

# DLTI Difference Equation

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# Causal LTI Systems (1)

$$a_N y[n-N] + \dots + a_1 y[n-1] + a_0 y[n] = b_M x[n-M] + \dots + b_1 x[n-1] + b_0 x[n]$$

$$\begin{aligned} & y[n+N] + a_1 y[n+N-1] + \dots + a_{N-1} y[n+1] + a_N y[n] \\ = & b_{N-M} x[n+M] + b_{N-M+1} x[n+M-1] + \dots + b_{N-1} x[n+1] + b_N x[n] \end{aligned}$$

$$\begin{aligned} & y[n+N] + a_1 y[n+N-1] + \dots + a_{N-1} y[n+1] + a_N y[n] \\ = & b_0 x[n+M] + b_1 x[n+M-1] + \dots + b_{N-1} x[n+1] + b_N x[n] \end{aligned}$$

$$y[n] + a_1 y[n-1] + \dots + a_{N-1} y[n-N+1] + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \dots + b_{N-1} x[n-N+1] + b_N x[n-N]$$

$$(E^N + a_1 E^{N-1} + \dots + a_{N-1} E + a_N) y(t) = (b_0 E^M + b_{N-M+1} E^{M-1} + \dots + b_{N-1} E + b_N) x(t)$$

$$Q(E)y(t) = P(E)x(t)$$

# Closed Form $h[n]$ (1)

$$y[n] + a_1 y[n-1] + \dots + a_{N-1} y[n-N+1] + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \dots + b_{N-1} x[n-N+1] + b_N x[n-N]$$

$$(E^N + a_1 E^{N-1} + \dots + a_{N-1} E + a_N) y(t) = (b_0 E^M + b_{N-M+1} E^{M-1} + \dots + b_{N-1} E + b_N) x(t)$$

$$Q(E)y(t) = P(E)x(t)$$

**$h[n]$  : system response to input  $\delta[n]$**

*When  $n < 0$ ,  $h[n] = 0$*

*When  $n > 0$ ,  $h[n]$  must be made up of characteristic modes*

*When the input is zero, only the characteristic modes can be sustained*

*When  $n = 0$ , it may have non-zero value  $A_0$*

$$h[n] = A_0 \delta[n] + y_c[n] u[n]$$

 *linear combination of the characteristic modes*

# Closed Form $h[n]$ (2)

$$y[n] + a_1 y[n-1] + \dots + a_{N-1} y[n-N+1] + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \dots + b_{N-1} x[n-N+1] + b_N x[n-N]$$

$$(E^N + a_1 E^{N-1} + \dots + a_{N-1} E + a_N) y(t) = (b_0 E^M + b_{N-M+1} E^{M-1} + \dots + b_{N-1} E + b_N) x(t)$$

$$Q(E)y(t) = P(E)x(t)$$

$$Q(E)y(t) = P(E)x(t) \quad \Rightarrow \quad Q(E)h(t) = P(E)\delta(t) \quad \text{causal } h[n]$$

$$h[n] = \frac{A_0 \delta[n] + y_c[n] u[n]}{\quad} \quad \text{initial condition} \quad h[-1] = h[-2] = \dots = h[-N] = 0$$

$$\downarrow$$

$$Q(E) \left( \frac{A_0 \delta[n] + y_c[n] u[n]}{\quad} \right) = P(E)\delta(t) \quad \text{y}_c \text{ is made up of characteristic modes}$$

$$\leftarrow Q(E)(y_c[n] u[n]) = 0$$

$$\downarrow$$

$$Q(E)(A_0 \delta[n]) = P(E)\delta(t)$$

$$A_0(\delta[n] + a_1 \delta[n-1] + \dots + a_{N-1} \delta[n-N+1] + a_N \delta[n-N]) = b_0 \delta[n] + b_1 \delta[n-1] + \dots + b_{N-1} \delta[n-N+1] + b_N \delta[n-N]$$

$$n=0 \quad A_0 a_N = b_N \quad A_0 = \frac{a_N}{b_N}$$

$$h[n] = \frac{b_N}{a_N} \delta[n] + y_c[n] u[n]$$

# Example (1)

$$y[n+2] - 0.6y[n+1] - 0.16y[n] = 5x[n+2]$$

$$(E^2 - 0.6E - 0.16)y[n] = 5E^2x[n]$$

initial condition  $y[-1] = 0, y[-2] = \frac{25}{4}$

input  $x[n] = 4^{-n}u[n]$

Characteristic polynomial

$$\gamma^2 - 0.6\gamma - 0.16 = (\gamma + 0.2)(\gamma - 0.8)$$

Characteristic Equation  $(\gamma + 0.2)(\gamma - 0.8) = 0$

Characteristic Roots  $\gamma = -0.2, \gamma = 0.8$

Zero Input Response  $y_0[n]$

$$y_0[n] = C_1(-0.2)^n + C_2(0.8)^n$$



$$y_0[n] = \frac{1}{5}(-0.2)^n + \frac{4}{5}(0.8)^n$$

$$y_0[-1] = -5C_1 + \frac{5}{4}C_2 = 0 \quad C_1 = \frac{1}{5}$$

$$y_0[-2] = 25C_1 + \frac{25}{16}C_2 = \frac{25}{4} \quad C_2 = \frac{4}{5}$$

## References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] B.P. Lathi, Linear Systems and Signals (2<sup>nd</sup> Ed)