G_{n0}(f) = G_n(f)\,|H(f)|^2 = \begin{cases} \frac{N_0}{2}|H(f)|^2 & for\ |f| < f_u \\ 0 & otherwise \end{cases}$$

Average output noise power $\qquad \sigma_0 = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H(f)|^2\, df$

Matched Filter (3)

instantaneous signal power a_i^2 ← $a(t) = \int_{-\infty}^{+\infty} S(f)H(f)e^{j2\pi f t}\, df$

average output noise power $\sigma_0 = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H(f)|^2\, df$

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{\left|\int_{-\infty}^{+\infty} H(f)S(f)e^{+j2\pi f T}\, df\right|^2}{N_0/2 \int_{-\infty}^{+\infty} |H(f)|^2\, df}$$

Does not depend on the particular shape of the waveform

Cauchy Schwarz's Inequality

$$\left|\int_{-\infty}^{+\infty} f_1(x) f_2(x)\, dx\right|^2 \le \int_{-\infty}^{+\infty} |f_1(x)|^2\, dx \int_{-\infty}^{+\infty} |f_2(x)|^2\, dx$$

'=' *holds when* $f_1(x) = k f_2^*(x)$

$$\left|\int_{-\infty}^{+\infty} H(f)S(f)e^{+j2\pi ft}\, dx\right|^2 df \le \int_{-\infty}^{+\infty} |H(f)|^2\, df \int_{-\infty}^{+\infty} |S(f)e^{+j2\pi f T}|^2\, df$$

$$|e^{+j2\pi f T}| = 1$$

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{\left|\int_{-\infty}^{+\infty} H(f)S(f)e^{+j2\pi f T}\, df\right|^2}{N_0/2 \int_{-\infty}^{+\infty} |H(f)|^2\, df} \le \frac{\int_{-\infty}^{+\infty} |H(f)|^2\, df \int_{-\infty}^{+\infty} |S(f)e^{+j2\pi f T}|^2\, df}{N_0/2 \int_{-\infty}^{+\infty} |H(f)|^2\, df} = \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2\, df$$

Matched Filter (4)

Two-sided power spectral density of input noise → $\dfrac{N_0}{2}$

Average noise power $\sigma_0 = \dfrac{N_0}{2} \int_{-\infty}^{+\infty} |H(f)|^2 \, df$

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{\left| \int_{-\infty}^{+\infty} H(f) S(f) e^{+j2\pi f T} \, df \right|^2}{N_0/2 \int_{-\infty}^{+\infty} |H(f)|^2 \, df}$$

Cauchy Schwarz's Inequality

$$\left(\frac{S}{N}\right)_T \leq \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 \, df$$

$$max \left(\frac{S}{N}\right)_T = \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 \, df = \frac{2E}{N_0}$$

input signal energy

power spectral density of input noise

does not depend on the particular shape of the waveform

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Matched Filter (5)

$$\left|\int_{-\infty}^{+\infty} H(f)S(f)e^{+j2\pi ft}\,dx\right|^2 df \le \int_{-\infty}^{+\infty}|H(f)|^2\,df \int_{-\infty}^{+\infty}|S(f)e^{+j2\pi fT}|^2\,df$$

$$\left(\frac{S}{N}\right)_T \le \frac{2}{N_0}\int_{-\infty}^{+\infty}|S(f)|^2\,df$$

$$max\left(\frac{S}{N}\right)_T = \frac{2}{N_0}\int_{-\infty}^{+\infty}|S(f)|^2\,df = \frac{2E}{N_0}$$

when complex conjugate relationship exists

$$H(f) = H_0(f) = kS^*(f)e^{-j2\pi fT}$$

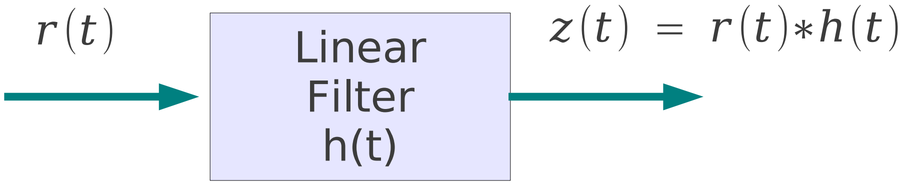
⟺

$$h(t) = h_0(t) = \begin{cases} ks(T-t) & 0 \le t \le T \\ 0 & elsewhere \end{cases}$$

$H_0(f)$ a filter transfer function that maximizes $\left(\frac{S}{N}\right)_T$

impulse response : delayed version of
the mirror image of the signal waveform

Convolution vs. Correlation Realization



$$z(t) = \int_0^t r(\tau) h(t-\tau)\, d\tau$$
$$= \int_0^t r(\tau) s(T-(t-\tau))\, d\tau$$
$$= \int_0^t r(\tau) s(T-t+\tau)\, d\tau$$
$$z(T) = \int_0^T r(\tau) s(\tau)\, d\tau$$

shift position

convolution $z(t) = \int_0^t r(\tau) s(T-t+\tau)\, d\tau$ $\qquad z(T) = \int_0^T r(\tau) s(\tau)\, d\tau$

a sine-wave amplitude modulated
by a linear ramp

correlation $z(t) = \int_0^t r(\tau) s(\tau)\, d\tau$ $\qquad z(T) = \int_0^T r(\tau) s(\tau)\, d\tau$

a linear ramp output

fixed position

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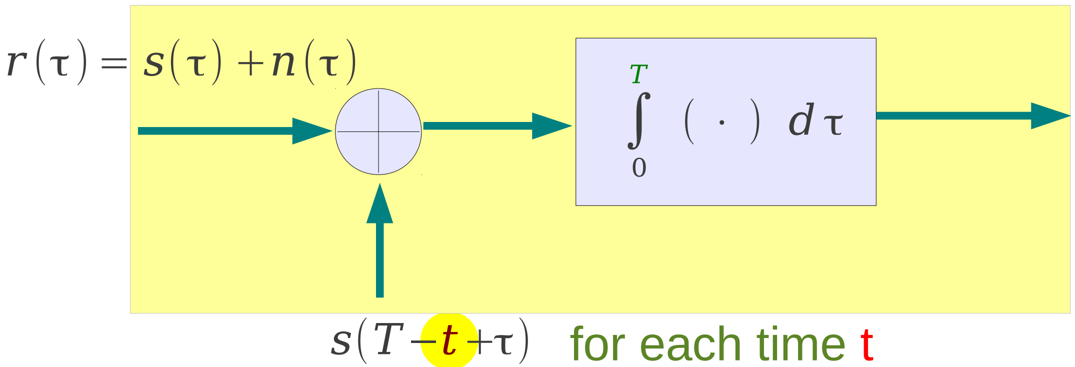
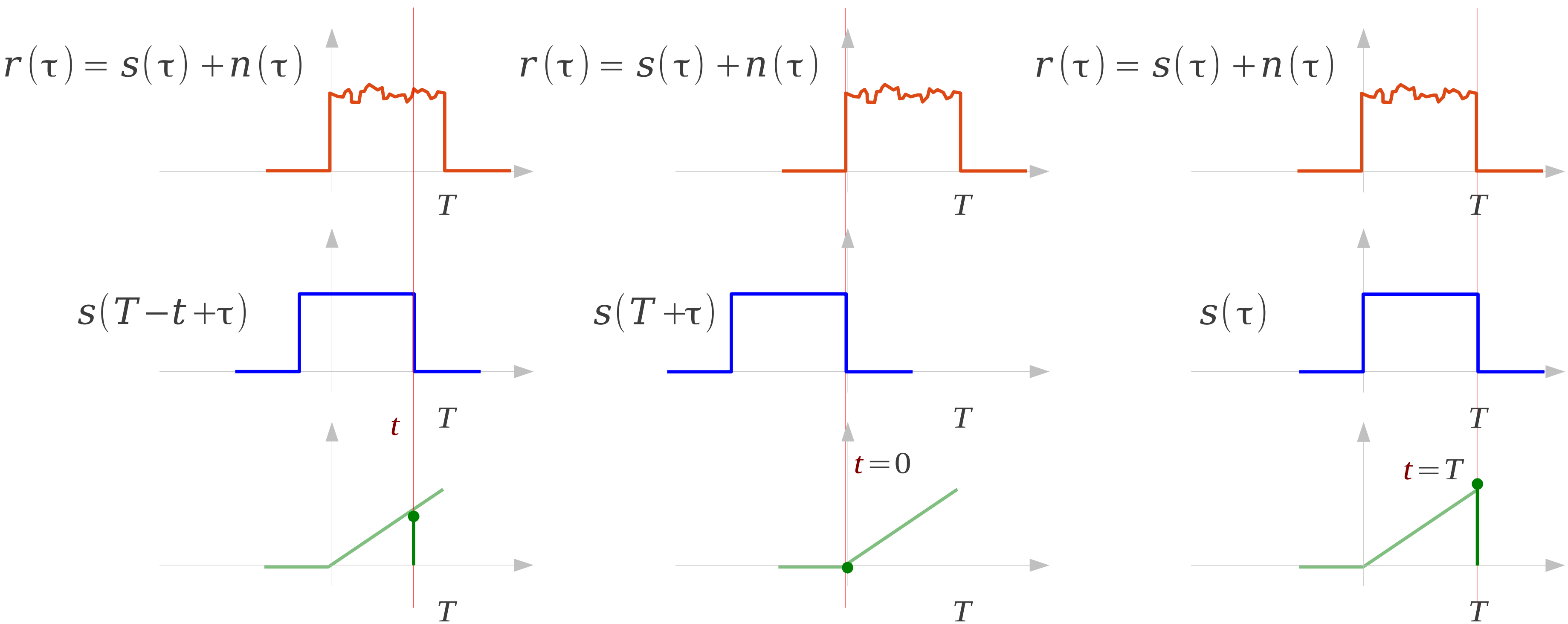
Convolution Realization



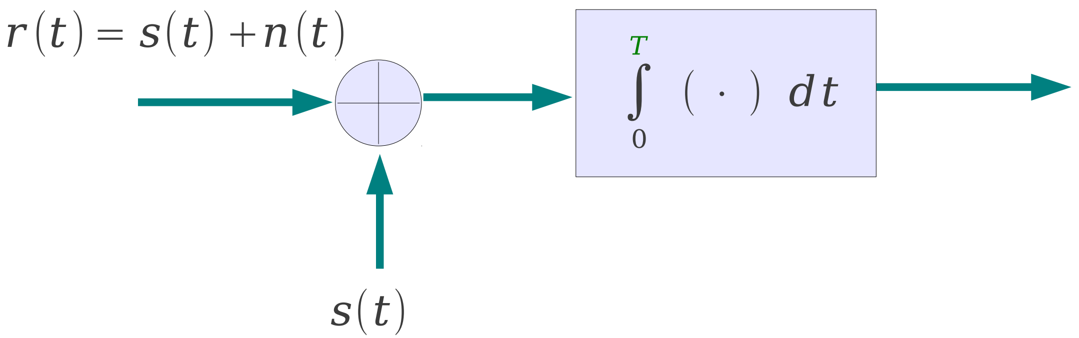
$$z(t) = \int_0^t r(\tau) h(t - \tau)\, d\tau$$

$$= \int_0^t r(\tau) s(T - (t - \tau))\, d\tau$$

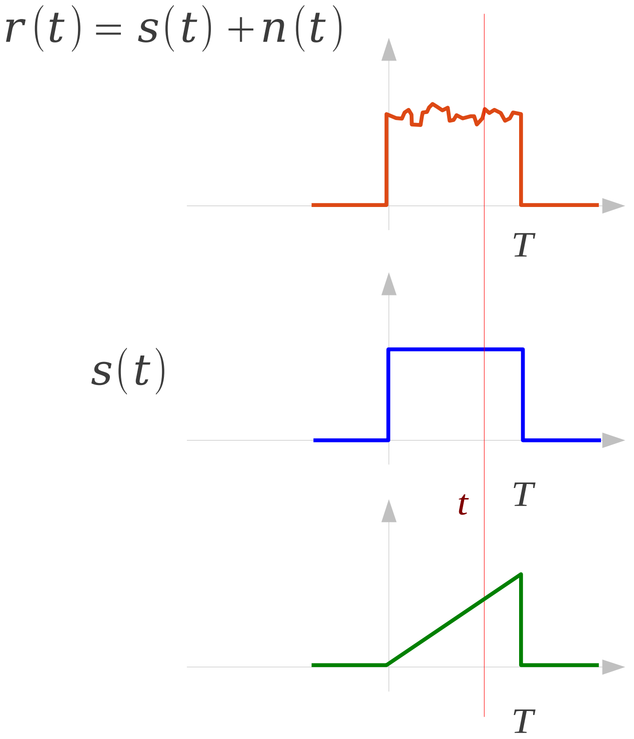
$$= \int_0^t r(\tau) s(T - t + \tau)\, d\tau$$



Correlation Realization (1)

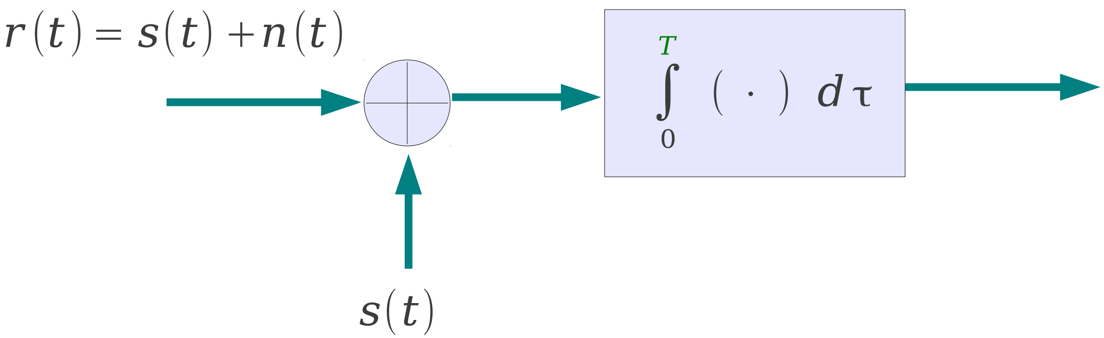


$$z(T) = \int_0^T r(t)s(t)\, dt$$



$$z(t) = \int_0^t r(\tau)s(\tau)\, d\tau$$

Correlation Realization (2)



$$z(T) = \int_0^T r(\tau) s(\tau)\, d\tau$$

$$r(t) = s(t) \qquad z(T) = \int_0^T s^2(\tau)\, d\tau = E$$

$$\sigma_0^2 = \boldsymbol{E}[n_o(t)] = \boldsymbol{E}\left[\int_0^T n(t) s(t)\, dt \int_0^T n(\tau) s(\tau)\, d\tau\right]$$

$$= \boldsymbol{E}\left[\iint_0^T n(t) n(\tau)\, s(t) s(\tau)\, dt\, d\tau\right]$$

$$= \iint_0^T \boldsymbol{E}[n(t) n(\tau)]\, s(t) s(\tau)\, dt\, d\tau$$

$$= \iint_0^T \frac{N_0}{2} \delta(t - \tau)\, s(t) s(\tau)\, dt\, d\tau$$

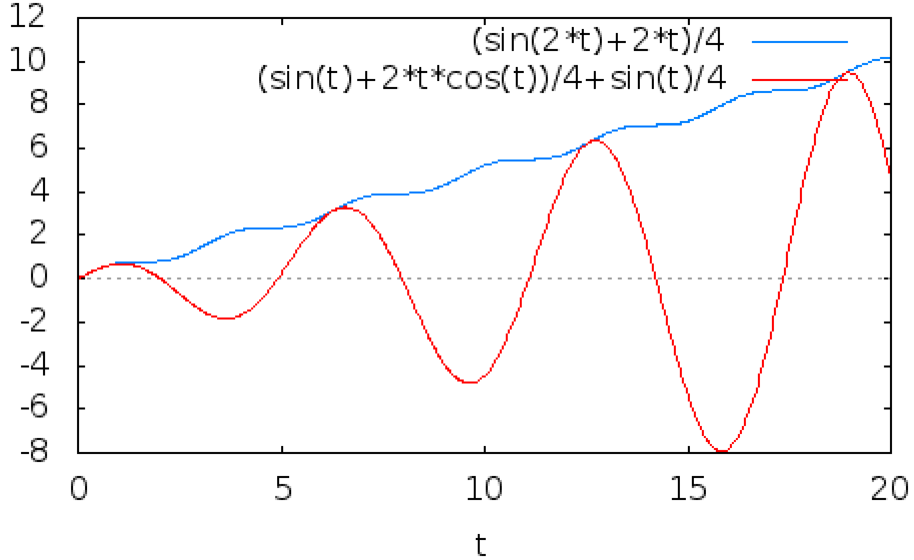
$$= \frac{N_0}{2} \int_0^T s^2(t)\, dt = \frac{N_0}{2} E$$

$$\frac{a_i^2(T)}{\overline{n_0^2(t)}} \qquad \left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2}$$

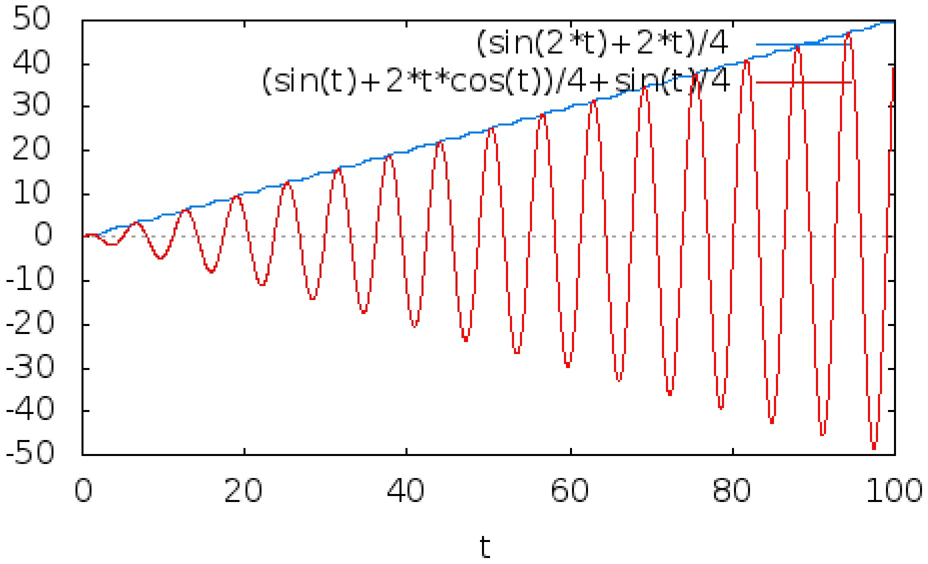
$$max \left(\frac{S}{N}\right)_T = \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2\, df = \frac{2E}{N_0}$$

Correlation and Convolution Examples (1)

correlation
convolution



correlation
convolution



z : integrate(cos(x)*cos(2*%pi - t +x), x, 0, t); convolution

(sin(t)+2*t*cos(t))/4+sin(t)/4

correlation

z : integrate(cos(x)*cos(x), x, 0, t);

(sin(2*t)+2*t)/4

Correlation and Convolution Examples (2)

$$s(t) \quad A\cos(\omega_0 t) \quad 0 \le t < T$$
$$\quad\quad 0 \quad elsewhere$$

$$z(t) = \int_0^t r(\tau)h(t-\tau)\,d\tau$$
$$= \int_0^t r(\tau)s(T-(t-\tau))\,d\tau$$
$$= \int_0^t r(\tau)s(T-t+\tau)\,d\tau$$

$$z(t) = \int_0^t r(\tau)s(T-t+\tau)\,d\tau$$

$$when \;\; r(t) = s(t)$$

$$z(t) = \int_0^t s(\tau)s(T-t+\tau)\,d\tau$$
$$= A^2\int_0^t \cos(\omega_0\tau)\cos(\omega_0(T-t+\tau))\,d\tau$$
$$= \frac{A^2}{2}\int_0^t \cos(\omega_0(T-t)) + \cos(\omega_0(T-t+2\tau))\,d\tau$$
$$= \frac{A^2}{2}\left[\cos(\omega_0(T-t)) - \frac{1}{2\omega_0}\sin(\omega_0(T-t+2\tau))\right]_0^T$$
$$= \frac{A^2}{2}\left[\cos(\omega_0(T-t))\tau - \frac{1}{2\omega_0}\sin(\omega_0(T-t+2\tau))\right]_0^T$$

$$= \frac{A^2}{2}\left[T - \frac{1}{2\omega_0}\sin(\omega_0(T-t+2\tau))\right]_0^T$$

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] B. Sklar, “Digital Communications: Fundamentals and Applications”