Signals and Spectra (1A)

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Energy and Power

Instantaneous Power

$$p(t) = x^2(t)$$ real signal

Energy dissipated during

$$(-T/2,\ +T/2)$$

$$E_x^T = \int_{-T/2}^{+T/2} x^2(t)\,dt$$

Affects the performance
of a communication system

Average power dissipated during

$$(-T/2,\ +T/2)$$

$$P_x^T = \frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)\,dt$$

The rate at which energy is dissipated

Determines the voltage

Energy and Power Signals (1)

Energy dissipated during

$$(-T/2,\ +T/2)$$

$$E_x^T = \int_{-T/2}^{+T/2} x^2(t)\,dt$$

Average power dissipated during

$$(-T/2,\ +T/2)$$

$$P_x^T = \frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)\,dt$$

Energy Signal

Nonzero but finite energy

$0 < E_x < +\infty$ for all time

$$E_x = \lim_{T\to+\infty}\int_{-T/2}^{+T/2} x^2(t)\,dt$$

$$= \int_{-\infty}^{+\infty} x^2(t)\,dt < +\infty$$

Power Signal

Nonzero but finite power

$0 < P_x < +\infty$ for all time

$$P_x = \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)\,dt$$

$$< +\infty$$

Energy and Power Signals (2)

Energy Signal

Nonzero but finite energy

$0 < E_x < +\infty$ for all time

$$E_x = \lim_{T \to +\infty} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

$$= \int_{-\infty}^{+\infty} x^2(t)\, dt < +\infty$$

Power Signal

Nonzero but finite power

$0 < P_x < +\infty$ for all time

$$P_x = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

$$< +\infty$$

$$P_x = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

$$= \lim_{T \to +\infty} \frac{B}{T} \to 0$$

$$E_x = \lim_{T \to +\infty} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

$$= \lim_{T \to +\infty} B \cdot T \to +\infty$$

Non-periodic signals
Deterministic signals

Periodic signals
Random signals

Energy and Power Spectral Densities (1)

Total Energy, Non-periodic

$$E_x^T = \int_{-\infty}^{+\infty} x^2(t)\, dt$$

Average power, Periodic

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

Parseval's Theorem, Non-periodic

$$= \int_{-\infty}^{+\infty} |X(f)|^2\, df$$

$$= \int_{-\infty}^{+\infty} \Psi(f)\, df$$

$$= 2\int_{0}^{+\infty} \Psi(f)\, df$$

Parseval's Theorem, Periodic

$$= \sum_{n=-\infty}^{+\infty} |c_n|^2$$

$$= \int_{-\infty}^{+\infty} G_x(f)\, df$$

$$= 2\int_{0}^{+\infty} G_x(f)\, df$$

Energy Spectral Density

$$\Psi(f) = |X(f)|^2$$

Power Spectral Density

$$G_x(f) = \sum_{n=-\infty}^{+\infty} |c_n|^2\, \delta(f - n f_0)$$

Energy and Power Spectral Densities (2)

Energy Spectral Density

$$\Psi(f) = |X(f)|^2$$

Power Spectral Density

$$G_x(f) = \sum_{n=-\infty}^{+\infty} |c_n|^2 \, \delta(f - n f_0)$$

Total Energy, Non-periodic

$$E_x^T = \int_{-\infty}^{+\infty} x^2(t)\, dt$$

$$= \int_{-\infty}^{+\infty} \Psi(f)\, d f$$

Parseval's Theorem, Non-periodic

Average power, Periodic

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

$$= \int_{-\infty}^{+\infty} G_x(f)\, d f$$

Parseval's Theorem, Periodic

Non-periodic power signal
(having infinite energy) ?

Energy and Power Spectral Densities (3)

Power Spectral Density

$$G_x(f) = \lim_{T \to \infty} \frac{1}{T} |X_T(f)|^2$$

Non-periodic power signal
(having infinite energy) ?

→ No Fourier Series

truncate $\left(-\frac{T}{2} \le t \le +\frac{T}{2}\right)$

$x(t) \Rightarrow x_T(t)$

→ Fourier Transform $X_T(f)$

$$P_x^T = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

$$= \int_{-\infty}^{+\infty} \lim_{T \to \infty} \frac{|X(f)|^2}{T}\, df$$

Power Spectral Density

$$G_x(f) = \sum_{n=-\infty}^{+\infty} |c_n|^2\, \delta(f - n f_0)$$

Average power, Periodic

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

$$= \int_{-\infty}^{+\infty} G_x(f)\, df$$

Parseval's Theorem, Periodic

Autocorrelation of Energy and Power Signals

Autocorrelation of an Energy Signal

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau)\,dt$$

$$(-\infty \le \tau \le +\infty)$$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \le R_x(0)$$

$$R_x(\tau) \Leftrightarrow \Psi(f)$$

$$R_x(0) = \int_{-\infty}^{+\infty} x^2(t)\,dt$$

Autocorrelation of a Power Signal

$$R_x(\tau) = \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{+T/2} x(t)x(t+\tau)\,dt$$

$$(-\infty \le \tau \le +\infty)$$

Autocorrelation of a Periodic Signal

$$R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x(t)x(t+\tau)\,dt$$

$$(-\infty \le \tau \le +\infty)$$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \le R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x^2(t)\,dt$$

Ensemble Average

Random Variable

$$m_x = \boldsymbol{E}\{X\}$$

$$= \int_{-\infty}^{+\infty} x\, p_X(x)\, dx$$

$$\boldsymbol{E}\{X^2\} = \sigma_x^2 + m_x^2$$

$$= \int_{-\infty}^{+\infty} x^2\, p_X(x)\, dx$$

Random Process

$$m_x(t_k) = \boldsymbol{E}\{X(t_k)\}$$

$$= \int_{-\infty}^{+\infty} x\, p_{X_k}(x)\, dx$$

for a given time t_k

$$R_x(t_1,\ t_2) = \boldsymbol{E}\{X(t_1)\, X(t_2)\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2\, p_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2$$

WSS (Wide Sense Stationary)

Random Process

$$m_x(t_k) = \boldsymbol{E}\{X(t_k)\}$$

$$= \int_{-\infty}^{+\infty} x\, p_{X_k}(x)\, dx$$

for a given time t_k

WSS Process by ensemble average

$$m_x(t_k) = \boldsymbol{E}\{X(t_k)\}$$

$$= m_x$$

constant for all times

Random Process

$$R_x(t_1, t_2) = \boldsymbol{E}\{X(t_1)\, X(t_2)\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 p_{X_1, X_2}(x_1, x_2)\, dx_1 dx_2$$

WSS Process

$$R_x(t_1, t_2) = \boldsymbol{E}\{X(t_1)\, X(t_2)\}$$

$$= R_x(t_1 - t_2)$$

depends on time differences

Ergodicity and Time Averaging

Random Process

$$m_x(t_k) = \boldsymbol{E}\{X(t_k)\}$$

$$= \int_{-\infty}^{+\infty} x\, p_{X_k}(x)\, dx$$

for a given time

$$R_x(t_1, t_2) = \boldsymbol{E}\{X(t_1)\, X(t_2)\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2\, p_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2$$

WSS Process by ensemble average

$$m_x(t_k) = \boldsymbol{E}\{X(t_k)\}$$

$$= m_x$$

$$R_x(t_1, t_2) = \boldsymbol{E}\{X(t_1)\, X(t_2)\}$$

$$= R_x(t_1 - t_2) = R_x(\tau)$$

Ergodic Process by time average

$$m_x(t_k) = \boldsymbol{E}\{X(t_k)\} =$$

$$m_x = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} X(t)\, dt$$

$$R_x(t_1, t_2) = \boldsymbol{E}\{X(t_1)\, X(t_2)\} =$$

$$= \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} X(t)\, X(t+\tau)\, dt$$

Autocorrelation of Power Signals

Autocorrelation of a Random Signal

$$R_x(\tau) = \boldsymbol{E}\{X(t)\,X(t+\tau)\}$$

Autocorrelation of a Power Signal

$$R_x(\tau) = \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{+T/2} x(t)x(t+\tau)\,dt$$

$$(-\infty \le \tau \le +\infty)$$

Autocorrelation of a Periodic Signal

$$R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x(t)x(t+\tau)\,dt$$

$$(-\infty \le \tau \le +\infty)$$

Random signal:

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \le R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \boldsymbol{E}\{X^2(t)\}$$

Power / periodic signal:

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \le R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x^2(t)\,dt$$

Autocorrelation of Random Signals

Autocorrelation of
a Random Signal

$$R_x(\tau) = \boldsymbol{E}\{X(t)\,X(t+\tau)\}$$

$$= \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} X(t)\,X(t+\tau)\,dt$$

if *ergodic* in the
autocorrelation function

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \le R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \boldsymbol{E}\{X^2(t)\}$$

Power Spectral Density of
a Random Signal

$$G_x(f) = \lim_{T \to +\infty} \frac{1}{T} |X_T(f)|^2$$

$$G_x(f) = G_x(-f)$$

$$G_x(f) \ge 0$$

$$G_x(f) \Leftrightarrow R_x(\tau)$$

$$P_x(0) = \int_{-\infty}^{+\infty} G_X(f)\,df$$

Ergodic Random Process

$m_X = \boldsymbol{E}\{X(t)\}$ DC level

m_X^2 normalized power in the dc component

$\boldsymbol{E}\{X^2(t)\}$ total average normalized power (mean square value)

$\sqrt{\boldsymbol{E}\{X^2(t)\}}$ rms value of voltage or current

σ_X^2 average normalized power in the ac component

σ_X

$m_x = m_X^2 = 0$ ➡ $\sigma_X^2 = \boldsymbol{E}\{X^2\}$ var = total average normalized power
= mean square value (rms^2)

σ_X rms value of the ac component

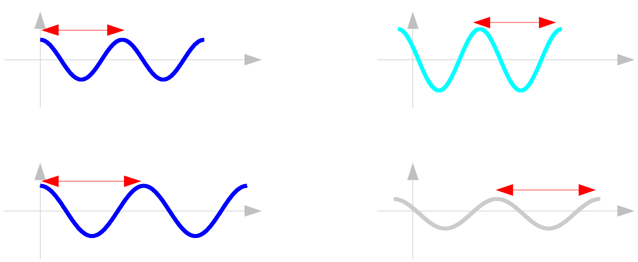
$m_X = 0$ rms value of the signal

Single Tone Input

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$x(t) = A e^{j\Phi} e^{j\omega t}$ → $h(t)$ → $y(t)$

$$\begin{cases} amplitude = A \\ phase = \Phi \\ frequency = \omega \end{cases}$$



$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} h(\tau)\, A e^{j\Phi} e^{j\omega(t-\tau)}\, d\tau \\ &= \int_{-\infty}^{+\infty} h(\tau)\, A e^{j\Phi} e^{j\omega t} e^{-j\omega\tau}\, d\tau \\ &= A e^{j\Phi} e^{j\omega t} \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau \\ &= A e^{j\Phi} e^{j\omega t}\, H(j\omega) \\ &= A e^{j\Phi} e^{j\omega t}\, \rho e^{j\theta} \end{aligned}$$

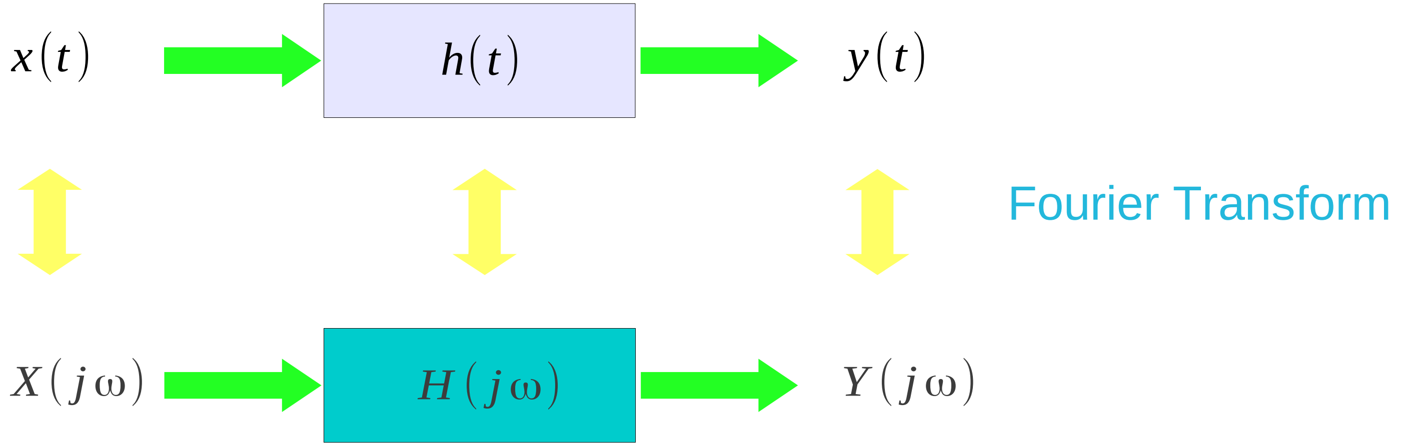
Fourier Transform

complex number given ω and t

changes the amplitude and the phase of the single tone input but *not the frequency*

Impulse Response & Frequency Response

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$



$$Y(j\omega) = H(j\omega) X(j\omega)$$

$$Y(j\omega) = \int_{-\infty}^{+\infty} y(\tau)\, e^{-j\omega\tau}\, d\tau \qquad H(j\omega) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

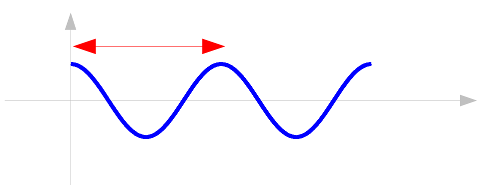
Linear System

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

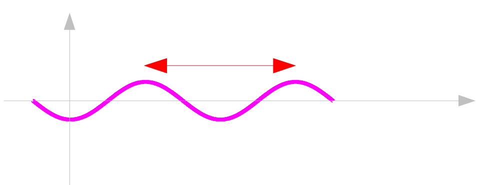
$x(t) \rightarrow$ $h(t)$ $\rightarrow y(t)$

$\delta(t) \rightarrow$ $h(t)$ $\rightarrow h(t)$

$A\, e^{j\Phi}\, e^{jwt} \rightarrow$ $h(t)$ $\rightarrow H(j\omega)\, A\, e^{j\Phi}\, e^{j\omega t}$

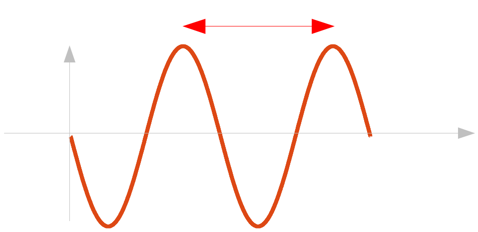


single frequency component : ω



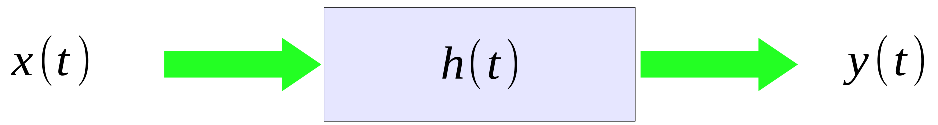
Frequency Response

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$



Transfer Function & Frequency Response

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$



Transfer Function — **Laplace Transform**

$$H(s) = H(\sigma + j\omega)$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

initial state
differential equation

$$\sigma = 0$$

Frequency Response — **Fourier Transform**

$$H(\omega) = H(j\omega)$$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau$$

steady state
frequency response

Linear System & Random Variables (1)

R.V $X(t)$ → $h(t)$ → R.V $Y(t)$

mean $E[X(t)]$ → $h(t)$ → $E[Y(t)] = \int_{-\infty}^{+\infty} E[X(\tau)]h(t-\tau)\,d\tau$

correlation $E[X(t_1)X(t_2)]$ → $h(t) * h^*(-t)$ → $E[Y(t_1)Y(t_2)]$

Fourier Transform

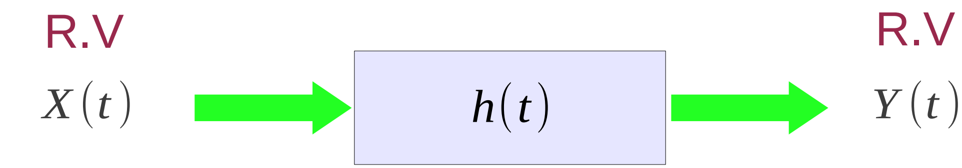
WSS

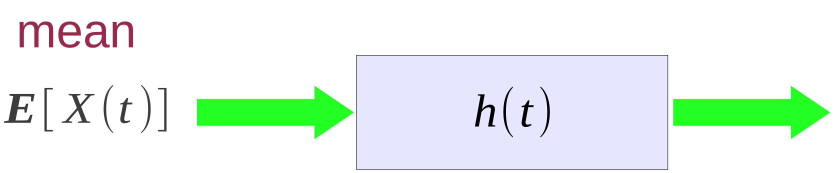
$S_{xx}(\omega)$ → $H(\omega)H^*(\omega)$ → $S_{YY}(\omega)$ → $= |H(\omega)|^2 S_{XX}(\omega)$

$= \int_{-\infty}^{+\infty} R_{xx}(\tau)e^{-j\omega\tau}\,d\tau$

$= \int_{-\infty}^{+\infty} R_{YY}(\tau)e^{-j\omega\tau}\,d\tau$

Linear System & Random Variables (2)





$$E[Y(t)] = \int_{-\infty}^{+\infty} E[X(\tau)]h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)E[X(t-\tau)]\,d\tau$$

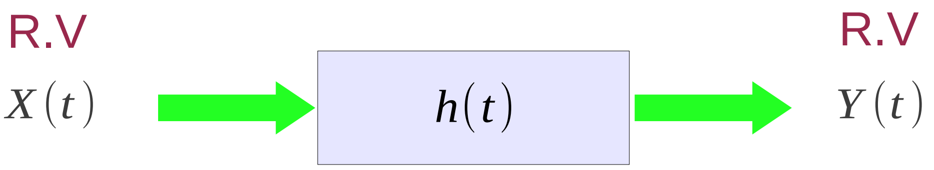
Ergodic WSS

$$E[X(t)] = m_x = \overline{X(t)} \qquad E[Y(t)] = m_y = \overline{Y(t)}$$

$$E[Y(t)] = \int_{-\infty}^{+\infty} h(\tau)E[X(t-\tau)]\,d\tau$$

$$\overline{Y(t)} = \int_{-\infty}^{+\infty} h(\tau)\overline{X(t)}\,d\tau = \overline{X(t)} \int_{-\infty}^{+\infty} h(\tau)\,d\tau = \overline{X(t)}\,H(0)$$

Linear System & Random Variables (3)



$$\rho(t) = h(t) * h^*(-t)$$

$$|H(\omega)|^2 = H(\omega) H^*(\omega)$$

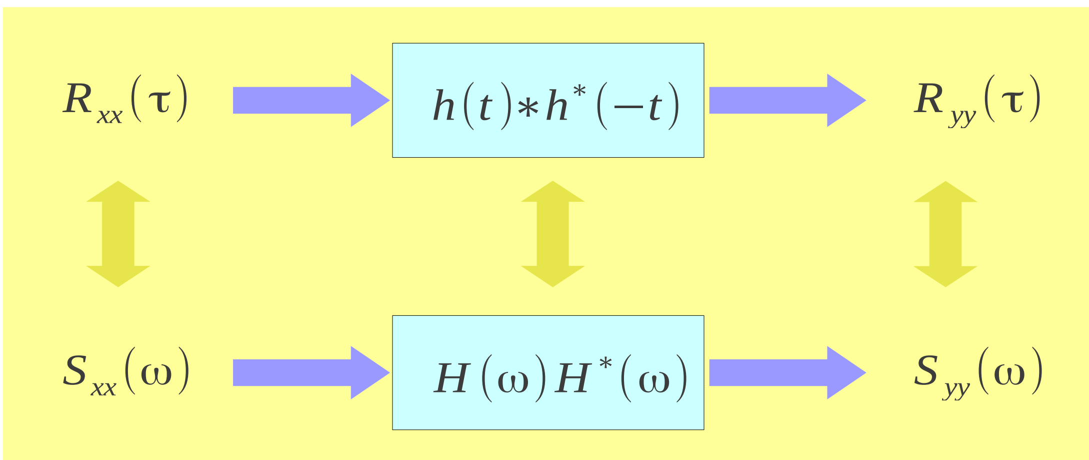
$$R_{xy}(\tau) = R_{xx}(\tau) * h^*(-\tau) \qquad R_{yy}(\tau) = R_{xy}(\tau) * h(\tau)$$

$$S_{xy}(\omega) = S_{xx}(\omega) H^*(\omega) \qquad S_{yy}(\omega) = S_{xy}(\omega) H(\omega)$$

$$R_{yy}(\tau) = R_{xx}(\tau) * \rho(\tau)$$

$$S_{yy}(\omega) = S_{xx}(\omega) |H(\omega)|^2$$

can be viewed as follows



Fourier Transform

Summary (1)

Non-periodic signals

Energy Signal

$$E_x^T = \int_{-T/2}^{+T/2} x^2(t)\, dt$$

Energy Spectral Density

$$\Psi(f) = |X(f)|^2$$

Total Energy

$$\int_{-\infty}^{+\infty} \Psi(f)\, df$$

Periodic signals

Power Signal

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

Power Spectral Density

$$G_x(f) = \sum_{n=-\infty}^{+\infty} |c_n|^2\, \delta(f - n f_0)$$

Average Power

$$\int_{-\infty}^{+\infty} G_x(f)\, df$$

Random signals

Power Signal

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

Power Spectral Density

$$G_x(f) = \lim_{T \to \infty} \frac{1}{T} |X_T(f)|^2$$

Average Power

$$\int_{-\infty}^{+\infty} G_x(f)\, df$$

Summary (2)

Energy Signal Autocorrelation

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau)\,dt$$

Power Signal Autocorrelation

$$R_x(\tau) = \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{+T/2} x(t)x(t+\tau)\,dt$$

Random Signal Autocorrelation

$$R_x(\tau) = \boldsymbol{E}\{X(t)\,X(t+\tau)\}$$

Non-periodic signals

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau)\,dt$$

Periodic signals

for a Periodic Signal

$$R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x(t)x(t+\tau)\,dt$$

Random signals

for a Ergodic Signal

$$R_x(\tau) = \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{+T/2} X(t)\,X(t+\tau)\,dt$$

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References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] B. Sklar, “Digital Communications: Fundamentals and Applications”